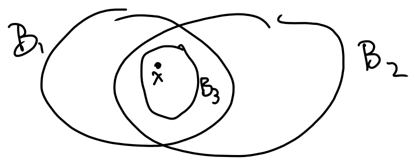
Any Basis defines a topology

Basis: $\underline{i}$ sets from $\mathcal{B}$ cover X.

$\underline{ii}$ If $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$



then $\exists B_3 \in \mathcal{B}$

with $x \in B_3 \subseteq B_1 \cap B_2$

$\underline{\text{Def}}$ Let $\mathcal{B}$ be a basis (satisfies 2 props)

then $\underline{\text{the topology generated by } \mathcal{B}}$ consists

of $\emptyset$, plus all unions of sets from $\mathcal{B}$.

On $\mathbb{R}$, if $\mathcal{B}$ = set of all open intervals,

then $\mathcal{B}$ generates the standard topology.

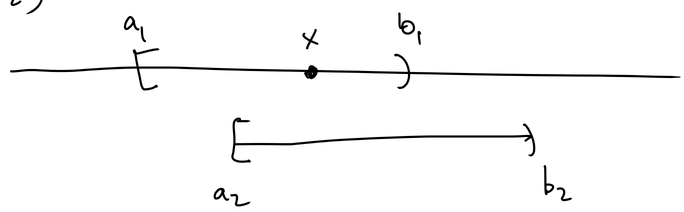
<u>The lower-limit topology</u>:

Let $\mathcal{B} = \{ [a, b) \mid a < b \}$

This is a basis: $\underline{i}$ any $x \in \mathbb{R}$ has $x \in [x, x+1)$

ii $\quad B_1 = [a_1, b_1)$

$B_2 = [a_2, b_2)$



then let $B_3 = [a_2, b_1)$

then $x \in B_3 \subseteq B_1 \cap B_2$.

This generates the lower-limit top. on $\mathbb{R}$.

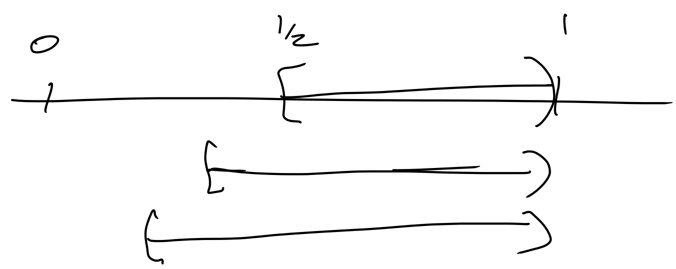
[Also there's the ~~upper-limit top.~~ using intervals like $(a, b]$]

In lower-limit $\mathbb{R}$, $[0, 1)$ is open.

also $[0, 1) \cup [2, 3)$ is open.

also $\bigcup_{n=2}^{\infty} [1/n, 1)$ is open

$= [0, 1)$.



in L.L top on $\mathbb{R}$,

$[0,1)$ is open,

also $(0,1)$ is open.

(but $[0,1]$ is not open

$(0,1]$ is not open)

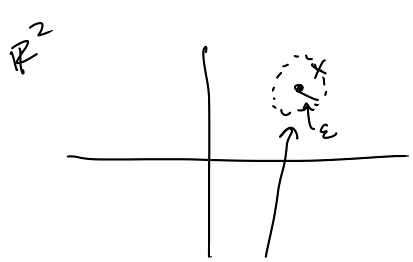
Last time: A is open iff $\forall x \in A$ $\exists$ a nbhd $\mathcal{U}$ with $x \in \mathcal{U} \subseteq A$.

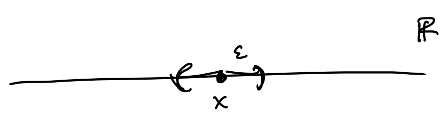
Actuactually, we can say stronger:

Thm A is open iff $\forall x \in A$ $\exists$ a basis nbhd B with $x \in B \subseteq A$.

Topologies in $\mathbb{R}^2$ (or $\mathbb{R}^n$)

To get the standard topology in $\mathbb{R}^2$, we need to decide on a basis.





