trivial top $\subset$ $\mathbb{R}_{fc}$ $\subset$ $\mathbb{R}$ (standard) $\subset$ discrete top

$\longleftarrow$ coarser $\qquad$ finer $\longrightarrow$

"$\subseteq$" means subset or equal

in the book $A \subset B$ means A is a subset or $A = B$.

Another topology on $\mathbb{R}$:

$$\mathcal{T} = \{\emptyset, \mathbb{R}\} \cup \{(b,\infty) \mid b \in \mathbb{R}\}$$

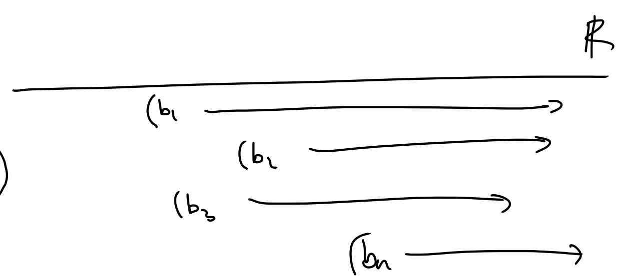
open sets are $\emptyset, \mathbb{R}$, or any open interval unbounded above.

Show it is a topology:

i) $\emptyset$ and $\mathbb{R}$ are open: by definition.

ii) (finite $\cap$) Let $U_1, \dots, U_n$ be open.

then $U_i = (b_i, \infty)$



Then $\bigcap_{i=1}^{n} U_i = (\max\{b_i\}, \infty)$

so $\bigcap_{i=1}^{n} U_i$ is open.

iii (unions) Let $\mathcal{U}_\alpha$ be open for $\alpha \in A$

then $\mathcal{U}_\alpha = (b_\alpha, \infty)$

then $\bigcup_{\alpha \in A} \mathcal{U}_\alpha = (\inf\{b_\alpha\}, \infty)$

as long as $\inf\{b_\alpha\}$ exists.

if $\inf\{b_\alpha\}$ does not exist, then $\bigcup_{\alpha \in A} \mathcal{U}_\alpha = \mathbb{R}$.

either way, $\bigcup_{\alpha \in A} \mathcal{U}_\alpha$ is open.

Basic terminology: In a top. space X, with $x \in X$,

a <u>neighborhood of</u> x is any open set $\mathcal{U}$ with $x \in \mathcal{U}$.

in $\mathbb{R}$ (standard),

$(-1, 1)$ is a nbhd of 0.

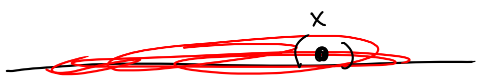
$(-1, 10)$ " " " " of 0.

$\mathbb{R}$ is a nbhd of 0.

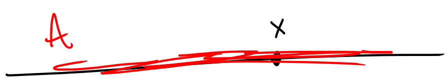
A theorem from $\mathbb{R}$:

<u>Thm</u> Let X be any top. space, and $A \subseteq X$.

then A is open iff $\forall x \in A$, $\exists\, \mathcal{U}$ a nbhd of x such that $\mathcal{U} \subseteq A$.



<u>Pf</u> <u>$\Rightarrow$</u> Assume A is open, let $x \in A$, will find $\mathcal{U}$ a nbhd of x with $\mathcal{U} \subseteq A$.



Let $\mathcal{U} = A$. then $\mathcal{U}$ is open since A is open, and $x \in \mathcal{U} = A$.

So $\mathcal{U}$ is a nbhd of x.

<u>$\Leftarrow$</u> Assume $\forall x \in A$, $\exists\, \mathcal{U}$ a nbhd of x. with $\mathcal{U} \subseteq A$.

Write $\mathcal{U}_x$ as the nbhd of x, $\mathcal{U}_x \subseteq A$.

WTS A is open.

$\bigcup\limits_{x \in A} \mathcal{U}_x$ contains every $x \in A$.

So $A \subseteq \bigcup \mathcal{U}_x$

since $\mathcal{U}_x \subseteq A$, $\bigcup \mathcal{U}_x \subseteq A$.

$\Rightarrow A = \bigcup \mathcal{U}_x$

So A is open. by unions property

A convenient way to define a topology is in terms of a <u>Basis for a topology</u>

a (smaller) set of open sets which are used to build any open set.

in $\mathbb{R}$ (standard), the open intervals form a basis for the topology, since any open set is a union of open intervals.

To work as a basis, the basis needs some nice intersections property.

<u>Def</u> Let X be a top. space with top $\mathcal{T}$.

A set $\mathcal{B} \subseteq \mathcal{T}$ is a <u>basis for a topology</u> when:

<u>i</u> $\forall x \in X$, $\exists B \in \mathcal{B}$ with $x \in B$.

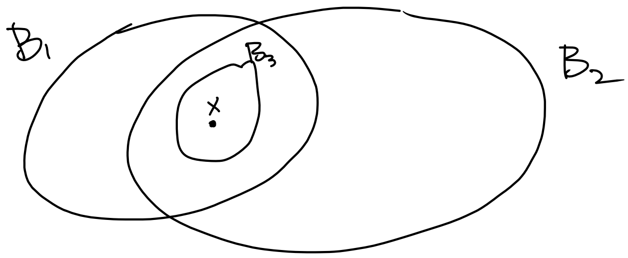
"$\mathcal{B}$ covers X"

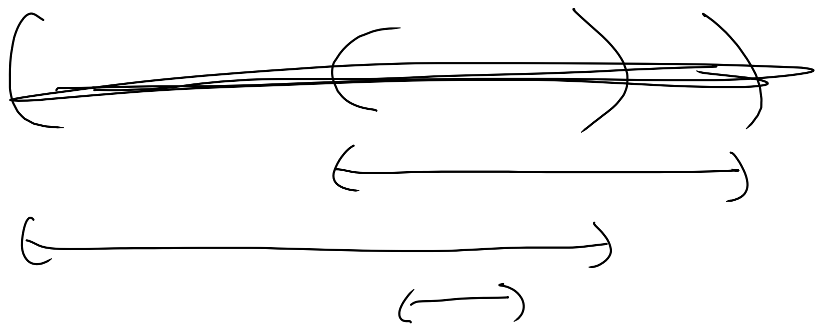
<u>ii</u> If $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$,

then $\exists B_3 \in \mathcal{B}$ with $x \in B_3$

and $B_3 \subseteq B_1 \cap B_2$.

"shrinkable"





Thm The open intervals in $\mathbb{R}$ are a basis for a topology.

Pf i Let $x \in \mathbb{R}$, will find a basis set B with $x \in B$.

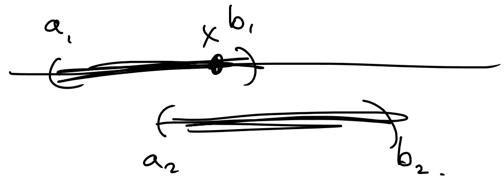
for any x, $x \in (-\infty, \infty)$

$$\left(x \in (x-1, x+1) \right)$$

ii Take B_1, B_2 open intervals, with $x \in B_1 \cap B_2$.

$B_1 = (a_1, b_1)$

$B_2 = (a_2, b_2)$



We need $x \in B_3 \subseteq B_1 \cap B_2$

use $B_3 = (a_2, b_1)$. Then $x \in B_3 \subset B_1 \cap B_2$ as desired.