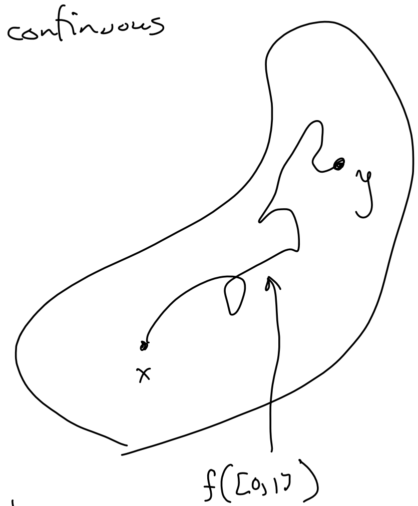
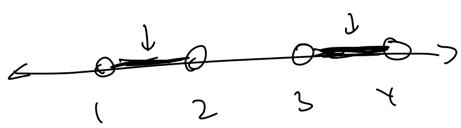
Path - Connectedness

Def A <u>path</u> from x to y is a continuous function $f : [0,1] \to X$ with $f(0) = x$ & $f(1) = y$

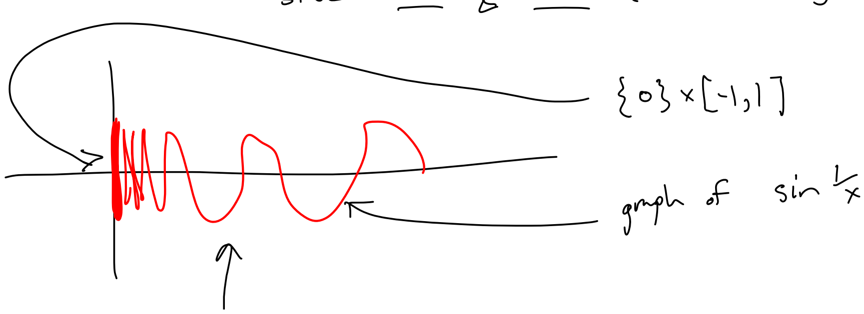


A set X is <u>path - connected</u> when $\forall_{x,y} \in X$, $\exists$ a path from x to y.

So $\mathbb{R}^n$, intervals in $\mathbb{R}$, balls in $\mathbb{R}^n$, a single pt are all path connected.



$(1,2) \cup (3,4)$ is not path connected, since <u>1.5</u> & <u>3.5</u> cannot be joined by a path.

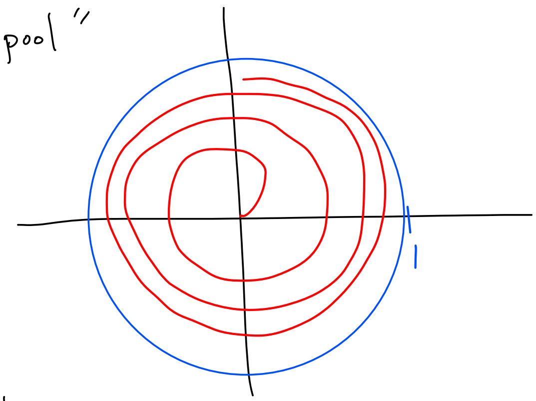


"The topologist's sine curve" it is connected but not path connected.

Also "the topologist's whirlpool"

$S^1 \cup \{(\frac{\theta}{\theta+1}, \theta)\}$

↑ polar coords

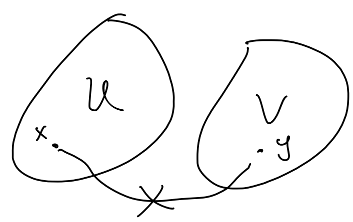


This is connected, but not path connected.

A space can be connected but not path connected.

Can it be path connected but not connected?

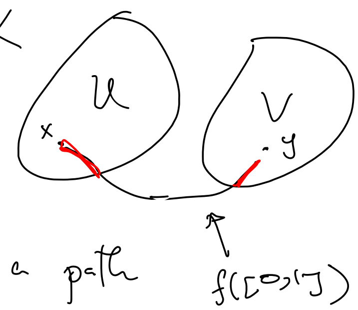
" This is impossible since the separation $X = U \cup V$ would also separate the path "



Thm If X is disconnected, then X is not path connected.

Pf Let $X = U \cup V$ be a separation,

To get a contradiction, assume X is path connected.



Take $x \in U$ & $y \in V$, then $\exists$ a path

$$f : [0,1] \to X \text{ from } x \text{ to } y.$$

Then $f([0,1])$ will be separated by

$$U \cap f([0,1]) \ \& \ V \cap f([0,1])$$

So $f([0,1])$ is disconnected.

