Limit pts & sequences

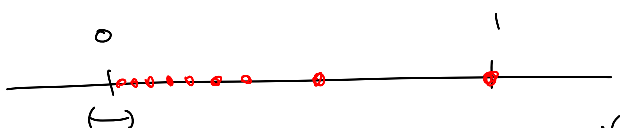
__Def__ $x_n \to x$ means : $\forall \mathcal{U}$ a nbhd of x,

$\exists N \in \mathbb{N}$ s.t.

$$n > N \Rightarrow x_n \in \mathcal{U}$$

" For any nbhd of x, eventually all sequence terms are inside the nbhd "

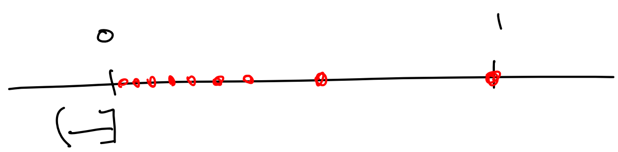
in $\mathbb{R}$, $1/n \to 0$



all terms eventually are inside here,

so $1/n \to 0$.

in $\mathbb{R}_u$ ← upper limit

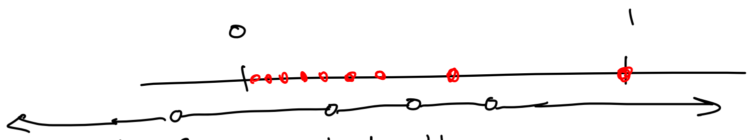


this time a nbhd of 0 excludes all the terms, so $1/n \not\to 0$.

In $\mathbb{R}_{fc}$ ← finite comp.

open sets look like 

in $\mathbb{R}_{fc}$, does $1/n \to 0$?



an nbhd of 0 looks like

Eventually, all sequence terms are inside the nbhd
(the nbhd is all of $\mathbb{R}$, except a few pts)

So in $\mathbb{R}_{fc}$, $1/n \to 0$.

<u>Weird</u>: Also, the seq terms eventually lie inside any nbhd of 1.

So also $\frac{1}{n} \to 1$ in $\mathbb{R}_{fc}$.

So in $\mathbb{R}_{fc}$, $\frac{1}{n} \to x \quad \forall x \in \mathbb{R}$.

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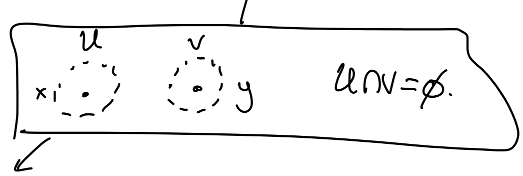
in some top spaces, the limit of a sequence is not unique.

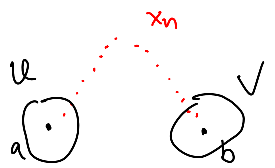


<u>Thm</u> If X is Hausdorff, then the limit of a conv. seq is unique.

Pf Let (x_n) be a sequence,

For a contradiction, assume $x_n \to a$

and also $x_n \to b$ where $a \neq b$.



Since X is Hausdorff, $\exists$ nbhds $\mathcal{U}$ of a & V of b, with $\mathcal{U} \cap V = \emptyset$

Since $x_n \to a$, all terms are eventually inside $\mathcal{U}$.

Also $x_n \to b$, . - - - - - - - - V.

So eventually, all terms are inside $\mathcal{U} \cap V$,

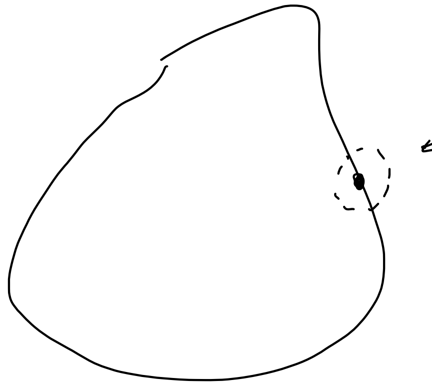
but $\mathcal{U} \cap V = \emptyset$! $\rightarrow\!\leftarrow$.

