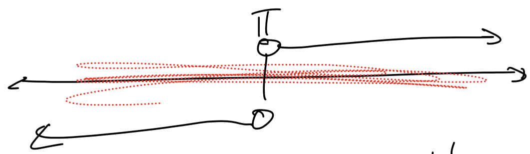
X is <u>totally</u> <u>disconnected</u> when the components are all single points.

$\mathbb{Z}$: 

$\mathbb{Q}$ 

$\mathbb{Q}$ is disconnected: $\big((-\infty,\pi)\cap\mathbb{Q}\big) \cup \big((\pi,\infty)\cap\mathbb{Q}\big)$

is a separation of $\mathbb{Q}$.

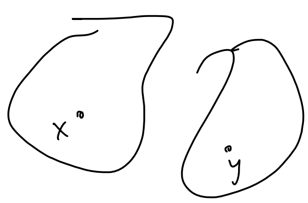
Actually it's <u>totally</u> disconnected:

Take $x,y \in \mathbb{Q}$, we'll show x & y are in different components when $x \neq y$.

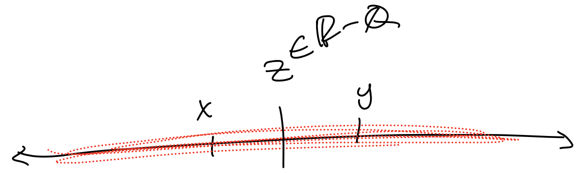
i.e. we'll find a separation

$$\mathbb{Q} = U \cup V$$

where $x \in U$, $y \in V$.



if $x,y \in \mathbb{Q}$ with $x<y$



$\exists\, z \notin \mathbb{Q}$ with $x<z<y$

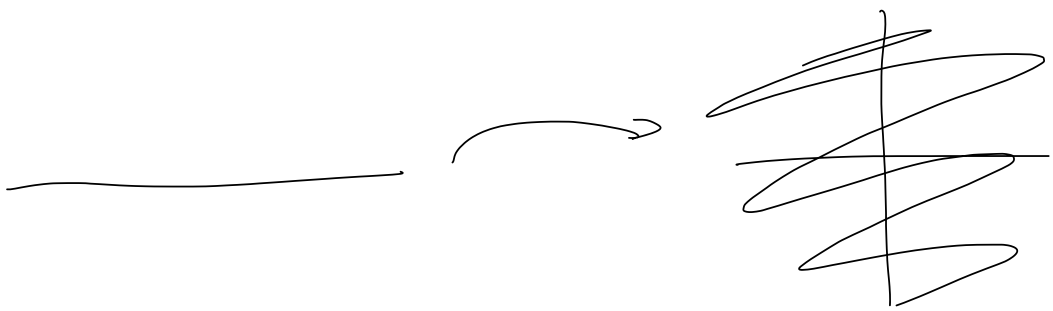
use $U = (-\infty, z) \cap \mathbb{Q}$
$V = (z,\infty) \cap \mathbb{Q}$

This is a separation, so x&y are in different components.

Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$? NO

hard to show using def. of homeo.

$f : \mathbb{R} \to \mathbb{R}^2$ bijection,
continuous f & f^{-1}



$\mathbb{R}$ & $\mathbb{R}^2$ are connected.

$\mathbb{R} - \{0\}$ is disconnected

but $\mathbb{R}^2 - \{\vec{0}\}$ is connected.

so $\mathbb{R} - \{0\} \not\cong \mathbb{R}^2 - \{0\}$

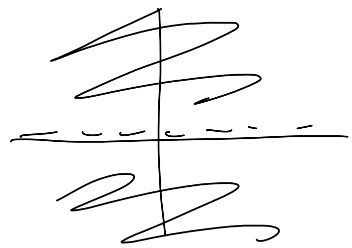
so $\mathbb{R} \not\cong \mathbb{R}^2$

Similarly $\mathbb{R} \not\cong \mathbb{R}^n$ for $n > 1$

how about $\mathbb{R}^2$ vs $\mathbb{R}^3$?

$\mathbb{R}^2 - \{x\text{-axis}\}$

is disconnected

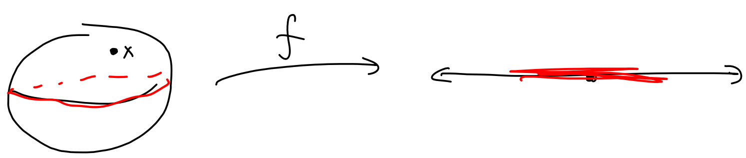


vs $\mathbb{R}^3 - \{x\text{-axis}\}$

is connected

Strange fact about maps

$f: S^2 \to \mathbb{R}$ (not homeos)



for any $x \in S^2$, $f(x)$ is a number.

(like on earth, temperature, or air pressure, or squirrel population density)

<u>Thm</u> If $f: S^2 \to \mathbb{R}$ is continuous, then $\exists\, x \in S^2$ where $f(x) = f(-x)$

the "antipodal" point

Pf Remember if $f: X \to Y$ and X is connected & f is continuous, then $f(X)$ is connected.

So $f(S^2)$ is connected.

Trick: Consider $g(x) = f(x) - f(-x)$

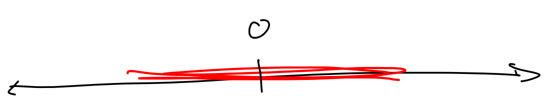
We'll show $\exists\, x$ with $g(x) = 0$

g is cont, so $g(S^2)$ is connected.

Note: $g(-x) = f(-x) - f(x) = -\left(f(x) - f(-x)\right) = -g(x)$

$g(-x)$ & $g(x)$ have opposite sign.

So $g(S^2)$ has some pos, some neg. values



Since $g(S^2)$ is connected, it must include 0.

This is a special case of the

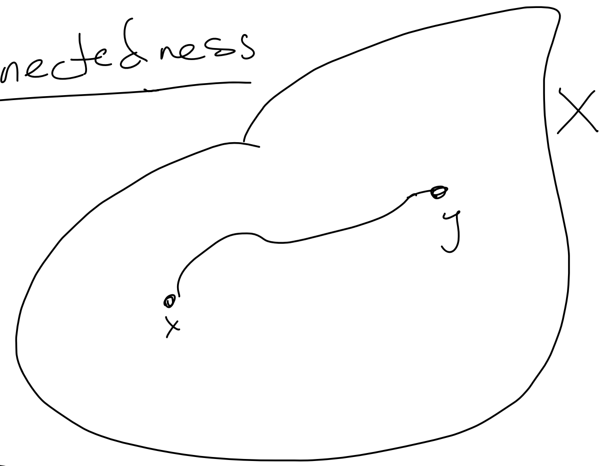
Borsuk – Ulam Theorem

for any $f: S^n \to \mathbb{R}^n$

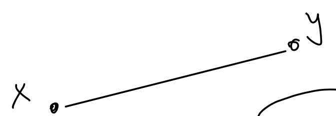
$\exists\, x \in S^n$ with $f(x) = f(-x)$

Path - Connectedness

A path from x to y:



or

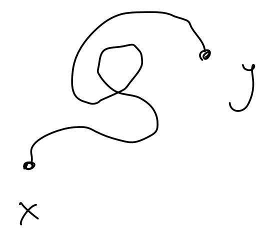


or

Def In X, a path from x to y is

a continuous function

$$f : [0,1] \to X$$



with $f(0) = x$ & $f(1) = y$.

X is path connected if: for any $x, y \in X$,

$\exists$ a path from x to y.