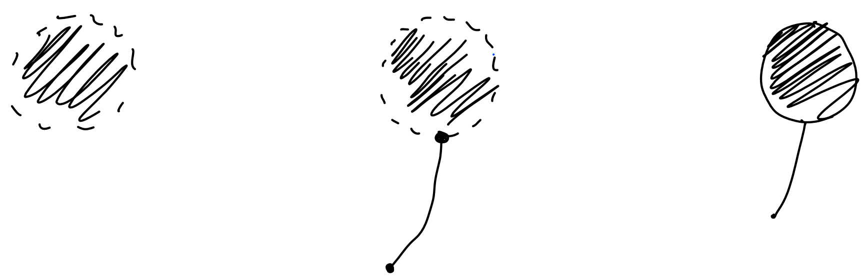
Fun fact: Interior & Closure of a complement

$$\mathrm{Int}(X-A) = ?$$



Int(A) A Cl(A)

$x \in \mathrm{Int}(A)$ means $\exists$ a nbhd $U \subseteq A$ $x \in U$.

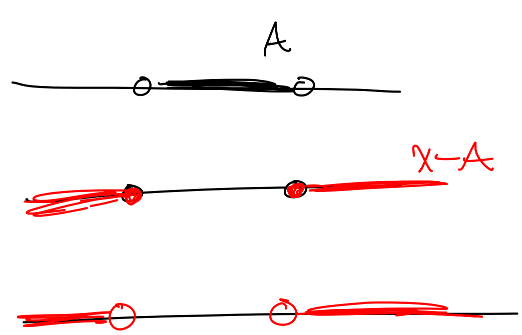
$x \in \mathrm{Cl}(A)$ means any nbhd U with $x \in U$ has some intersection with A.

$\mathrm{Int}(X-A)$

if $A = (0,1)$

$X - A = (-\infty, 0] \cup [1, \infty)$

$\mathrm{int}(X-A)$



$$\boxed{\mathrm{Int}(X-A) = X - \mathrm{Cl}(A)}$$

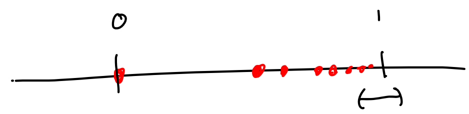
Int of a comp is a comp of the closure.

also $\boxed{Cl(X - A) = X - Int(A)}$

Limit Points

In analysis: Closed means "contains all its limit points"

limit pt means:

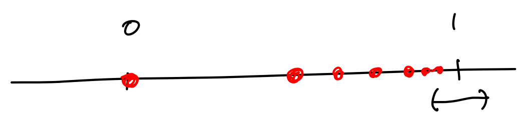


$A = \{1 - 1/n\}$

1 is a limit pt of A because any ε-nbhd of 1 touches pts of A.

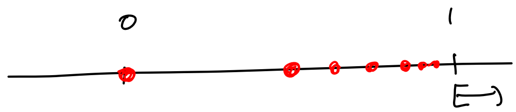
Def Let $A \subseteq X$, $x \in X$. We say x is a limit pt of A when: any nbhd U of x intersects A at some pt other than x itself.

So let $A = \{1 - 1/n\}$



Any nbhd of 1 ($\mathbb{R}$-standard) contains red pts, so 1 is a limit pt of A.

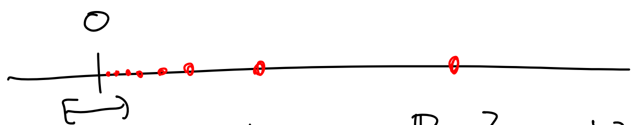
What about $\mathbb{R}_\ell$ ← lower-limit top. on $\mathbb{R}$.



must any nbhd of 1 include red pts? NO!

$[1, 2)$ is a nbhd of 1,
but excludes all pts of A.
So 1 is not a limit pt of A in $\mathbb{R}_\ell$.

$A = \{1/n\}$



is 0 a limit pt in $\mathbb{R}_\ell$? Yes!

- limit pts & closed sets
- limit pts & lims of sequences

<u>Closed sets:</u>

in $\mathbb{R}$-anal the closure is the original set, plus all limit pts

For a set A, let A' = set of limit pts.

<u>Thm</u> $Cl(A) = A \cup A'$

Pf $\supseteq$ WTS $A \cup A' \subseteq Cl(A)$

$A \subset Cl(A)$, so we need only show $A' \subset Cl(A)$

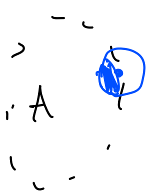
If x is a limit pt, this means any nbhd of x intersects A at some pt other than x.

