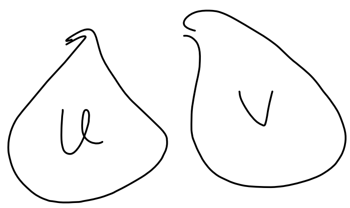
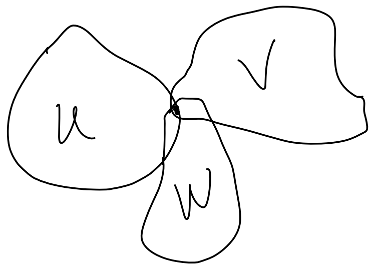
Connectedness:



not connected

Last time:
If C_1 & C_2 are connected
and $C_1 \cap C_2 \neq \emptyset$
then $C_1 \cup C_2$ is connected.



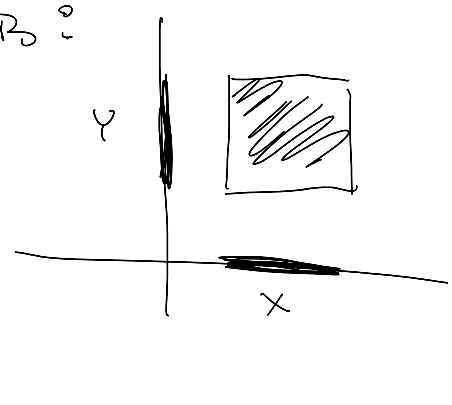
More generally we can have any number of sets:

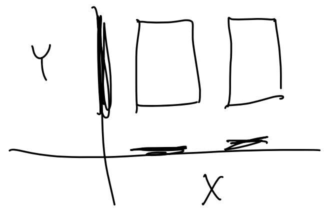
<u>Thm</u> If C_α are connected for all $\alpha \in A$,

and $\bigcap_{\alpha \in A} C_\alpha \neq \emptyset$, then

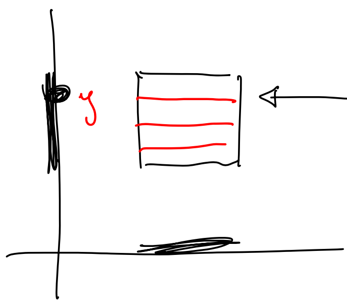
$\bigcup_{\alpha \in A} C_\alpha$ is connected.

Products of connected sets:





Thm If X & Y are connected, then $X \times Y$ is connected.

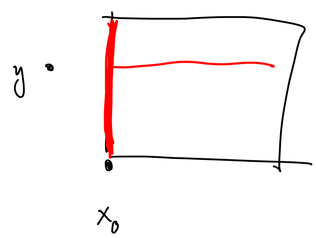


Write the product as a union

each bar looks like

$X \times \{y\}$ is connected, since it looks the same as X.

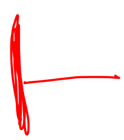
Use crosses:



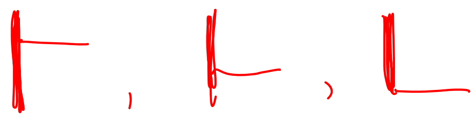
this is

$$(\{x_0\} \times Y) \cup (X \times \{y\})$$

these are connected, and they intersect at (x_0, y)

So  is connected.

