Functions!

We will finally say what it means for 2 top. spaces to be "the same"

Top spaces have continuous functions

In $\mathbb{R}$: f is continuous means: $\forall a \in \mathbb{R}$,

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $\quad 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

rephrase without using $|\ldots - \ldots|$

Thm from $\mathbb{R}$-analysis:

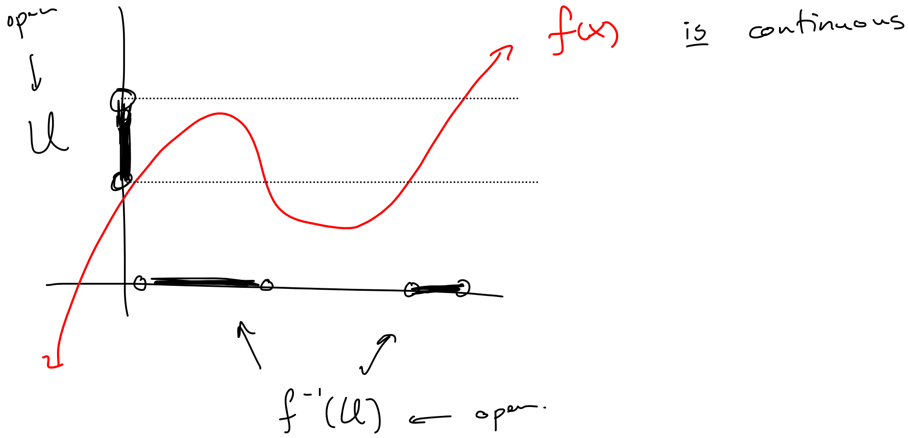
Thm f is continuous iff whenever $U \subseteq \mathbb{R}$ is open,

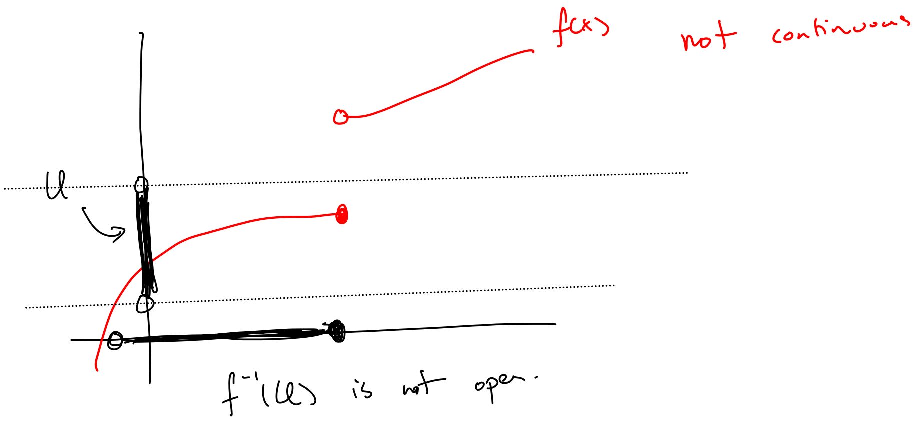
then $f^{-1}(U) \subseteq \mathbb{R}$ is open.

We use this as our definition.

Def Let X, Y be top. spaces, and a function $f: X \to Y$. Then f is continuous iff: if $U \subseteq Y$ is open, then $f^{-1}(U) \subseteq X$ is open

(U open $\Rightarrow f^{-1}(U)$ open)

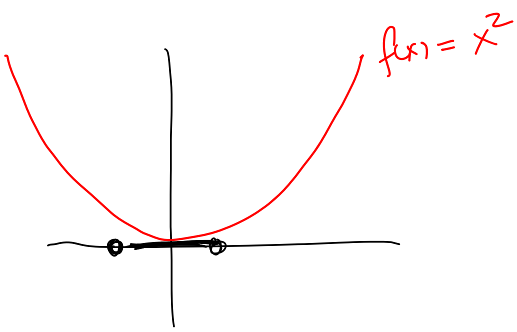




Continuous means $\underline{\mathcal{U} \text{ open} \Rightarrow f^{-1}(\mathcal{U}) \text{ open}}$