So $x \in Cl(A)$. So $A' \subset Cl(A)$.

$\subseteq$ WTS $Cl(A) \subset A \cup A'$

Take some $x \in Cl(A)$, WTS $x \in A$ or $x \in A'$

$x \in Cl(A)$ means any nbhd of x intersects with A.



if x is in the intersection, then $x \in A$

if x is not in the intersection, then $x \in A'$.

So $x \in A$ or $x \in A'$ as desired.

Thm A is closed iff A contains all its limit points

Pf A is closed iff $A = Cl(A)$

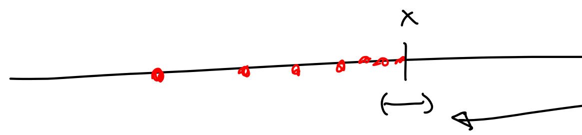
i.e. $A = A \cup A'$

i.e. $A' \subset A$ $\leftarrow$ A contains all limit pts.

Lims of sequences

in analysis: $a_n \to x$ means: $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$n > N \implies |a_n - x| < \varepsilon.$$



eventually, all pts of the sequence lie inside the ε-nbhd

Def Let $x_n \in X$ be a sequence, $x \in X$.

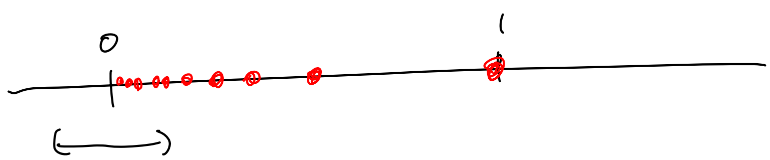
We say $x_n \to x$ when: $\forall \mathcal{U}$, $\mathcal{U}$ a nbhd of x,

$\exists N \in \mathbb{N}$ s.t.:

$$n > N \implies x_n \in \mathcal{U}.$$

"eventually, all pts of the seq lie inside $\mathcal{U}$"

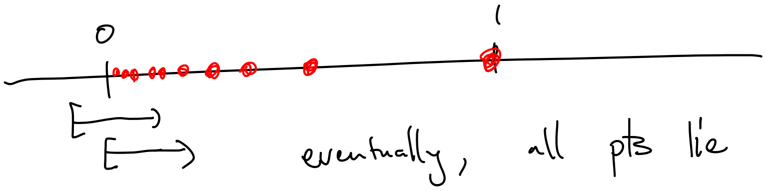
In $\mathbb{R}$ (standard), $(1/n) \to 0$



For any nbhd of 0, eventually all red dots are inside the nbhd.

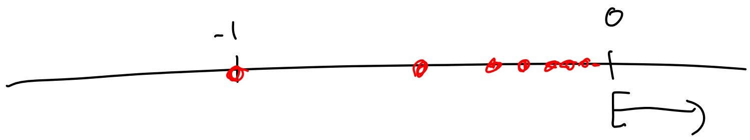
So $1/n \to 0$.

In $\mathbb{R}_\ell$, $(1/n) \to 0$



eventually, all pts lie in the nbhd.

In $\mathbb{R}_\ell$, $(-1/n)$

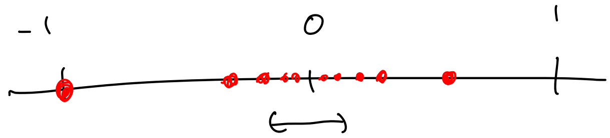


$[0,1)$ is a nbhd of 0, excluding all seq terms.

so $(-1/n) \not\to 0$.

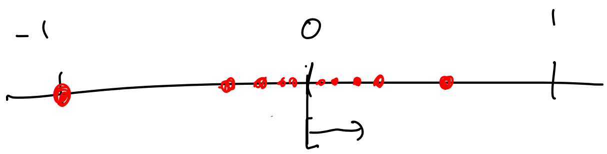
Actually in $\mathbb{R}_\ell$, $(-1/n)$ diverges.

What about $\dfrac{(-1)^n}{n}$



in $\mathbb{R}$: this $((-1)^n/n) \to 0$.

in $\mathbb{R}_\ell$:



The nbhd $[0,1)$ excludes all odd # terms.

so it's not true that all terms are eventually in the nbhd.

so $\frac{(-1)^n}{n} \not\to 0$ in $\mathbb{R}_\ell$.