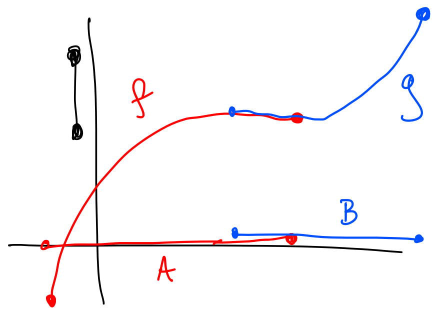
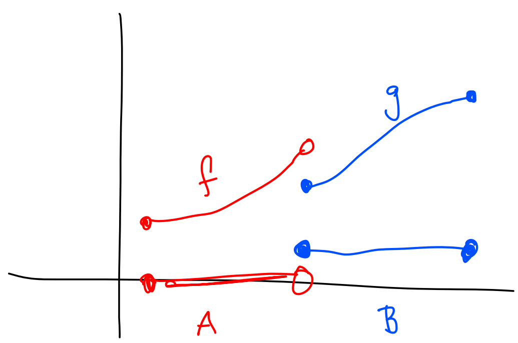
Patching Continuous fcns :



If f & g are equal on $A \cap B$, then f & g can be "pasted" to make a contin. fcn on $A \cup B$.

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

A & B must be closed:



these can't be pasted together.

Thm (Pasting Lemma) Let X & Y be top. spaces, let $X = A \cup B$ where A & B are both closed.

Let $f: A \to Y$ & $g: B \to Y$ be continuous; with $f(x) = g(x)$ $\forall x \in A \cap B$.

Then:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$ is continuous.

Pf We'll use the "closed" version of continuity.
(f is cts when: $\mathcal{U}$ closed $\Rightarrow f^{-1}(\mathcal{U})$ closed)

Let $C \subseteq Y$ be closed, we'll show $h^{-1}(C)$ is closed in X.

Note that: $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$

Since f is cts & C is closed, $f^{-1}(C)$ is closed in $A \subseteq X$

Also g - - - - - - - - , $g^{-1}(C)$ is closed in $B \subseteq X$.

$f^{-1}(C)$ is closed in A means: $f^{-1}(C) = D \cap A$ for D closed in X.

$g^{-1}(C)$ - - - - B - - - $g^{-1}(C) = E \cap B$ for E closed in X.

So $h^{-1}(C) = (\underset{\text{closed in } X}{D} \cap \underset{\text{closed in } X}{A}) \cup (\underset{\text{closed in } X}{E} \cap \underset{\text{closed in } X}{B})$

So $h^{-1}(C)$ is closed since it's a $\cup$ & $\cap$ of closed sets.

Homeomorphisms

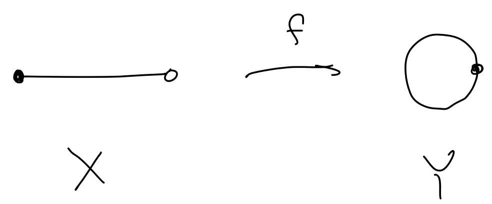
The basic idea of two top. spaces being "the same"

X & Y are "the same" when

$\exists\ f: X \to Y$ such that:

- f is a bijection ← (1-1 & onto) injective & surjective
- f is continuous
- f^{-1} is continuous

bijection & cts is not enough:

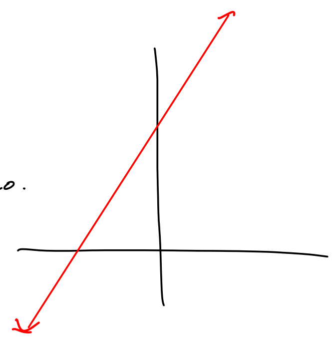


this is continuous & bijective, but not a homeomorphism

Def A function $f: X \to Y$ is a homeomorphism if f is a bijection, f is continuous, and f^{-1} is continuous.

In this case we say X & Y are homeomorphic or topologically equivalent.

Simple examples:



Ex $f(x) = 3x+4$ is a homeo.

It's a bijection because it has an inverse function:

$$y = 3x+4 \rightarrow x = \frac{y-4}{3}$$

so $f^{-1}(x) = \frac{x-4}{3}$

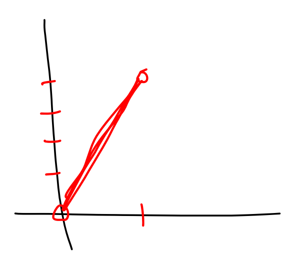
Both f & f^{-1} are continuous, so it's a homeo.

$f: \mathbb{R} \rightarrow \mathbb{R}$ so $\mathbb{R}$ is homeo. to $\mathbb{R}$

Any two open intervals are homeomorphic:

ex: $\underset{X}{\overset{\shortparallel}{(0,1)}}$ is homeo. to $\underset{Y}{\overset{\shortparallel}{(0,4)}}$

We need to create $f: X \rightarrow Y$



$$f(x) = 4x$$

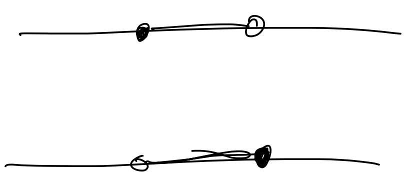
it's a bijection: $f^{-1}(x) = \frac{1}{4}x$

Both are continuous.

Any two open intervals are top. equiv.
Also any two closed intervals are top. equiv.
Also any intervals $[a,b)$ & $[c,d)$
are top. equiv.

How about $[a,b)$ vs $(c,d]$

like $[0,1)$ vs $(0,1]$



Use
$f(x) = 1-x$
this is a bijection, f & f^{-1} are cts.

So any half-open intervals will be homeomorphic.

What about $[a,\infty)$ or (a,∞)
or $\mathbb{R}$?