$f: X \to Y$ is a homeomorphism when:

f is a bijection,

f is continuous, & f^{-1} is continuous.

Then we say X & Y are topologically equiv.

or homeomorphic.

write $X \cong Y$

in $\mathbb{R}$, (standard)

any open intervals are homeomorphic to each other

also closed — — — — — — —

also any intervals like $[a,b)$ or $(a,b]$

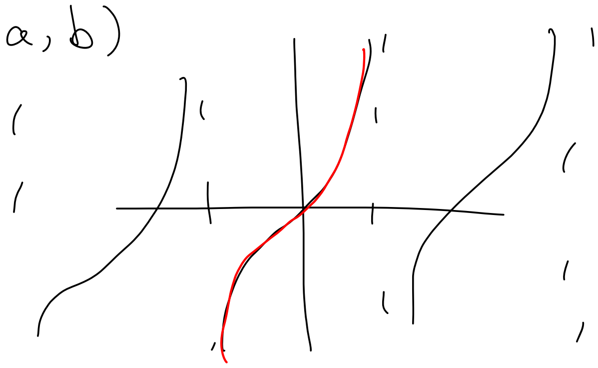
are all homeomorphic.

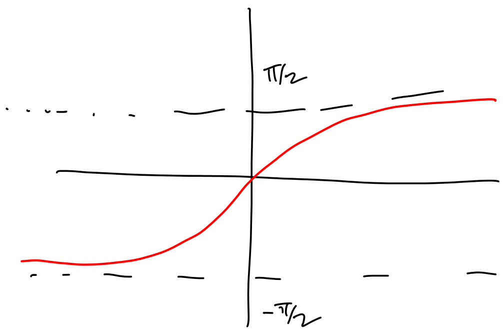
What about unbounded intervals?

Actually all of $\mathbb{R}$ is homeo to any open interval.

$\mathbb{R} \cong (a,b)$

$f(x) = \arctan x$





here, $f: \mathbb{R} \to (-\pi/2, \pi/2)$

is a bijection,

f & f^{-1} are continuous.

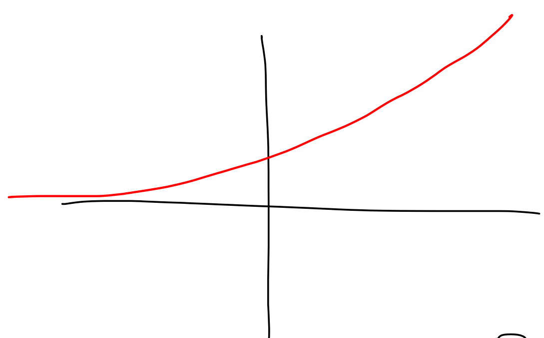
So f is a homeo,

So $\mathbb{R} \cong (-\pi/2, \pi/2)$

So $\mathbb{R}$ is homeo. to any bounded open interval.

What about $\mathbb{R}$ vs $(0, \infty)$

We need a function $f: \mathbb{R} \to (0, \infty)$



use $f(x) = e^x$.

