A closed set is one whose complement is open.

Facts: $\emptyset$ and X are always closed

(Since these are complements of X & $\emptyset$, which are open)

- Any intersection of closeds is closed.
- Any finite union of closeds is closed.

In $\mathbb{R}$, any single point is a closed set.

not always true in a top space:

Last time example $X = \{a, b, c, d\}$, $\mathcal{T} = \{\dots\}$

the closed sets included $\{a\}$,

but $\{b\}, \{c\}, \{d\}$ were not closed.

A single-point set is closed in a certain nice type of top. space:

<u>A Hausdorff Space</u>

Def A top. space X is a Hausdorff Space

if: $\forall_{x \neq y} \in X$, $\exists$ nbhds U, V
$x \in U$, $y \in V$ and $U \cap V = \emptyset$. "separation axiom"

picture:

Any 2 points can be separated by open sets.

"housed off"

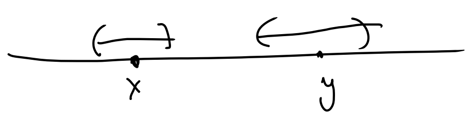
Hausdorff (1868-1942)

Created a set of axioms for topology

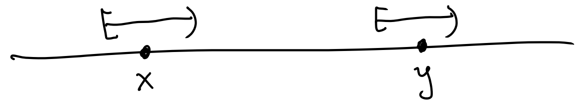
$\mathbb{R}^n$ is Hausdorff:

pts can be separated by nbhd.

$\mathbb{R}^1$



$\mathbb{R}$, lower limit top



is Hausdorff.

Discrete top if we have $x, y \in X$, $x \neq y$.

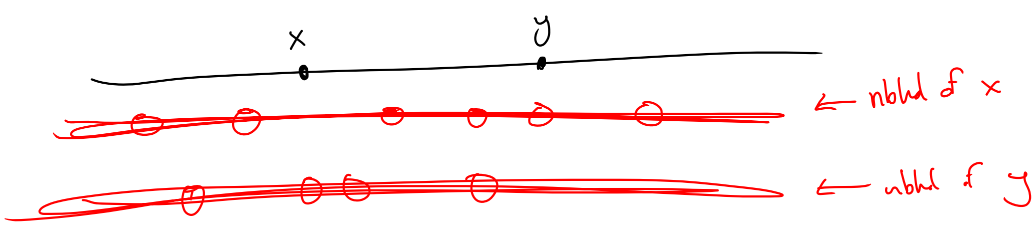
we can take $\mathcal{U} = \{x\}$

$V = \{y\}$

So $\mathcal{U}$ is a nbhd of x
V . - - - - y, and $\mathcal{U} \cap V = \emptyset$.

So the Discrete top is Hausdorff.

$\mathbb{R}_{fc}$ (finite complement) is not Hausdorff.



Any 2 open sets in $\mathbb{R}_{fc}$ will have many points in common.

So $\mathcal{U} \cap V = \emptyset$ is impossible.

Generally, Hausdorff spaces are nice

similar to $\mathbb{R}^n$

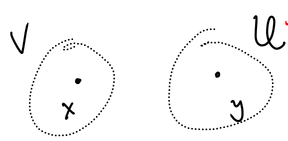
non-Hausdorff are weird

<u>Thm</u> If X is Hausdorff and $x \in X$, then $\{x\}$ is closed.

<u>Pf</u> We'll show $X - \{x\}$ is open.

Take $y \in X - \{x\}$, we'll find a nbhd U around y s.t. $U \subseteq X - \{x\}$.

So $y \neq x$, we want a nbhd U with $y \in U$ and $x \notin U$.



Since X is Hausdorff, $\exists U, V$ with $y \in U$, $x \in V$ and $U \cap V = \emptyset$.

So $x \notin U$ (since $x \in V$ and $U \cap V = \emptyset$)

so $y \in U$ and $x \notin U$. Sheeeam.

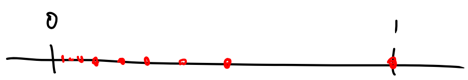
Closure of a set

The closure is when you make the set bigger to make it closed.

$\overline{A}$ is the closure of A.

in $\mathbb{R}$: $\overline{(0,1)} = [0,1]$

$\overline{\{1/n\}} = \{1/n\} \cup \{0\}$



$\overline{\mathbb{Q}} = \mathbb{R}$

$\overline{[0,1]} = [0,1]$

So we always have $A \subseteq \overline{A}$ and $\overline{A}$ is closed

we want $\overline{A}$ to be the "smallest" closed set containing A.

intersection of all closed sets

Def In a top space X, let $A \subseteq X$. Then the closure of A, written $\overline{A}$ is defined as the intersection of all closed sets containing A.