Closed Sets

Closed is not the opposite of open.

In analysis, "closed" means it contains all its limit points.

A thm: the complement of an open is closed.

Def Let X be a top. space, $A \subseteq X$ a subset.

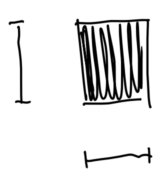
Then A is closed when $X - A$ is open.

In $\mathbb{R}$, closed sets look like closed intervals, or individual points, or weider stuff like Cantor set.

In $\mathbb{R}^2$: a closed set might look like

a closed ball $\overline{B}(x,\varepsilon) = \{ p \in \mathbb{R}^n \mid d(x,p) \leq \varepsilon \}$

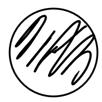
or a closed rectangle



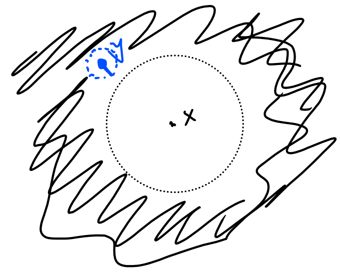
$[a,b] \times [c,d]$

Let's prove $\overline{B}(x,\varepsilon)$ is closed in $\mathbb{R}^n$.

We must show $\mathbb{R}^n - \overline{B}(x,\varepsilon)$ is open.



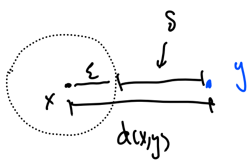
$\overline{B}(x,\varepsilon)$



$\mathbb{R}^n - \overline{B}(x,\varepsilon)$

We need to find $\delta > 0$ s.t.

$B(y,\delta)$ is contained in $\mathbb{R}^n - \overline{B}(x,\varepsilon)$.

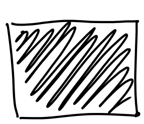


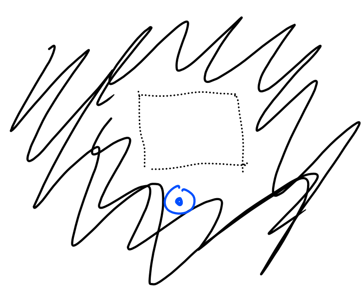
let $\delta = d(x,y) - \varepsilon$

then all points in $B(y,\delta)$ are outside $\overline{B}(x,\varepsilon)$.

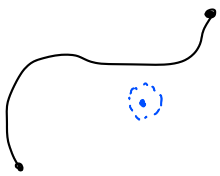
So $B(y,\delta) \subset \mathbb{R}^n - \overline{B}(x,\varepsilon)$

Also a closed rectangle $[a,b] \times [c,d]$ is closed.



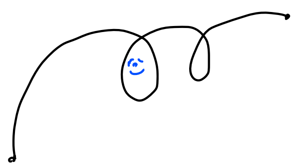


Is it closed?

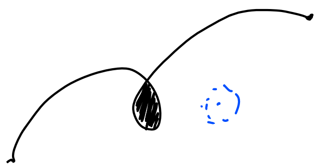


is closed.

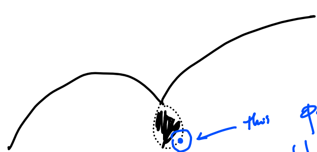
(any pt outside the set has a nbhd outside the set)



is closed



is closed

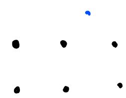


this point has no nbhd contained in the complement.

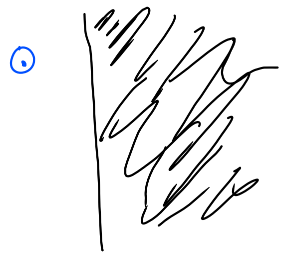
So complement is not open,

so the set is not closed.

is a closed set.

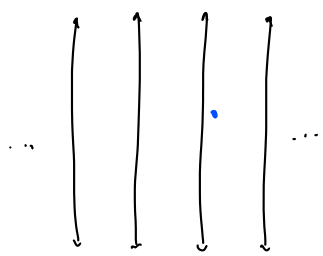


Is closed.

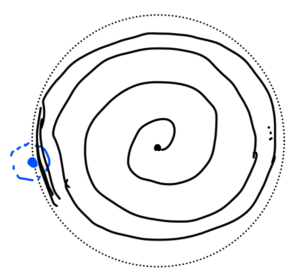


Is closed

"closed half-plane"



is closed.



Not closed.

points on the limiting circle have all nbhds touching the spiral.

An abstract example: $X = \{a, b, c, d\}$

$\mathcal{T} = \{\emptyset, \{b\}, \{a,b\}, \{c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$

Which sets are closed?

complements of each open set.

$X - \emptyset = X = \{a, b, c, d\}$ is closed ✓

$X - \{b\} = \{a, c, d\}$ is closed

$X - \{a,b\} = \{c, d\}$ "

$X - \{c,d\} = \{a, b\}$ "

$X - \{b,c,d\} = \{a\}$ is closed

$X - \{a,b,c,d\} = \emptyset$ is closed

$\{a,b\}$ is open and closed

$\{c,d\}$ also.

$\{c\}$ is not open or closed.