Test next week! (Friday)

Homeomorphisms

Def $f: X \to Y$ is a homeomorphism when

f is a bijection (1-1 & onto)

f is continuous

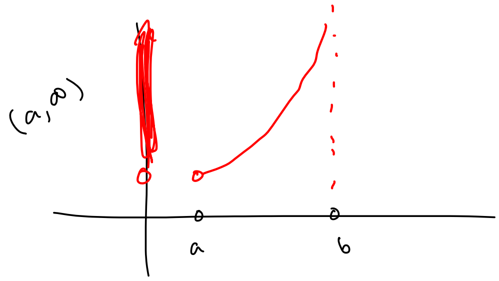
f^{-1} is continuous.

Then X & Y are top. equivalent or homeomorphic

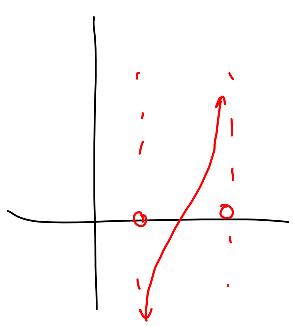
Any (a, b) is homeo to (c, d)

and also (a, ∞)

etc. also $\mathbb{R}$



(a,b)

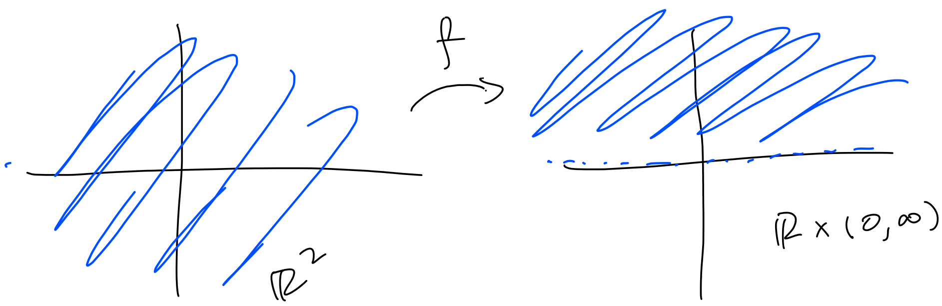


Homeos in $\mathbb{R}^2$:

$$\mathbb{R} \overset{\text{homeo}}{\cong} (0, \infty)$$

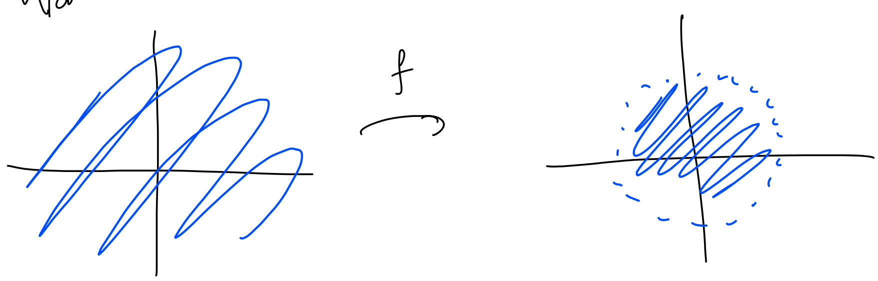
so automatically

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \cong \mathbb{R} \times (0, \infty)$$



$$f(x, y) = (x, e^y)$$

Also

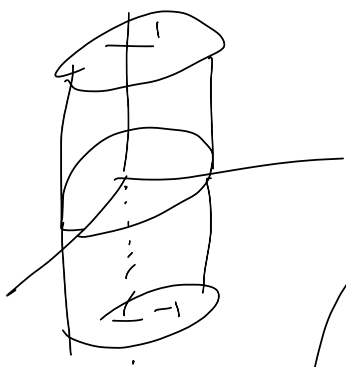


$$f(r, \theta) = \left(\frac{r}{r+1}, \theta\right)$$

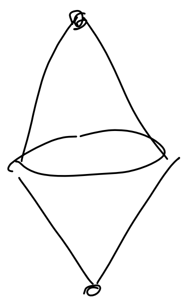
so $\mathbb{R}^2 \cong \{(r, \theta) \mid r < 1\}$

A quotient:

Let $X = S^1 \times [-1, 1]$



Let X^* be the quotient obtained by identifying the top circle to a point and the bottom circle to a point.



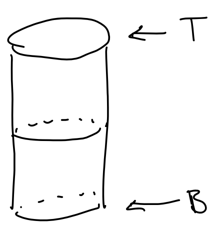
like this,

it's homeomorphic to S^2 (sphere in $\mathbb{R}^3$)

We need a function

$f: X^* \to S^2$ which is bijection and f & f^{-1} cont.

