A topology on X is a collection $\mathcal{T}$ of subsets called "open" satisfying:

i) $\emptyset$ & X are open

ii) Any finite intersection of open sets is open

iii) Any union of open sets is open.

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if $\mathcal{T} = \{\emptyset, X\}$, this is the trivial topology
"no set is open"

if $\mathcal{T} = \mathcal{P}(X) \leftarrow$ all subsets

the discrete topology.

Simple abstract examples:

$X = \{a, b, c\}$,

and $\mathcal{T} = \{\emptyset, \{a\}, \{a,b,c\}\}$

is $\mathcal{T}$ a topology?

i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

ii Is any intersection of sets in $\mathcal{T}$ also in $\mathcal{T}$?

$\emptyset \cap \{a\} = \emptyset \in \mathcal{T}$ ✓

$\emptyset \cap \{a,b,c\} = \emptyset \in \mathcal{T}$ ✓

$\{a\} \cap \{a,b,c\} = \{a\} \in \mathcal{T}$ ✓

$\emptyset \cap \{a\} \cap \{a,b,c\} = \emptyset \in \mathcal{T}$ ✓

ii holds.

<u>iii</u> unions: $\emptyset \cup \{a\} = \{a\} \in \mathcal{T}$

$\emptyset \cup \{a,b,c\} = \{a,b,c\} \in \mathcal{T}$

$\{a\} \cup \{a,b,c\} = \{a,b,c\} \in \mathcal{T}$

$\emptyset \cup \{a\} \cup \{a,b,c\} = \{a,b,c\} \in \mathcal{T}$

iii holds.

So $\mathcal{T}$ <u>is</u> a topology.

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on $\{a,b,c\}$

\# 1) $\{\emptyset, \{a\}, \{b\}, \{a,b,c\}\}$ ← <u>not</u> $\{a\} \cup \{b\} = \{a,b\} \notin \mathcal{T}$

\# 2) $\{\emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}\}$ <u>is</u> a topology.

\# 3) $\{\emptyset, \{a\}, \{b\}, \{a,c\}, \{a,b,c\}\}$

$\{a\} \cup \{b\} = \{a,b\} \notin \mathcal{T}$.

A non-standard topology on $\mathbb{R}$.

"The finite complement topology"

A set $U \subseteq \mathbb{R}$ is open in the fin. comp topology when $\mathbb{R} - U$ is finite. (or $U = \emptyset$)

$$\mathcal{T} = \{ \mathcal{U} \mid \mathbb{R} - \mathcal{U} \text{ is finite} \} \cup \{ \emptyset \}$$

$\mathbb{R}$ with fin. comp. top. is written $\mathbb{R}_{fc}$.

<u>Thm</u> $\mathbb{R}_{fc}$ is a topolgical space.

<u>Pf</u> (verify i, ii, iii for this topology)

<u>i</u> $\emptyset$ is open by declaration.

$\mathbb{R}$ is open since: $\mathbb{R} - \mathbb{R} = \emptyset$ which is finite.

<u>ii</u> (finite intersections) Let $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n$ be open.

Since $\mathcal{U}_i$ is open, we have $\underline{\mathbb{R} - \mathcal{U}_i \text{ is finite.}}$

WTS $\mathbb{R} - (\mathcal{U}_1 \cap \dots \cap \mathcal{U}_n)$ is finite.

$$\mathbb{R} - (\mathcal{U}_1 \cap \dots \cap \mathcal{U}_n) = \underbrace{(\mathbb{R} - \mathcal{U}_1)}_{\uparrow \text{ finite}} \cup \dots \cup \underbrace{(\mathbb{R} - \mathcal{U}_n)}_{\uparrow \text{ finite}}$$

so $\mathbb{R} - (\mathcal{U}_1 \cap \dots \cap \mathcal{U}_n)$ is finite.

Then $\mathcal{U}_1 \cap \dots \cap \mathcal{U}_n$ is open.

$$\boxed{\begin{aligned} X - (A \cup B) &= (X - A) \cap (X - B) \\ X - (A \cap B) &= (X - A) \cup (X - B) \end{aligned}}$$

iii (unions) Let U_α be open for all $\alpha \in A$.

"the index set"

So $\mathbb{R} - U_\alpha$ is finite for each α.

WTS $\bigcup_{\alpha \in A} U_\alpha$ is open, i.e. $\mathbb{R} - \left(\bigcup_{\alpha \in A} U_\alpha \right)$ is finite.

We have: $$\mathbb{R} - \left(\bigcup_{\alpha \in A} U_\alpha \right) = \bigcap_{\alpha \in A} \underbrace{\mathbb{R} - U_\alpha}_{\text{finite.}}$$

Any $\bigcap$ of finite sets is finite, so

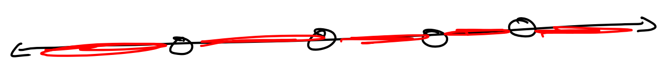
$\bigcap_{\alpha \in A} \mathbb{R} - U_\alpha$ is finite. Shown.

On $\mathbb{R}$, we have $\mathbb{R}$ (standard)

or $\mathbb{R}_{fc}$

or the trivial or discrete.

An open set in $\mathbb{R}_{fc}$ looks like



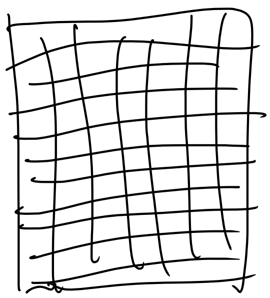
This is also open in $\mathbb{R}$ (standard)

But not all standard open intervals are open in $\mathbb{R}_{fc}$.

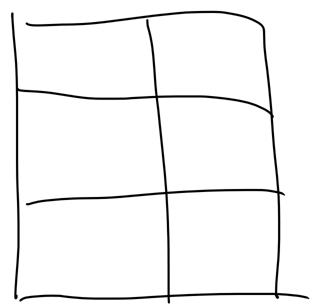
The standard top is "bigger" than $\mathbb{R}_{fc}$.

We say standard top. is "finer" than $\mathbb{R}_{fc}$.

$\mathbb{R}_{fc}$ is "coarser" than standard.



"fine"



"coarse"

If we have topologies τ_1 & τ_2

and $\tau_1 \subset \tau_2$

then τ_2 is finer than τ_1,

τ_1 is coarser than τ_2.

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trivial $\subset$ $\mathbb{R}_{fc}$ $\subset$ $\mathbb{R}$ (standard) $\subset$ discrete

$\xleftarrow{\text{coarser}}$ $\xrightarrow{\text{finer}}$