Closure, Interior, & Boundary

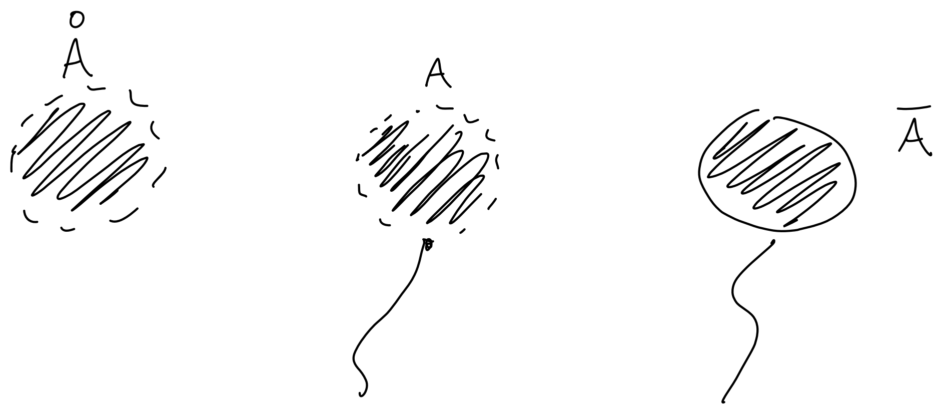
x is in the boundary of A when any nbhd of x

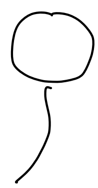
intersects with both A and $X - A$.

"any nbhd of x is half-in, half-out"



a boundary point.

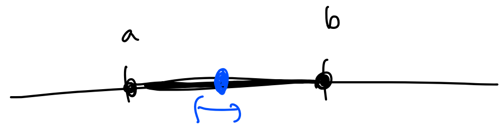


∂A :  (just outer circle)

<u>Def</u> For $A \subset X$, <u>the boundary of A</u> is

$$\partial A = Cl(A) - Int(A)$$

in $\mathbb{R}$: $\quad [a,b]$



in terms of nbhds: points where every nbhd is half-in/half-out.

this is a, b.

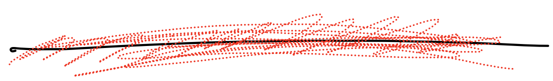
$$\partial[a,b] = \{a,b\}.$$

$$\partial[a,b] = Cl[a,b] - Int[a,b]$$
$$= [a,b] - (a,b) = \{a,b\}$$

in $\mathbb{R}$, $\quad (a,b)$

$$\partial(a,b) = \{a,b\}$$

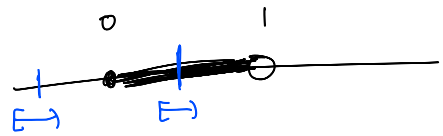
in $\mathbb{R}$, $\quad A = \mathbb{Q}$

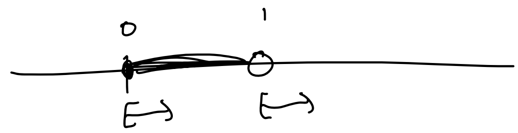


$$\partial\mathbb{Q} = Cl(\mathbb{Q}) - Int(\mathbb{Q})$$
$$= \mathbb{R} - \emptyset = \mathbb{R}$$

any real # is on $\partial\mathbb{Q}$.

in $\mathbb{R}_\ell$ $\quad A = [0, 1)$



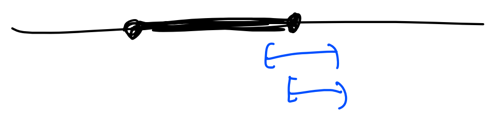


0 is all in,
1 is all out.

So $\partial A = \emptyset$.

$$\partial A = Cl(A) - Int(A) = A - A = \emptyset$$

in $\mathbb{R}_\ell$, $A = [0, 1]$



$\partial [0,1] = \{1\}$