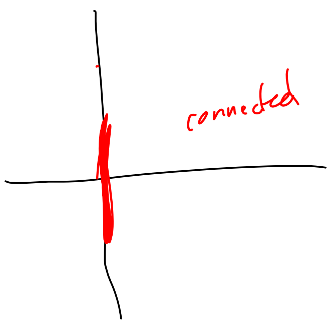
Take the union of all sets like

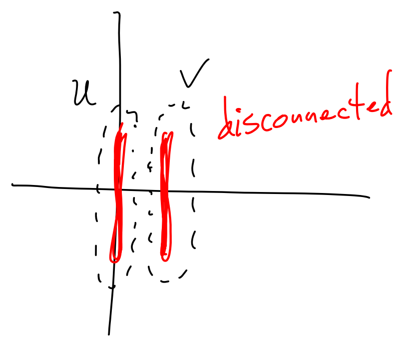


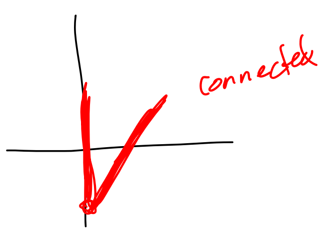
The union of all these is $X \times Y$,

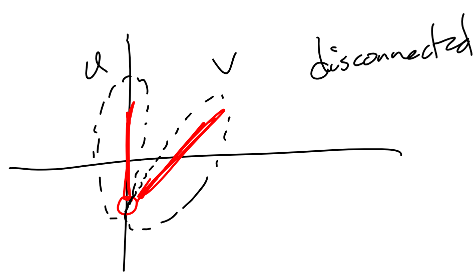
their $\cap$ is non-empty:

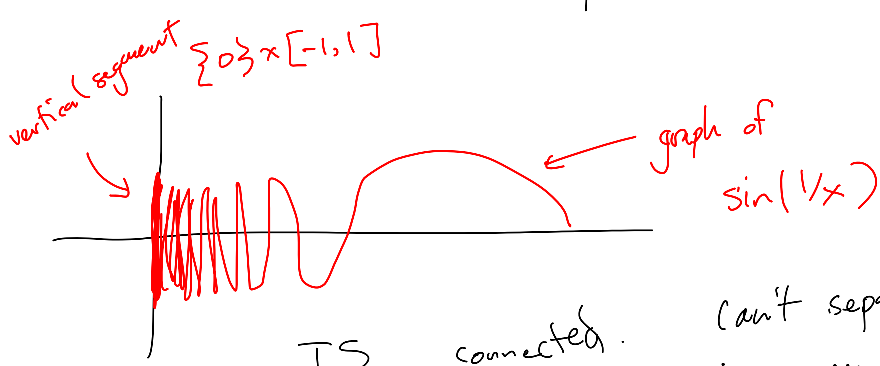
So $X \times Y$ is connected.







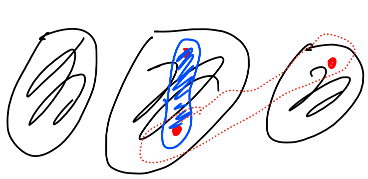




<u>IS</u> connected. Can't separate it using open U & V.

Components of a space

"the pieces of a disconnected space"



← this has 3 components

X Hard to say what a component is.

We can define: "x & y are in the same component"

write $\boxed{x \sim_c y \text{ when } \exists \text{ a connected set } A \text{ with } x, y \in A.}$

$\sim_c$ is an equivalence relation

Def A component of X is an equivalence class of points of X under the relation $\sim_c$.

equiv:

Def A component of X is a maximal connected subset.

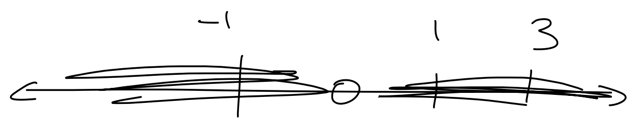
any larger set would be disconnected.

To show x & y are in the same component, just give a connected set that they're both in.

To show x & y are in different components we can use:

<u>Lem</u> x & y are in different components iff $\exists$ a separation $X = U \cup V$ where $x \in U$ & $y \in V$.

$X = \mathbb{R} - \{0\}$:



1 & 3 are in the same component,

since: 1 & 3 are both in $[1, 3]$, which is connected.

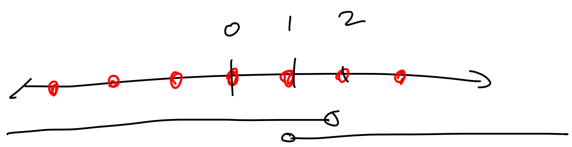
1 & -1 are in different components,

since: $X = \underline{(-\infty, 0)} \cup \underline{(0, \infty)}$

is a separation of X and $1 \in (0, \infty)$ and $-1 \in (-\infty, 0)$.

The components are $(-\infty, 0)$ & $(0, \infty)$

$\mathbb{Z}$:



1 & 2 are not in the same component since:

$$X = \underbrace{(-\infty, 2)\cap\mathbb{Z}}_{1 \in} \cup \underbrace{(1,\infty)\cap\mathbb{Z}}_{2 \in}$$

Each point by itself is a component.

<u>Def</u> If every component is a single point, we say X is <u>totally disconnected</u>.