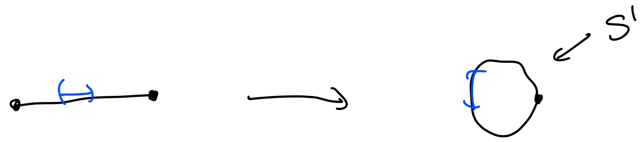
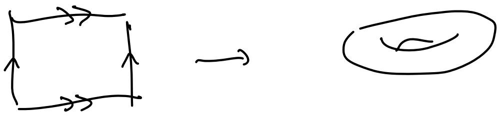
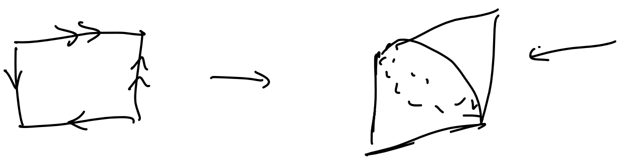
The Quotient Topology

$[0,1]$:

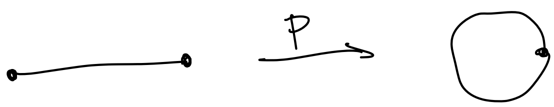






like S^2 the sphere.

Generally, start with X, glue parts to obtain A.



$X = [0,1]$ $\qquad$ $A = S^1$

The gluing is described by a function $p: X \to A$ called the quotient map.

For $p: [0,1] \to S^1$, we can use

$$p(t) = (\cos(2\pi t), \sin(2\pi t))$$

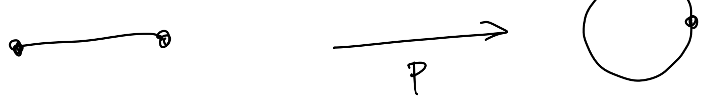
(usually we won't write formulas for p)

p must be <u>onto</u> (surjective)

<u>Def</u> For a top space X, and any set A with a surjection $p: X \to A$, this creates a topology on A called the <u>quotient topology</u> defined by:

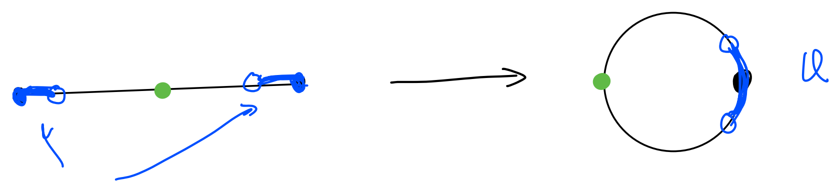
$$\boxed{\begin{array}{l} \mathcal{U} \text{ is open in } A \\ \text{iff } \; p^{-1}(\mathcal{U}) \text{ is open in } X. \end{array}}$$

$X = [0,1]$ with subspace top from $\mathbb{R}$ $\qquad$ $A = S^1$



open in X, $\quad$ so $\mathcal{U}$ is open in A.

not open in X, $\quad$ so $\mathcal{U}$ is not open in A.



$p^{-1}(\mathcal{U})$ is open in $[0,1]$ with the subspace top. So $\mathcal{U}$ is open in S^1.

S^1, using quot top from $[0,1]$ is the same as the subspace top from $\mathbb{R}^2$.

Thm For $p: X \to A$ onto, the quot top. on A is a topology.

Pf $\underline{i}$ ($\emptyset$ & A are open) $\qquad X \xrightarrow{p} A$

$\underline{\emptyset}$: is $\emptyset$ open in A?

is $p^{-1}(\emptyset)$ open in X?

$p^{-1}(\emptyset) = \emptyset$ is open in X (since X is a top. space)

so $\emptyset$ is open in A.

$\underline{A}$: $p^{-1}(A) = X$ since p is onto

so $p^{-1}(A)$ is open in X,

so A is open in A.

ii (finite $\cap$s)

Take $U_1, \ldots, U_n$ open in A,

WTS $\bigcap_{i=1}^{n} U_i$ is open in A.

Since U_i is open in A,

then $p^{-1}(U_i)$ is open in X.

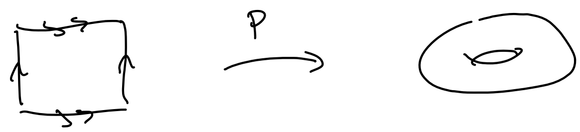
So $\bigcap_{i=1}^{n} p^{-1}(U_i)$ is open in X (fin. $\cap$s prop in X)

but $\underbrace{\bigcap_{i=1}^{n} p^{-1}(U_i)}_{\text{open in } X} = p^{-1}\left(\bigcap_{i=1}^{n} U_i\right)$

So $p^{-1}\left(\bigcap_{i=1}^{n} U_i\right)$ is open in X,

so $\bigcap_{i=1}^{n} U_i$ is open in A.

iii (unions) Same as above.



p as a function is complicated.

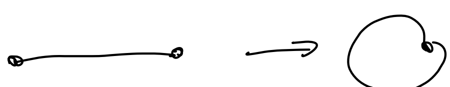
Easier to express using <u>equivalence relations</u>

In X, we imagine some points are equivalent to some others.

$[0,1]$

we are regarding $0 \sim 1$, no other points are equivalent.
"glue them together"

If X is a top space with some equiv. relation, let X^* be the set of equivalence classes.



X

here, X^* consists of $\{0,1\}$, plus all $\{x\}$ when $0<x<1$

Then the quotient map p is just

$$p : X \longrightarrow X^*$$

where on a point, $p(x) = [x]$

the class of x.