The open ε-ball around x

in $\mathbb{R}^2$, it's like a disk without the boundary.

$$B(x,\varepsilon) = \{ p \in \mathbb{R}^n \mid d(x,p) < \varepsilon \}$$

the Euclidean distance.

$$d((x_1,\dots,x_n),(p_1,\dots,p_n)) = \sqrt{(x_1-p_1)^2 + \dots + (x_n-p_n)^2}$$

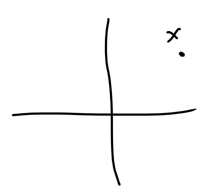
in $\mathbb{R}^2$, $B(x,\varepsilon)$ looks like a disk without bdry

in $\mathbb{R}^3$, — — — —— ball - - -

in $\mathbb{R}^1$, it's the ε-nbhd of x.

In $\mathbb{R}^2$, we can use this basis:

$$\mathcal{B} = \{ B(x,\varepsilon) \mid x \in \mathbb{R}^2, \varepsilon > 0 \}$$



Show it's really a basis.

<u>i</u> (covers $\mathbb{R}^2$) Let $x \in \mathbb{R}^2$,

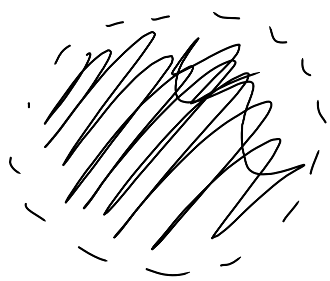
then $x \in B(x,1)$.

ii (shrinky)

If $x \in B_1 \cap B_2$, there is some $B_3 = B(x, \varepsilon)$ with $x \in B_3 \subseteq B_1 \cap B_2$.

So ε-balls form a basis for a top on $\mathbb{R}^2$.

This is the standard topology on $\mathbb{R}^2$ (or $\mathbb{R}^n$)

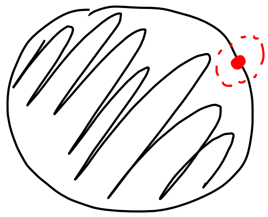


is open

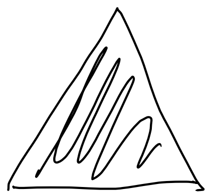
is open

(any pt x has a nbhd inside the set)

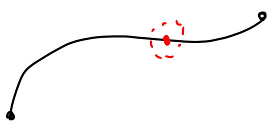
is open!



not open — the boundary pts have no balls inside the set.

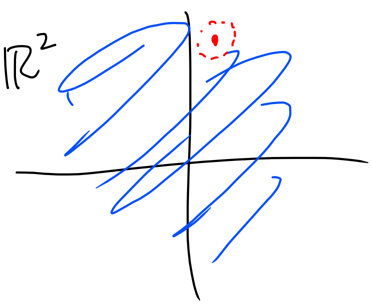


not open



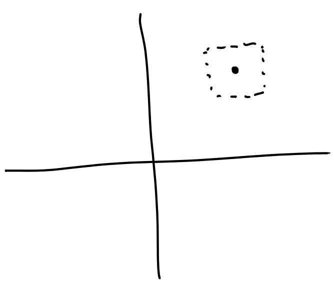
not open.

$\mathbb{Q}^2 \subseteq \mathbb{R}^2$



$\mathbb{Q}^2$ is not open.

What if instead we used boxes?

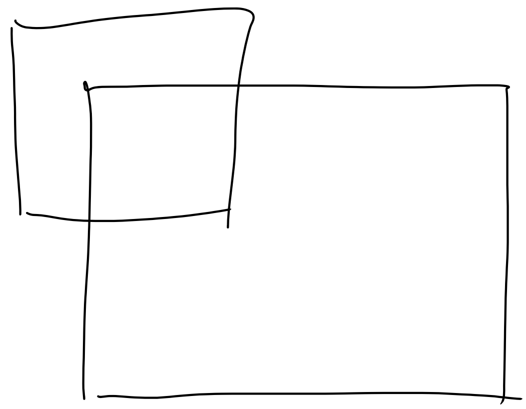


in $\mathbb{R}$, we used (a,b)

in $\mathbb{R}^2$, maybe we should use

$(a,b) \times (c,d)$

This is still a basis:

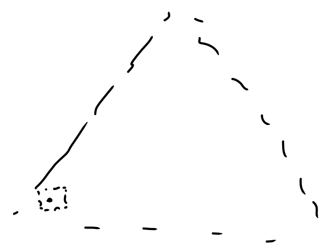


intersections property holds.

The boxes make a basis.

These also generate the standard top.

still open



We can generate the standard top

using  or  or