this is cont with cont. inverse

So this is a homeo

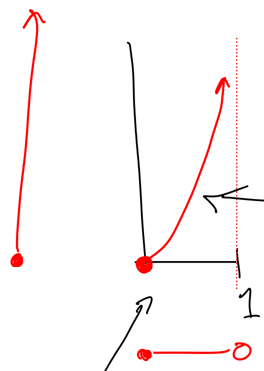
So $\mathbb{R} \cong (0, \infty)$

and $\mathbb{R} \cong (-\infty, 0)$ use. $f(x) = -e^x$

$\mathbb{R} \cong (a, \infty)$ and $\mathbb{R} \cong (-\infty, a)$

What about $[0, \infty)$

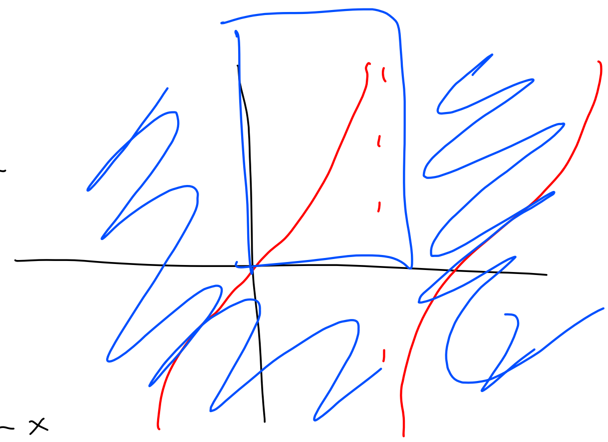
should be $[0,\infty) \cong [0,1)$



$f: [0,1) \to [0,\infty)$

this is a bijection with continuous & cont. inverse.

this is a piece of $\tan x$



The restriction of $\tan x$ to $[0, \pi/2)$ makes a homeo $[0,\pi/2) \to [0,\infty)$

In $\mathbb{R}$, standard top,

Categories of homeomorphic sets:

$[\ \mathbb{R},\ (a,b),\ (a,\infty),\ (-\infty,a)\ ]$

$[\ [a,b]\ ]$

$[\ [a,b),\ (a,b],\ [a,\infty),\ (-\infty,a]\ ]$

$\mathbb{R} \not\cong [a,b]$ (but hard to prove)

<u>Thm</u> $\cong$ is an equivalence relation.

<u>Pf</u> Will show refl, sym, trans

<u>Refl</u> WTS $X \cong X$

we want $f: X \to X$ cont. bij with cont. inv.

Use $f(x) = x$ "the identity function"

this is a homeo as desired.

<u>Sym</u> WTS $X \cong Y \Rightarrow Y \cong X$.

let $X \cong Y$, so $\exists f: X \to Y$, f is a homeo.

Since f is homeo, we can use f^{-1}

$f^{-1}: Y \to X$ is a bijection.

f^{-1} & $\underset{\substack{\shortparallel \\ f}}{(f^{-1})^{-1}}$ are cont. since f is homeo.

so $Y \cong X$.

<u>Trans</u> WTS $X \cong Y$ & $Y \cong Z \Rightarrow X \cong Z$.

Assume $\exists f: X \to Y$ a homeo
$\exists g: Y \to Z$ a homeo.

We can use $\underline{g \circ f} : X \to Z$

$g \circ f$ is a bijection since $f \& g$ are bijections

$g \circ f$ is cont. since $g \& f$ are cont.

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is cont. since $f^{-1} \& g^{-1}$ are cont.

so $g \circ f$ is a homeo so $X \cong Z$.

Homeomorphisms act nice on open sets.

For $f: X \to Y$ a homeo:

f is cont: $\mathcal{U}$ open $\Rightarrow f^{-1}(\mathcal{U})$ open.

f^{-1} is cont: $\mathcal{U}$ open $\Rightarrow \underbrace{(f^{-1})^{-1}(\mathcal{U})}_{f(\mathcal{U})}$ open.

$f(\mathcal{U})$ open.

f is a homeo means:

$\mathcal{U}$ open $\overset{\Longleftrightarrow}{\text{iff}}$ $f(\mathcal{U})$ open.

A homeo makes an exact correspondence

between all the open sets in X,
and - - - - - - Y.

So, any property of X defined using open sets must be the same for Y.

any topological property

Cor If $X \cong Y$ and X is Hausdorff, then Y is Hausdorff.

We can show $X \not\cong Y$ by demonstrating some specific top. property that's different in X & Y.

e.g. $\mathbb{R} \not\cong \mathbb{R}_{fc}$

Hausd. not Hausd.

$\mathbb{R} \not\cong \mathbb{R}_\ell$

in $\mathbb{R}$, the only clopen sets are $\mathbb{R}$ & $\emptyset$.

in $\mathbb{R}_\ell$, any interval $[a, b)$ is clopen.

So $\mathbb{R}$ & $\mathbb{R}_\ell$ are not homeo.