X



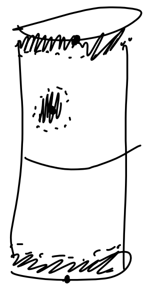
in X^*, it's a quotient according the partition:

$A_{x,y,z} = \{(x, y, z) \mid z \neq -1, z \neq 1\}$

$T = \{(x, y, 1) \mid (x, y) \in S^1\}$

$B = \{(x, y, -1) \mid (x, y) \in S^1\}$

in X^*, open sets look like



$$f: X^* \to S^2$$

(r, θ, z)

In cylindrical coords:

$$f(r, \theta, z) = \left(r\sqrt{1-z^2}, \theta, z\right)$$

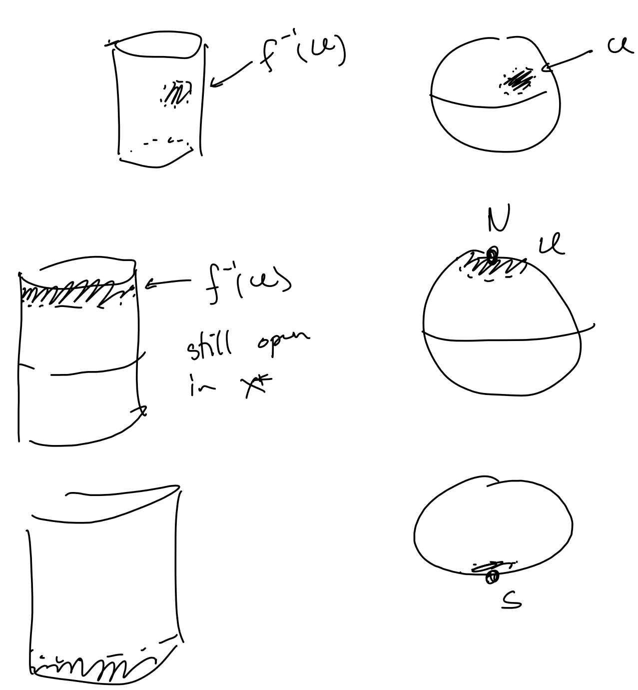
If the domain is X, it's not 1-1,
since the upper circle goes to a single point.

To make it 1-1, we use X^*

So $f: X^* \to S^2$ is 1-1 & onto.

is f continuous?

WTS if U is open, then $f^{-1}(U)$ is open.



This idea shows f is continuous.

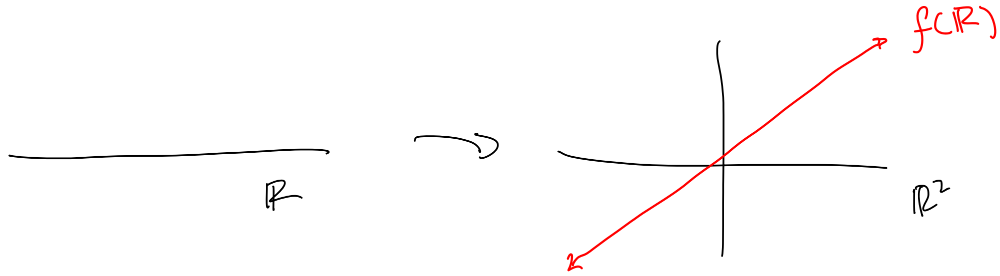
Similarly, f^{-1} is cont.

So $f: X^* \to S^2$ is a homeo.

It's useful to discuss functions which are almost-homeo, but not onto.

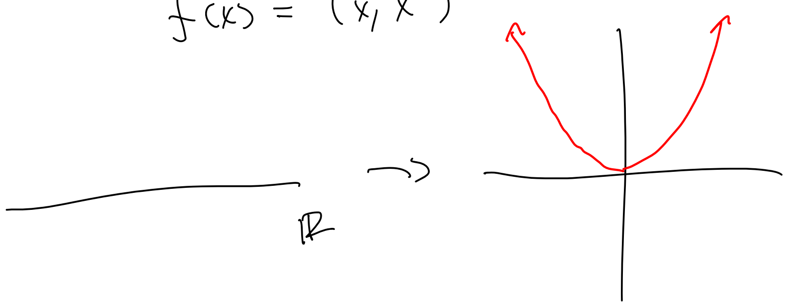
<u>Def</u> If $f: X \to Y$ is 1-1 and f & f^{-1} are continuous, then f is an <u>embedding</u> of X in Y

an embedding of $\mathbb{R}$ into $\mathbb{R}^2$ looks like:



f is homeo "onto its image"

