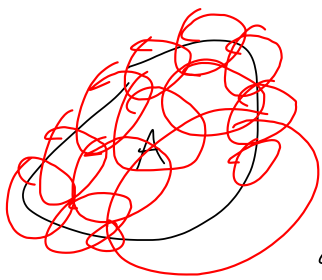
Compactness

Given $A \subseteq X$, an open cover of A is a collection of open sets $\mathcal{O}$ covering A

i.e. $$A \subseteq \bigcup_{U \in \mathcal{O}} U$$

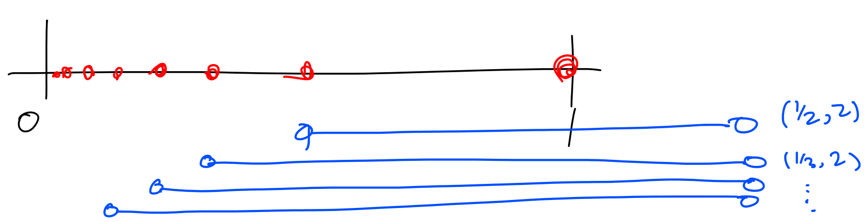


A finite subcover of $\mathcal{O}$ is a choice of only finitely man sets in $\mathcal{O}$ which still cover A.

Def A is compact when any open cover has a finite subcover.

(A is not compact means there is some open cover with no finite subcover.)

∞-ly many sets covering A, where you really need ∞-ly many.



$A = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$

Use $\mathcal{O}$ as the open intervals : $(\frac{1}{n}, 2)$

Sets from $\mathcal{O}$ are open & cover A;

but we need ∞-ly many to cover A.

So $\mathcal{O}$ has no finite subcover, so

A is not compact.

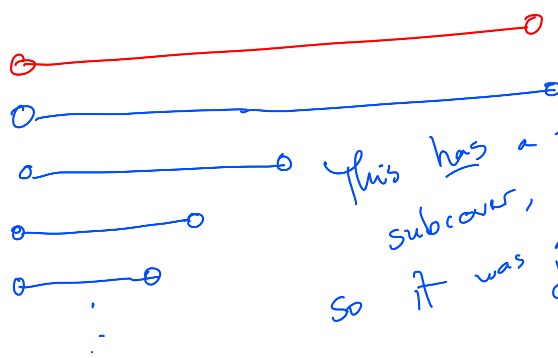
Pro-tip: choose open sets "stretching out to the boundary"

$A = (0,1)$ is not compact.

Let $\mathcal{O}$ be the cover using intervals

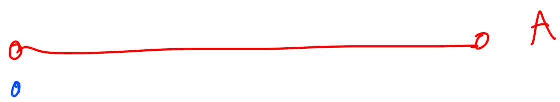
like : $(0, 1/n)$

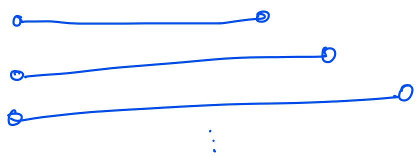
Bad idea



This has a finite subcover, so it was a bad choice.

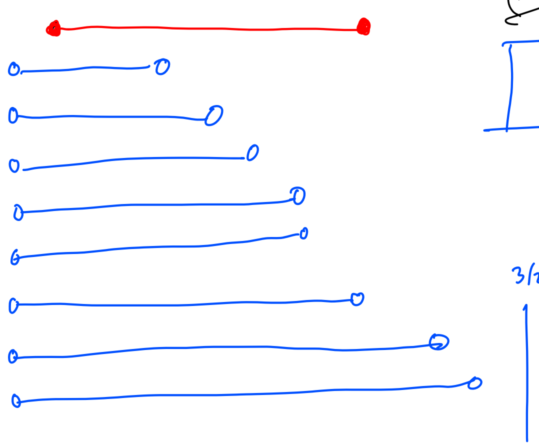
Try $\mathcal{O} = \{ (0, 1 - 1/n) \mid n \in \mathbb{N} \}$





This is an open cover with no finite subcover.

$A = [0, 1]$



$\cancel{(-1/2, 1-1/n)}$

$\boxed{[-1/2, 3/2 - 1/n)}$

has a finite subcover.

for $n = 4$, it's $(-1/2, 3/2 - 1/4)$

Actually A <u>is</u> compact.

