Metric Spaces

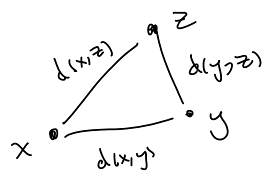
A set with some <u>metric</u> ← a distance function.

The metric looks like $d(x,y) = \underline{\quad\text{a real \#}\quad}$

satisfying some properties:

<u>Def</u> For a set X, a function $d: X \times X \to \mathbb{R}$ is a <u>metric</u> when:

- "positive-definite" • $d(x,y) \geq 0 \ \forall x,y \in X$, and $d(x,y) = 0 \Rightarrow x = y$
- "symmetric" • $d(x,y) = d(y,x)$
- "Δ-ineq" • $d(x,y) + d(y,z) \geq d(x,z)$

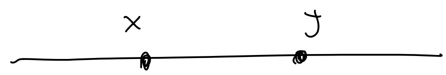


This d could be the ordinary distance function in $\mathbb{R}^2$, OR any other thing satisfying the 3 properties.

If X has a metric d, we say (X, d) is a <u>metric space</u>

Q's: Is every metric space a top space?
Is every top. space a metric space.

Metric spaces



$\mathbb{R}$ is a metric space, using

$$d(x,y) = |x-y|$$

is it a metric?

pos. def $\quad |x-y| \geq 0$ and if $|x-y|=0$, then $x=y$

Symmetric $\quad |x-y| = |y-x|$

Δ-ineq $\quad |x-y| + |y-z| \geq |x-z|$

also true.

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in $\mathbb{R}$, $\quad d_{disc}(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ "the discrete metric"

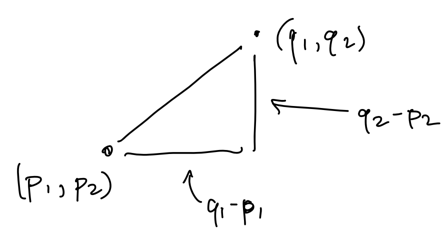
pos. def $\quad d(x,y) \geq 0$ and $d(x,y)=0 \Rightarrow x=y$

sym $\quad d(x,y) = d(y,x)$

Δ-ineq $\quad$ WTS $\quad d(x,y) + d(y,z) \geq d(x,z)$

if x,y,z all different, then it says $2 \geq 1$

other cases similar.

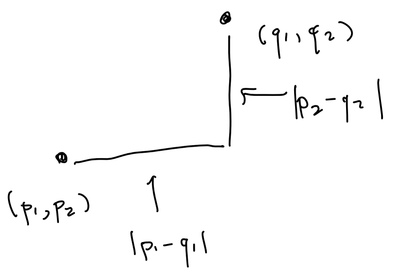


$\mathbb{R}^2$ The standard metric (Euclidean metric)

$$d(p,q) = d((p_1,p_2),(q_1,q_2)) = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}$$

This is a metric!

Ex $d_T((p_1,p_2),(q_1,q_2)) = |p_1-q_1| + |p_2-q_2|$



This is distance if you can only move horz & vert.

"the taxicab metric"

"the Manhattan metric"

show it's a metric $d_T((\overset{p}{p_1,p_2}),(\overset{q}{q_1,q_2})) = |p_1-q_1| + |p_2-q_2|$

pos def WTS $d(p,q) \geq 0$ and $d(p,q)=0 \Rightarrow p=q$

$d(p,q) \geq 0$ $d(p,q) = |\sim| + |\sim| \geq 0$ (sum of abs vals)

$d(p,q)=0 \Rightarrow p=q$ Assume $d(p,q)=0$

$|p_1-q_1| + |p_2-q_2| = 0$

so $|p_1-q_1| = 0$ and $|p_2-q_2| = 0$

so $p_1=q_1$ and $p_2=q_2$ so $(p_1,q_1)=(p_2,q_2)$

so $p=q$

Sym WTS $d(p,q) = d(q,p)$

$|p_1-q_1| + |p_2-q_2| \longleftrightarrow |q_1-p_1| + |q_2-p_2|$

the same!

Δ-ineq WTS $d(p,q) + d(q,r) \geq d(p,r)$

$$d(p,q) + d(q,r) = |p_1-q_1| + |p_2-q_2| + |q_1-r_1| + |q_2-r_2|$$

$$= \underbrace{|p_1-q_1| + |q_1-r_1|} + \underbrace{|p_2-q_2| + |q_2-r_2|}$$

$$\geq |p_1-r_1| + |p_2-r_2| = d(p,r)$$

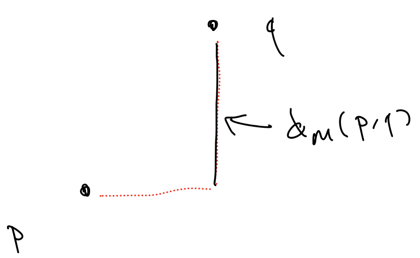
$|x-y| + |y-z| \geq |x-z|$

Another: $d_M((p_1,p_2),(q_1,q_2)) = \max\left(|p_1-q_1|, |p_2-q_2|\right)$

"The maximum metric"

horiz. dist

vert. dist



This satisfies the 3 properties.

So d_M is a metric.

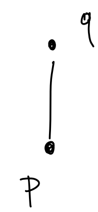
$$d_{min}(p,q) = \min\left(|p_1 - q_1|, |p_2 - q_2| \right)$$

is not a metric!

(it is symmetric)

<u>pos. def</u> $\quad d(p,q) \geq 0$? yes

$d(p,q) = 0 \Rightarrow p = q$



here, $|p_1 - q_1| = 0$

so $d_{min}(p,q) = 0$

but $p \neq q$.

So d_{min} is not pos def.