Don't say $\mathcal{U}$ open $\Rightarrow f(\mathcal{U})$ open

this isn't true for cont. functions



$\mathcal{U} = (-1, 1)$ open

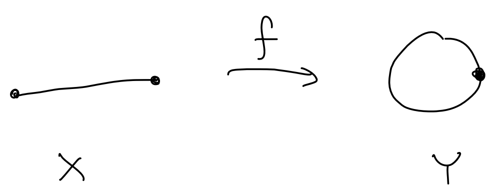
$f(\mathcal{U}) = [0, 1)$ not open.

cts means $\mathcal{U}$ open $\Rightarrow f^{-1}(\mathcal{U})$ is open

it's good enough to assume $\mathcal{U}$ is a basis open set.

Ex $X = [0,1]$ with subspace from $\mathbb{R}$

$Y = S^1$ circle, subspace from $\mathbb{R}^2$

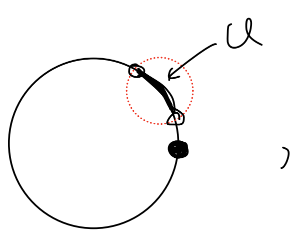


wraps the interval around so $f(0) = f(1)$.

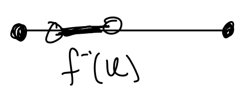
Show f is continuous. ($\mathcal{U}$ open in $Y \Rightarrow f^{-1}(\mathcal{U})$ open in X)

Let $\mathcal{U} \subset Y$ be a basis open set. WTS $f^{-1}(\mathcal{U})$ is open in X.

$\mathcal{U}$ looks like

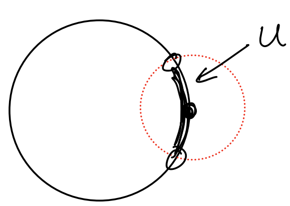


, then $f^{-1}(\mathcal{U})$ looks like:

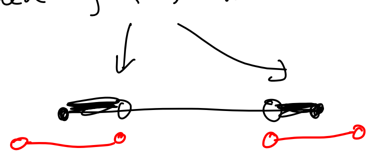


Here $f^{-1}(\mathcal{U})$ is open in $[0,1]$ as desired.

OR



then $f^{-1}(\mathcal{U})$ looks like:



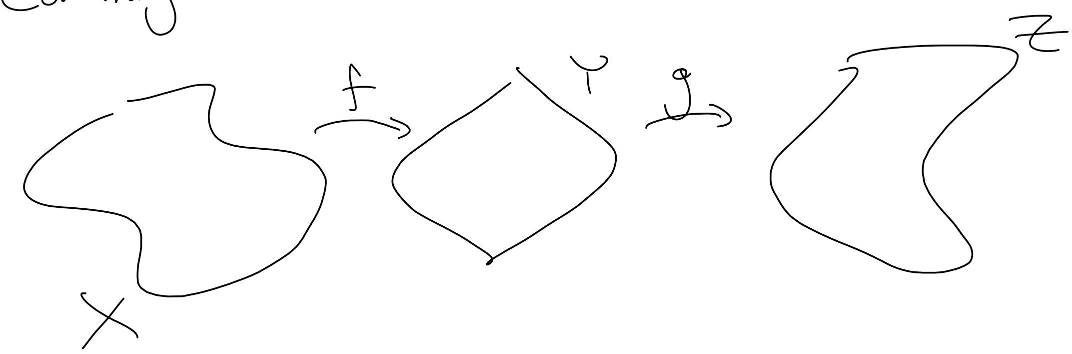
Here $f^{-1}(\mathcal{U})$ is open in $[0,1]$ as desired.

2 Thms about continuous fcns.

<u>#0: (on HW)</u> f is continuous iff

U closed $\Rightarrow f^{-1}(U)$ closed.

Combining continuous functions:



<u>Thm</u> Let X, Y, Z be top spaces, let

$f: X \to Y$ be continuous

$g: Y \to Z$ be continuous.

