Closure & Interior

$Cl(\{1,2,3\}) = \{1,2,3\}$

$Cl(\mathbb{Q}) = \mathbb{R}$

Def When $A \subseteq X$ with $\bar{A} = X$ then we say A is **dense** in X.

$Cl(A)$ is "the smallest closed set containing A"

Def $Cl(A)$ **the closure of A** is the intersection of all closed sets containing A.

Thm $A = \bar{A}$ iff A is closed.

"closure is the smallest closed set containing A"

Thm **i** $\bar{A}$ is closed

ii If $A \subseteq C$ and C is closed, then $\bar{A} \subseteq C$. ← "$\bar{A}$ is smaller than any closed set containing A"

<u>Pf</u> <u>i</u> $\bar{A}$ is an intersection of closed sets,

so it's closed.

<u>ii</u> Let $A \subseteq C$ with C closed.

So C is a closed set containing A,

$\bar{A}$ is the $\cap$ of <u>all</u> closed sets containing A,

so $\bar{A} \subseteq C$.

A dual notion: The <u>interior</u> of A.

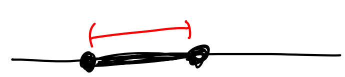
$\mathrm{Int}(A)$ or $\mathring{A}$

$\mathrm{Cl}(A)$ is the smallest closed set containing A

$\mathrm{Int}(A)$ is the biggest open set inside A.

<u>Def</u> $\mathrm{Int}(A)$ is the union of all open sets $\mathcal{U}$ with $\mathcal{U} \subseteq A$.

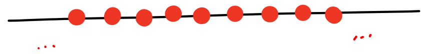
in $\mathbb{R}$, standard top:



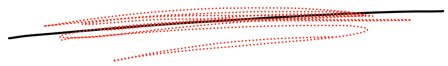
$\mathrm{Int}\,[0,1] = (0,1)$

$\mathring{\mathbb{R}} = \mathbb{R}$ ← $\mathbb{R}$ is already open

$\mathring{\mathbb{Z}} = \emptyset$



$\mathring{\mathbb{Q}} = \emptyset$



Thm i. $\mathring{A}$ is open

ii. If $U \subseteq A$ and U is open, then $U \subseteq \mathring{A}$.

Thm $A = \mathring{A}$ iff A is open.

Pf $\Rightarrow$ Assume $A = \mathring{A}$. WTS A is open.

$A = \mathring{A}$ and $\mathring{A}$ is open, so A is open.

$\Leftarrow$ Assume A is open. WTS $A = \mathring{A}$.

We'll show $\underbrace{A \subseteq \mathring{A}}$ and $\underbrace{\mathring{A} \subseteq A}$.

A is open, so $A \subseteq \mathring{A}$ by ii above.

since $\mathring{A}$ is a union of subsets of A.

Ex) in $\mathbb{R}_\ell$ ← lower-limit topology.

$[a,b)$ is open.

in $\mathbb{R}_\ell$, let $A = [0,1)$ this is open, so

$$\mathring{A} = [0,1).$$

For $\bar{A}$: $\mathbb{R} - A$ will be closed:

$$\mathbb{R} - A = (-\infty, 0) \cup [1, \infty)$$

open in $\mathbb{R}_\ell$ open in $\mathbb{R}_\ell$

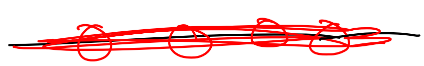
So $\mathbb{R} - A$ is open, so A is closed.

So $\bar{A} = A = [0,1)$.

In $\mathbb{R}_\ell$, a set like $[a,b)$ is clopen.

$\mathbb{R}_{fc}$, $A = [0,1)$

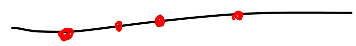
Int (A): biggest open set inside A.

open in $\mathbb{R}_{fc}$ looks like 

No (Nontrivial) open set is inside A;

so $\mathring{A} = \emptyset$

$Cl(A)$: smallest closed set containing A.

closed in $\mathbb{R}_{fc}$ looks like  a finite set.

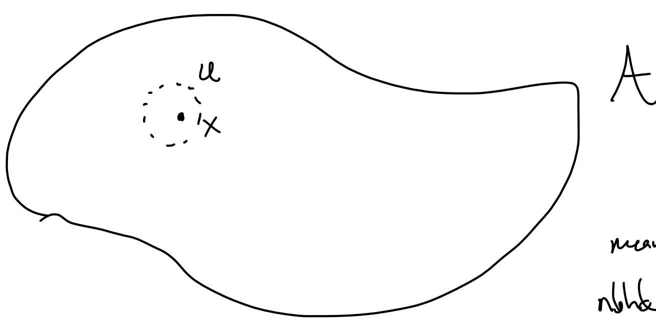
A 

no finite set contains A, so the

$\bar{A} = \mathbb{R}$.

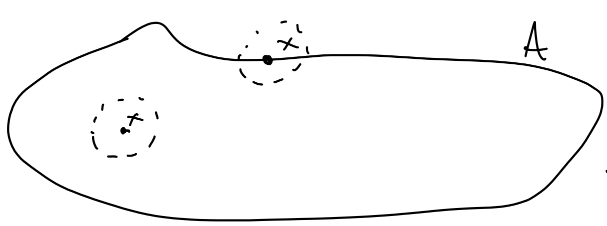
How to tell if some x is in $\mathring{A}$ or $\bar{A}$?

$x \in Int(A)$ means $x \in \mathcal{U}$ for $\mathcal{U} \subseteq A$, $\mathcal{U}$ open.



$x \in Int(A)$ means x is in some nbhd inside A.

$x \in Cl(A)$



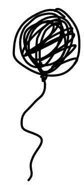
$x \in Cl(A)$ any nbhd of x intersects A.

For any set we have a <u>set sandwich</u>

$$\mathring{A} \subset A \subset \overline{A}$$

in $\mathbb{R}^2$

A: 

$\overline{A}$: 

$\mathring{A}$:  no string!