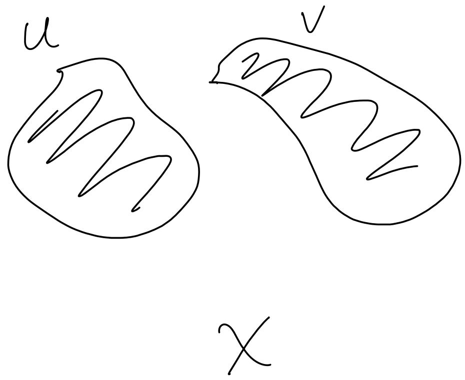


disconnected means $\exists$ a separation:

$X = U \cup V$

$U \cap V = \emptyset$

$U \neq \emptyset$, $V \neq \emptyset$,

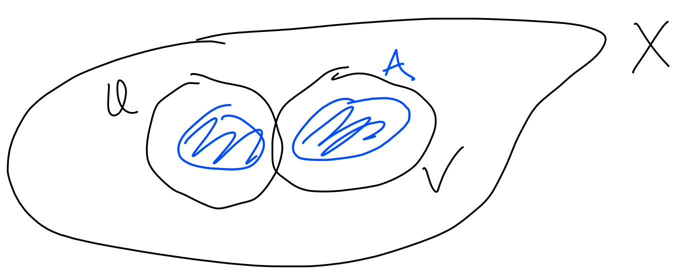
U & V are open

If $f: X \to Y$ is continuous & X is connected, then $f(X)$ is connected. ($f(X) \subset Y$ is connected)

Connectedness of a subset $A \subseteq X$.

$A \subset X$ is connected when A is connected as a space, using subspace topology.

What this means:



U, V are a separation of $A \subset X$ when:

$A \subseteq U \cup V$

$U \cap V \cap A = \emptyset$

$U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$

U & V are open.

<u>Thm</u> If $f: X \to Y$ is continuous:

X is connected $\Rightarrow$ $f(X)$ is connected.

<u>Pf</u> We'll prove the contrapositive:

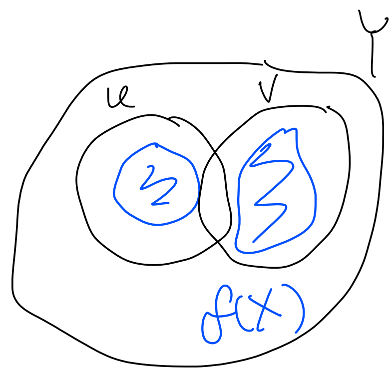
$f(X)$ is disconnected $\Rightarrow$ X is disconnected.

Since $f(X)$ is disconnected, $\exists\ U, V$ open sets

$f(X) \subset U \cup V$

with $U \cap V \cap f(X) = \emptyset$

$U \cap f(X) \neq \emptyset \quad V \cap f(X) \neq \emptyset$

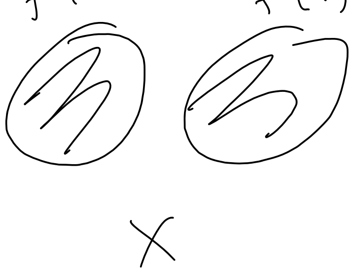


We'll build a separation of X,

using continuity of f.

Since f is continuous, $f^{-1}(U)$ is open

and $f^{-1}(V)$ is open.



We'll show $f^{-1}(U)$ & $f^{-1}(V)$ is a separation of X.