Then $g \circ f : X \to Z$ <u>is continuous</u>.

<u>Pf</u> Let $U \subseteq Z$ be open, will show $(g \circ f)^{-1}(U)$ is open.

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$$(g \circ f)^{-1}(U) = f^{-1}\left(\underline{g^{-1}(U)}\right)$$

since g is cts, $g^{-1}(U)$ is open.

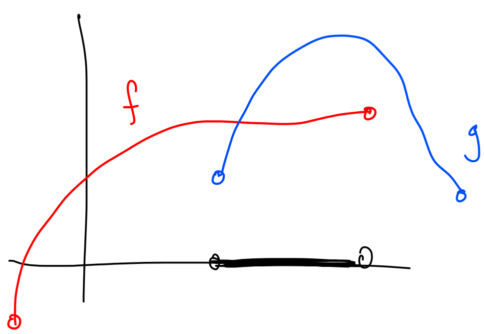
since f is cts, this means

$$f^{-1}(g^{-1}(U)) \text{ is open.}$$

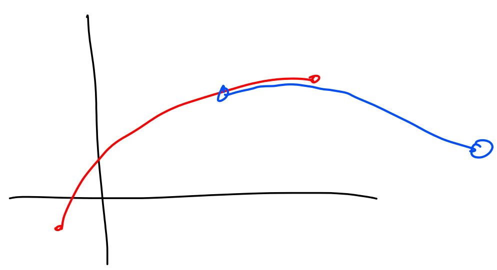
So $(g \circ f)^{-1}(U)$ is open. Shown.

Thm #2 "The pasting lemma"

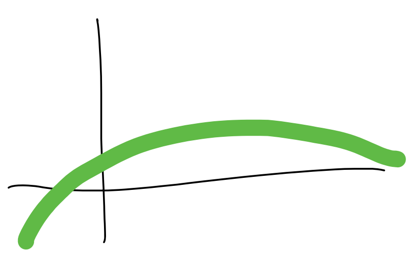
When 2 cont. functions have overlapping domains.



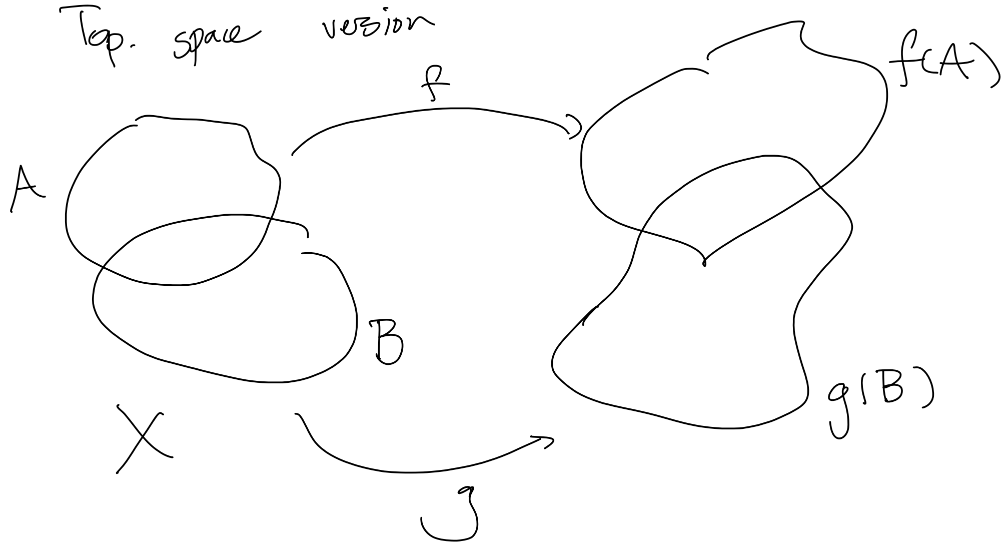
can't be pasted



they can be "pasted" into a big function



Top. space version



if f & g are cts and they agree on $A \cap B$, then they can build a fcn on $A \cup B$ which is continuous.