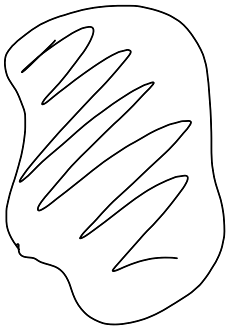
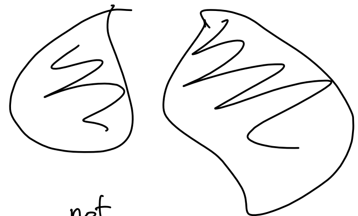
Connected Sets

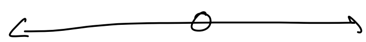


is connected



not connected.

$\mathbb{R}$ is connected



$\mathbb{R} - \{0\}$ is disconnected

is connected

$[1,2] \cup [3,4]$ is disconnected.

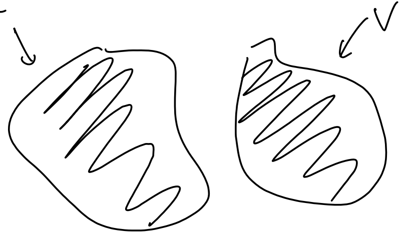
Disconnected is easier to define:

there is a separation into 2 components

Basically its $X = U \cup V$, where:

U

V



X

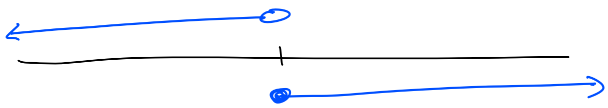
U & V are disjoint: $U \cap V = \emptyset$.

U & V are nonempty: $U \neq \emptyset$, $V \neq \emptyset$

U & V are both open

$$\mathbb{R} = \overset{U}{(-\infty, 0)} \cup \overset{V}{[0, \infty)}$$

Not allowed as a separation

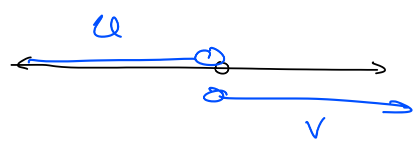


Def U & V are a separation of X when:

$X = U \cup V$, U & V are nonempty, disjoint open sets.

Def X is connected when there is no separation of X. X is disconnected when there is a separation of X.

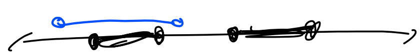
Ex] $\mathbb{R} - \{0\}$ is disconnected



A separation is: $U = (-\infty, 0)$

$V = (0, \infty)$

$X = [1,2] \cup [3,4]$



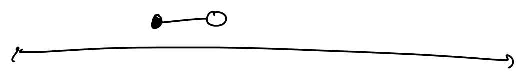
is disconnected. Let $U = [1,2]$

$V = [3,4]$

these are open in subspace top.

since $U = (0, 3) \cap X$ so U is open in subspace top.

$\mathbb{R}_\ell$ is <u>disconnected</u>!



a separation: $\mathbb{R}_\ell = \underset{U}{(-\infty, 0)} \cup \underset{V}{[0, \infty)}$

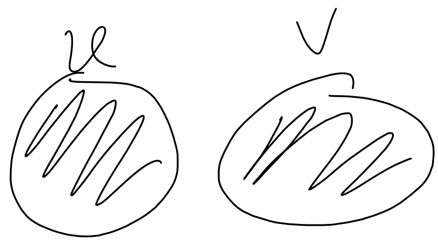
both open in $\mathbb{R}_\ell$

$\mathbb{Q}$

$$\mathbb{Q} = \underbrace{\left((-\infty, \pi) \cap \mathbb{Q}\right)}_{U} \cup \underbrace{\left((\pi, \infty) \cap \mathbb{Q}\right)}_{V}$$

Connectedness & clopen sets.

if X is disconnected:



U, V are each open.

is U closed?

i.e. is $X - U$ open?

X

$X - U = V$ and V is open,

so U is also closed.

So if $X = \mathcal{U} \cup V$ is a separation,
both $\mathcal{U}$ & V are clopen.

<u>Thm</u> X is disconnected <u>iff</u>

$\exists$ a clopen set $A \subset X$
with $A \neq \emptyset$, $A \neq X$.

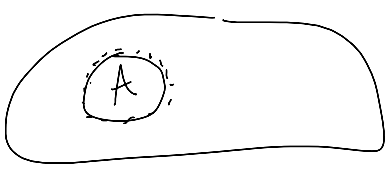
<u>Pf</u> $\Rightarrow$ is above.

If $X = \mathcal{U} \cup V$ is a separation,
then $\mathcal{U}$ is clopen.

<u>$\Leftarrow$</u> Assume $A \subset X$ is a proper nonempty clopen subset.

WTS X is disconnected.

For our separation, use



X

$\mathcal{U} = A$

$V = X - A$

So $X = \mathcal{U} \cup V$

$\mathcal{U}$ is open, also V is open since it's the complement of a closed.

$\mathcal{U} \cap V = \emptyset$ since V is the comp. of $\mathcal{U}$.

$\mathcal{U} \neq \emptyset$ & $V \neq \emptyset$ since A is a nonempty proper subset.

Sbam.

$\mathbb{R}_\ell$ is disconnected since

$[0,1)$ is clopen.

Any space X with discrete top:

all sets are clopen, so it is disconnected.

$\mathbb{R}_{fc}$ finite complement top.

open means complement is finite

closed means complement is infinite

So there is no proper nonempty clopen subset.

So $\mathbb{R}_{fc}$ <u>is</u> connected!

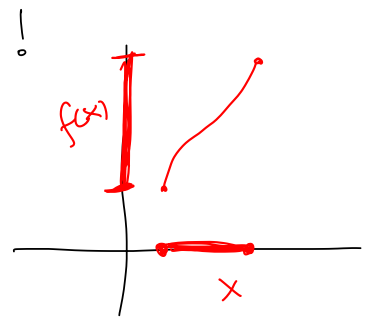
If X is connected & $f: X \to Y$ is continuous, is Y connected?

NO, since X might miss some of Y

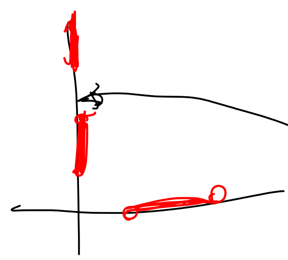
$$X = [0,1] \xrightarrow{id} Y = [0,1] \cup [3,4]$$

Thm If X is connected and $f: X \to Y$ is continuous, then $f(X)$ is connected. "IVT"

Important in calc!



can't get



Range must contain all values "in between"

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