$f^{-1}(U)$ & $f^{-1}(V)$ are open ✓

$X = f^{-1}(U) \cup f^{-1}(V)$ since ✓

$f(X) \subset U \cup V$

$f^{-1}(U) \cap f^{-1}(V) = \emptyset$: since ✓

$U \cap V \cap f(X) = \emptyset$,

$f^{-1}(U)$ & $f^{-1}(V)$ are not empty,

since $U \cap f(X) \neq \emptyset$

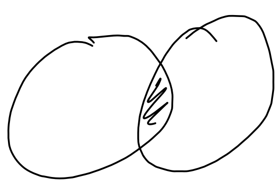
and $V \cap f(X) \neq \emptyset$. ✓

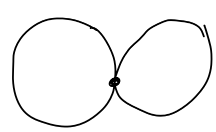
If C_1 & C_2 both connected,

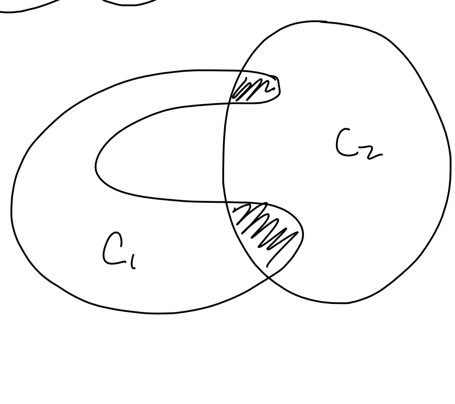
is $C_1 \cup C_2$ connected? ← No, but sometimes... if $C_1 \cap C_2 \neq \emptyset$, then yes

$C_1 \cap C_2$?

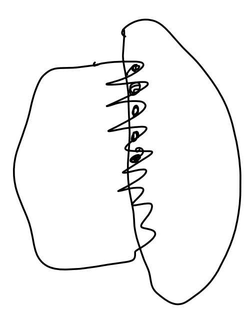
$C_1 \times C_2$?

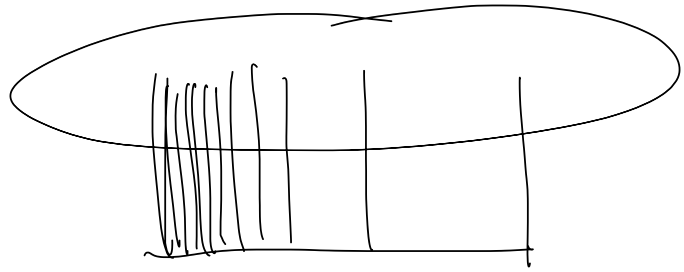




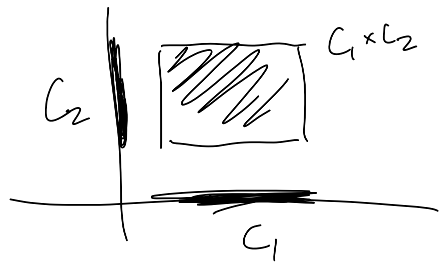


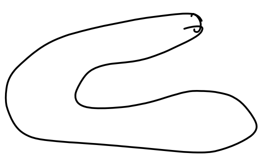
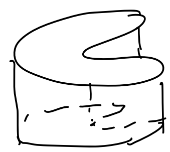
$C_1 \cap C_2$ is disconnected





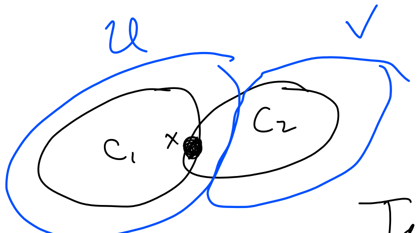
$C_1 \times C_2$



 $\times [0,1] =$ 

Thm If C_1 & C_2 are connected and $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is connected.

Pf To get a contradiction, assume $C_1 \cup C_2$ is disconn.



So $\exists$ open sets $\mathcal{U}$ & V, that separate $C_1 \cup C_2$.

Take $x \in C_1 \cap C_2$

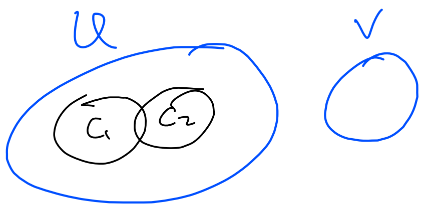
Since C_1 is connected, $\mathcal{U}$ & V cannot separate C_1, so $C_1 \subset \mathcal{U}$ or $C_1 \subset V$, not both.

Also $C_2 \subset \mathcal{U}$ or $C_2 \subset V$, not both.

$x \in C_1 \cap C_2$, WLOG assume $x \in \mathcal{U}$.

Since $x \in C_1$, this means $C_1 \subset \mathcal{U}$.

And $x \in C_2$, so $C_2 \subset \mathcal{U}$.



this is not allowed

Since C_1 & C_2 are in $\mathcal{U}$,

$$C_1 \cap C_2 \cap V = \emptyset$$

this is illegal if V is part of a separation of $C_1 \cup C_2$.