This is a contradiction since $[0,1]$ is connected & f is continuous, so $f([0,1])$ is connected

So not connected $\Rightarrow$ not path connected.

Thm If X is path connected,

then X is connected.

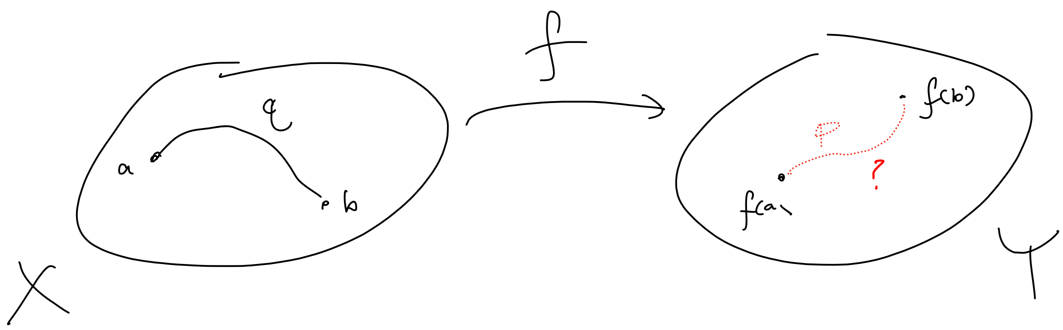
Most thms about "connected" are also true of "path connected"

<u>Thm</u> If C_1 & C_2 are path-conn & $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is path-conn.

<u>Thm</u> If X is path connected & $f: X \to Y$ is continuous, then <u>$f(X)$ is path connected.</u>

<u>Pf</u> Choose $f(a)$ & $f(b) \in f(X)$ (so $a, b \in X$), we'll find a path from $f(a)$ to $f(b)$.

i.e. $p: [0,1] \to Y$ with $p(0) = f(a)$
$p(1) = f(b)$



Since X is path connected,

$\exists$ a path q from a to b

($q(0) = a$ & $q(1) = b$)

Let $p = f \circ q$ this is continuous since f & q are continuous.

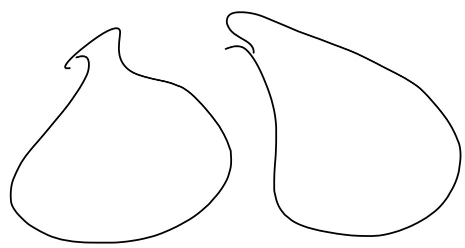
$p(0) = f(q(0)) = f(a)$

$p(1) = f(q(1)) = f(b)$

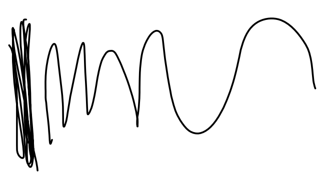
So p is a path from $f(a)$ to $f(b)$.

Shem.

We can also discuss path-components.



this has 2 path compoents.
(also 2 components)



has 1 component,
but 2 path components
(the vertical part
& the curve)

Compactness

From $\mathbb{R}$-analysis definition was:

X is compact iff every sequence in X
has a conv. subseq and its limit is in X.

Heine-Borel Thm

X is compact iff X is closed & bounded

requires a metric space

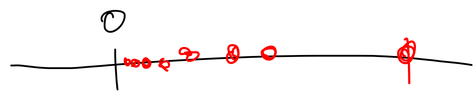
Def X is compact when every open cover of X has a finite subcover

An open cover is a collection of open sets U_α where

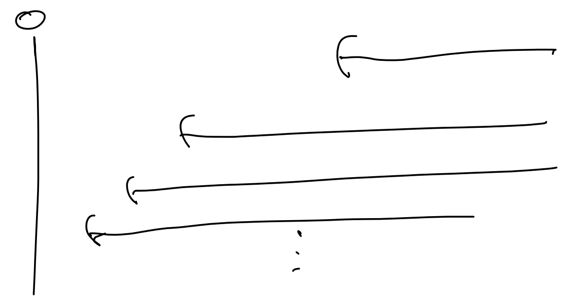
$$X = \bigcup_{\alpha \in A} U_\alpha$$

A subcover is just some, but not all of the U_α's which still covers X.

$X = \{ 1/n \mid n \in \mathbb{N} \}$



an open cover: let $\mathcal{U}_n = (\frac{1}{n}, \infty)$



this has no finite subcover.