Set theory

Prove: $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

We'll show $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$

and also $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$

First: Take $x \in (A \cup B) \cap C$, WTS $x \in (A \cap C) \cup (B \cap C)$

Since $x \in (A \cup B) \cap C$, so $x \in A \cup B$ and $x \in C$

so ($x \in A$ or $x \in B$) and $x \in C$.

so ($x \in A$ and $x \in C$) or ($x \in B$ and $x \in C$)

so $x \in A \cap C$ or $x \in B \cap C$

so $x \in (A \cap C) \cup (B \cap C)$ as desired.

Also: Take $x \in (A \cap C) \cup (B \cap C)$, WTS $x \in (A \cup B) \cap C$.

Since $x \in (A \cap C) \cup (B \cap C)$, this means $x \in A \cap C$ or $x \in B \cap C$.

Case 1: if $x \in A \cap C$ Then $A \cap C \subseteq (A \cup B) \cap C$

so if $x \in A \cap C$ then $x \in (A \cup B) \cap C$ as desired.

Case 2: if $x \in B \cap C$ Then $B \cap C \subseteq (A \cup B) \cap C$

so if $x \in B \cap C$ then $x \in (A \cup B) \cap C$.

Sheeran.

$$X-(A\cup B) = (X-A)\cap(X-B)$$

$$f(A\cup B) = f(A)\cup f(B)$$

$$f^{-1}(A\cup B) = f^{-1}(A)\cup f^{-1}(B)$$

Take $x\in X-(A\cup B)$ so $x\in X$ and $x\notin A\cup B$

so $x\in X$ and $(x\notin A$ and $x\notin B)$

so $(x\in X$ and $x\notin A)$ and $(x\in X$ and $x\notin B)$

so $x\in X-A$ and $x\in X-B$

so $x\in (X-A)\cap(X-B)$

Take $y\in f(A\cup B)$, so $\exists x\in A\cup B$ with $f(x)=y$

so $x\in A$ and $f(x)=y$ or $x\in B$ and $f(x)=y$

so $y\in f(A)$ or $y\in f(B)$

so $y\in f(A)\cup f(B)$.

$$f^{-1}(A\cup B) = f^{-1}(A)\cup f^{-1}(B)$$

Take $x\in f^{-1}(A\cup B)$ so $f(x)\in A\cup B$

so $f(x)\in A$ or $f(x)\in B$

so $x\in f^{-1}(A)$ or $x\in f^{-1}(B)$

So $x \in f^{-1}(A) \cup f^{-1}(B)$

Topologies

Point-set topology is about connections and "nearness"

We discuss this via open sets

in $\mathbb{R}$, an open set $\mathcal{U} \subseteq \mathbb{R}$ means

$$\forall x \in \mathcal{U}, \quad \exists \varepsilon > 0 \text{ s.t. } (x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}.$$

Thm (from $\mathbb{R}$)

Any union of open sets is open

Any finite intersection of open sets is open.

intersection of finitely many sets.

P.S. Topology is all about different abstract definitions of "open set"

We'll consider <u>any</u> way of describing open sets as long as the $\cup$ & $\cap$ properties hold.

<u>Def</u> Let X be any set, and let $\mathcal{T}$ be a collection of subsets called "open sets".

Then $\mathcal{T}$ is <u>a topology on X</u> when:

<u>i</u> $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$. "$X$ and $\emptyset$ are open"

<u>ii</u> Any finite intersection of sets in $\mathcal{T}$ is a set in $\mathcal{T}$.

<u>iii</u> Any union of sets in $\mathcal{T}$ is a set in $\mathcal{T}$.

If X has some topology $\mathcal{T}$, then these form a <u>topological space</u>

So $\mathbb{R}$, using ε-nbhds to define open sets, is a topological space.

Called "$\mathbb{R}$ with the standard topology"

2 stupid examples of topologies

For any set X, let $\mathcal{T} = \{\emptyset, X\}$

This is a topology, called <u>the trivial topology</u>

Also for any X, let $\mathcal{T} = \mathcal{P}(X)$

all subsets are open.

This is a topology, called <u>the discrete topology</u>