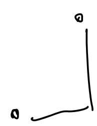
Metric Spaces

X with a metric $d: X \times X \to \mathbb{R}$

pos def $\quad d(x,y) \geq 0$ and $d(x,y) = 0 \Rightarrow x = y$

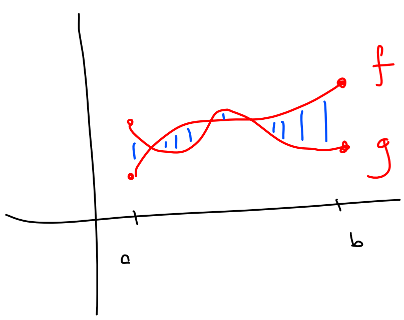
sym $\quad d(x,y) = d(y,x)$

Δ-ineq $\quad d(x,y) + d(y,z) \geq d(x,z)$



$$d_T(p,q) = |p_1 - q_1| + |p_2 - q_2|$$

For a closed interval $[a,b]$, let $C[a,b]$ be the set of all continuous functions $f: [a,b] \to \mathbb{R}$



$f, g \in C[a,b]$

Define:

$$d(f,g) = \int_a^b |f(x) - g(x)|\, dx$$

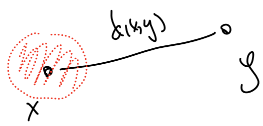
This is a metric. So $(C[a,b], d)$ is a metric space.

Metricspaces vs Top. spaces

A standard way to create a topology in a metric space.

We need to create "small neighborhoods" using the metric

We can define



$$B(x,\varepsilon) = \{ y \in X \mid d(x,y) < \varepsilon \}$$

"The open ball around x of radius ε"

"The open ε-ball"

The closed ε-ball is

$$\overline{B}(x,\varepsilon) = \{ y \in X \mid d(x,y) \leq \varepsilon \}$$

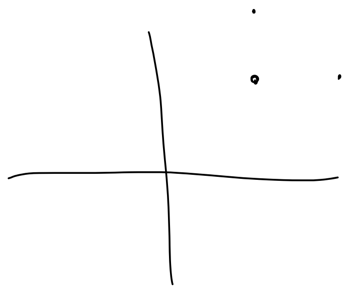
Balls can look different in different metrics:

<u>Standard metric in $\mathbb{R}^2$</u>

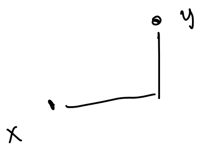
$B(x,1)$

a disc

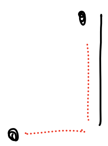
d_T Taxicab



$B(x,1)$



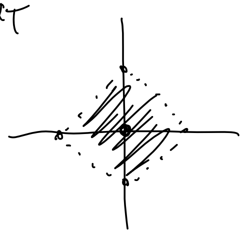
d_M



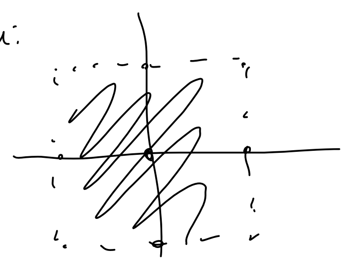
$d_{disc}(x,y) = 1$ always (unless $x=y$)

$B(x,1)$ & $\overline{B}(x,1)$

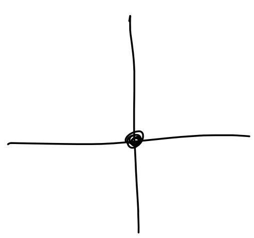
d_T



d_M:

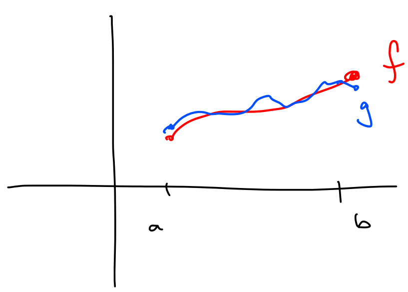


d_{disc}:



$B(x,1) = \{x\}$ $\qquad$ $B(x,2) = \mathbb{R}^2$

$\overline{B}(x,1) = \mathbb{R}^2$



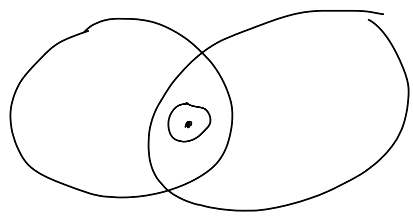
$B(f,1)$ is all "nearby" functions

the area difference is < 1.

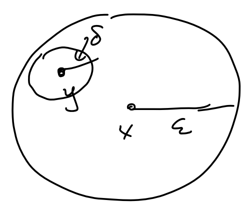
We'll use the ε-balls to build a topology,

We'll define a top. using the ε-balls as basis nbhds.

(involves the basis set shrinking property)

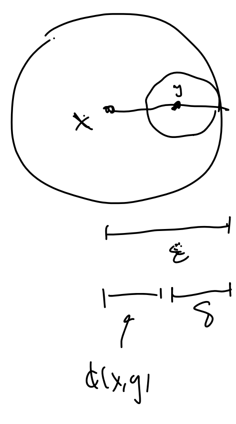


Lemma Given $x, y \in X$ a metric space, if $y \in B(x,\varepsilon)$, then $\exists \delta$ with $B(y,\delta) \subseteq B(x,\varepsilon)$



How to choose exactly what δ is?

Pf



We should choose

$$\delta = \varepsilon - d(x,y)$$

Now we'll show $B(y,\delta) \subseteq B(x,\varepsilon)$

Choose $z \in B(y,\delta)$, wts $z \in B(x,\varepsilon)$.

ie. $d(y,z) < \delta$ WTS $d(x,z) < \varepsilon$

Δ-ineq says: $d(x,y) + d(y,z) \geq d(x,z)$

$$d(x,z) \le d(x,y) + d(y,z)$$

$$< d(x,y) + \delta = \cancel{d(x,y)} + \varepsilon - \cancel{d(x,y)}$$

so $d(x,z) < \varepsilon$ Shown