Math 3385: Homework #7 answers

#5.25a

In $\mathbb{R}$, let $A = (0, 1)$. Then we have $d(\{0\}, A) = 0$ even though $0 \notin A$.

#5.30

Remember that the definition of injective is: f is injective when $f(a) = f(b)$ implies $a = b$.

So assume for some $a, b \in X$ that $f(a) = f(b)$, and we will show that $a = b$. Since $f(a) = f(b)$ we have $d_Y(f(a), f(b)) = 0$, and by our assumption this means that $d_X(a, b) = d_Y(f(a), f(b)) = 0$. And since $d_X(a, b) = 0$ this means that $a = b$ as desired.

#6.3

We'll prove the contrapositive, which would be: "X is disconnected if and only if there is some nonempty proper subset of X with empty boundary."

$\Rightarrow$: Assume that X is disconnected, and we'll find a nonempty proper subset with empty boundary. Let $X = U \cup V$ be a separation, and we'll show that U is the set we're looking form. Since U and V make a separation, U is a proper nonempty clopen subset. This means that U equals its closure (since it's closed), and also equals its interior (since it's open). Thus $\partial U = \bar{U} - \mathring{U} = \emptyset$ as desired.

$\Leftarrow$: Assume there is a nonempty proper subset $A \subset X$ with empty boundary. Then $\partial A = \bar{A} - \mathring{A} = \emptyset$, so $\bar{A} \subseteq \mathring{A}$. But $\mathring{A} \subseteq A \subseteq \bar{A}$ for any set, so this means that $\bar{A} = \mathring{A} = A$, which means that A is clopen.

Then $X = A \cup (X - A)$ is a separation of X, so X is disconnected.

#6.6

We'll do the PPX_p part. We want to show that every subset containing p is connected, so to obtain a contradiction assume we have some $A \subset X$ containing p with a separation $A \subseteq U \cup V$. This means that U and V are both open in PPX_p, which means that they both contain the special point p. The separation also means that $A \cap U \cap V = \emptyset$, and since U and V both contain p, this means that $p \notin A$, which contradicts our assumption that A contains p.