$$f(x) = (x, x^2)$$

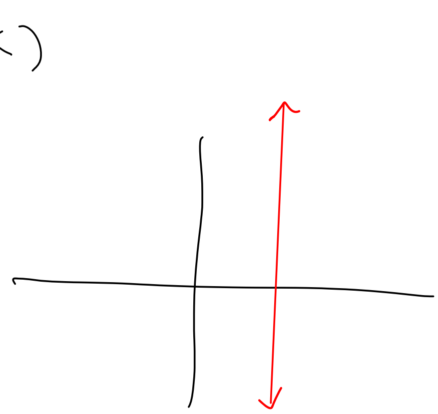


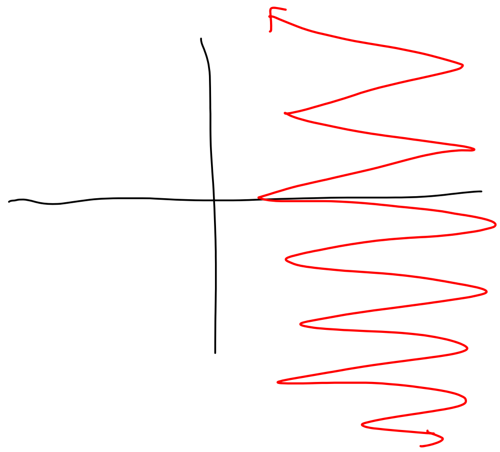
Since f is 1-1 and f & f^{-1} are continuous,

automatically if $f : X \to Y$ is an embedding,

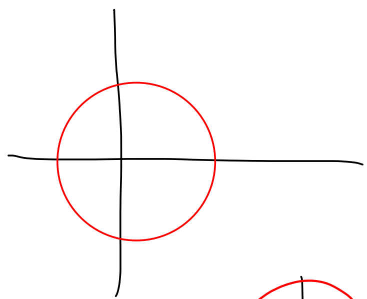
then $X \cong f(X)$

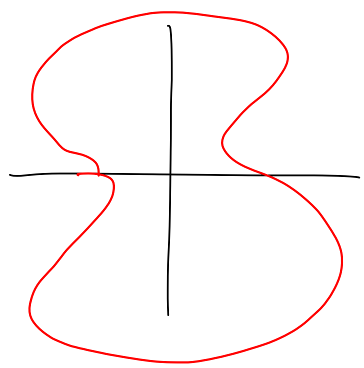
Another embedding $\mathbb{R} \to \mathbb{R}^2$

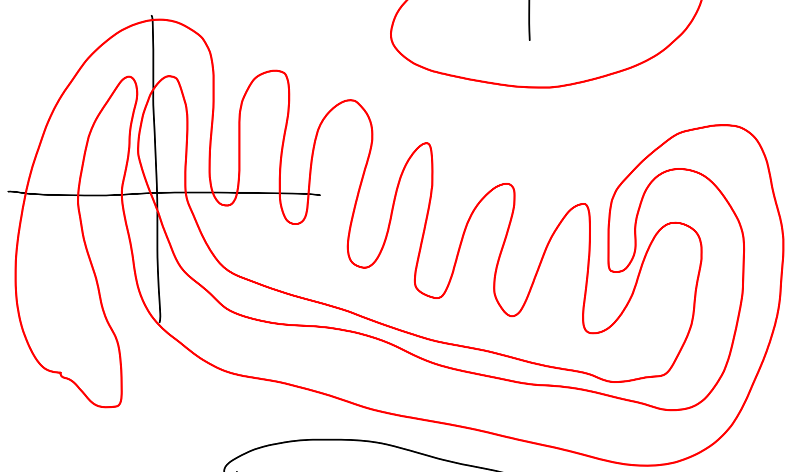


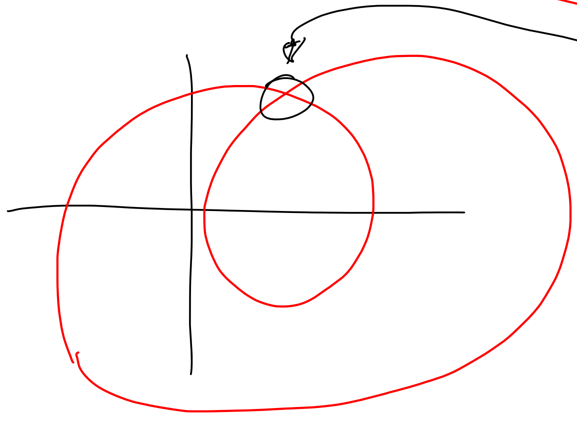


Embeddings of S^1 into $\mathbb{R}^2$:









not an embedding

not 1-1 here

$S^1 \to \mathbb{R}^3$ embeddings: