Math 3385: Homework #6 answers

#4.22

There are 4 different topologies:

1. $\{\emptyset, \{a, b\}\}$ (the trivial topology)
2. $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ (the discrete topology)
3. $\{\emptyset, \{a\}, \{a, b\}\}$ (the discrete topology)
4. $\{\emptyset, \{b\}, \{a, b\}\}$ (the discrete topology)

The last two are homeomorphic using a function that carries a to b and b to a. The other two are not homeomorphic to any of the others.

#4.25a

I want a function whose graph goes down to $-\infty$ on the left side, but increases to a horizontal asymptote at $y = a$ on the right side. A good first attempt is $-e^{-x}$, which does what we want but approaches the x-axis as a horizontal asymptote. To get it to approach $y = a$, use $f(x) = -e^{-x} + a$.

#5.3b

Let's looks at some ball $B(p, \epsilon)$, where $p = (p_1, p_2) \in \mathbb{R}^2$.

First we'll think about when $\epsilon \leq 1$. All points having different x-coordinate from p will have their distance equal to 1, so these points are outside of the ball. So the ball contains ONLY those points with the same x-coordinate as p, whose y-coordinate is within ϵ of p. This ball is the same as a small basis neighborhood in the "vertical line topology" on $\mathbb{R}^2$.

If $\epsilon > 1$, then the ball $B(p, \epsilon)$ is all of $\mathbb{R}^2$. This is weird, but it is clear from the definition that $d(p, q) \leq 1$ for all p and q. So if $\epsilon > 1$, then all points are within distance ϵ from p, so $B(x, \epsilon) = \mathbb{R}^2$.

#5.16

Positive definite: The Hamming distance is the number of places where the two words differ, so it is always greater or equal to zero. For the other part, note that $D_H(x, y) = 0$ means that x and y differ in zero places, so $x = y$.

Symmetric: From the definition it's clear that $D_H(x, y) = D_H(y, x)$. This is just the number of places that the two words differ.

Triangle inequality: Let $x, y, z \in V^n$ be words, and we want to show that

$$D_H(x, z) \leq D_H(x, y) + D_H(y, z).$$

The right side is the number of differences between x and y, plus the number of differences between y and z. This must be bigger than the number of differences between x and z, since any difference between x to z must be accounted as either a difference between x and y, or between y and z.