(Hard to prove using covers - need to prove <u>all</u> open covers have a finite subcover)

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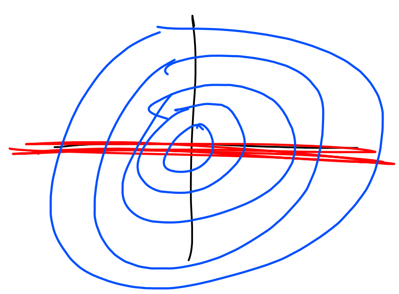
in $\mathbb{R}^2$

$A =$ the x-axis



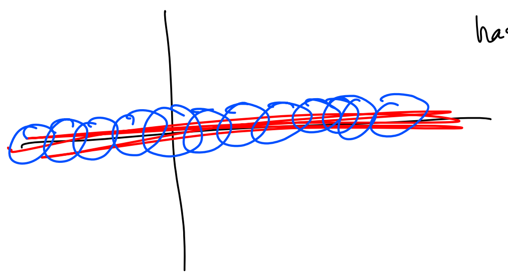
use $\mathcal{O} = \{ B(\vec{0}, n) \mid n \in \mathbb{N} \}$

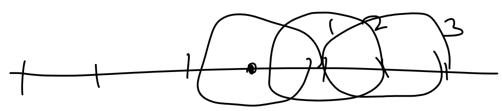
this covers the x-axis, but we need ∞-ly many to cover it.

So $\nexists$ a finite sub cover.

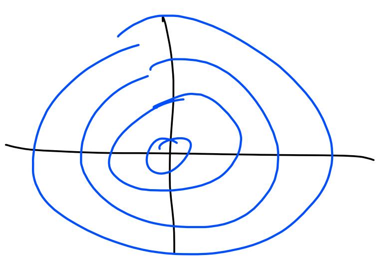
OR: use $\mathcal{O} = \{ B(n, 1.) \mid n \in \mathbb{N} \}$

has no finite subcover.





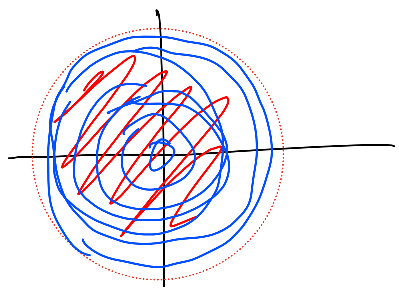
In $\mathbb{R}^2$ (or any $\mathbb{R}^n$), any set stretching out to ∞ (any unbounded set) can be covered by $\mathcal{O} = \{ B(0, n) \}$ with no finite subcover.



So in $\mathbb{R}^n$, any unbounded set is not compact.

<u>Thm</u> In $\mathbb{R}^n$, compact $\Rightarrow$ bounded.

In $\mathbb{R}^2$: the unit open disc is not compact!



use $\mathcal{O} = \{ B(\vec{0}, 1-1/n) \}$

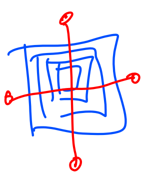
has no finite subcover.

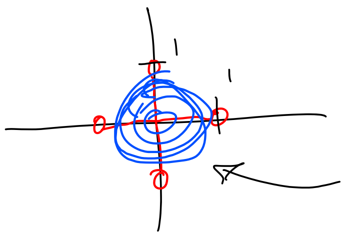
Show it's not compact:

$(3, 5) \subseteq \mathbb{R} \leftarrow \mathcal{O} = \{ (3, 5-1/n) \}$

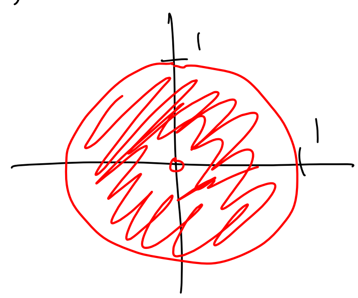
$(3+1/n, 5)$

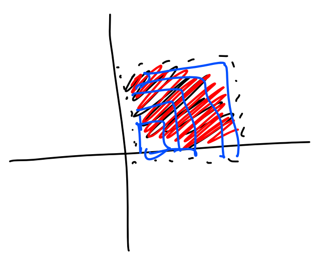
$(-1+1/n, 1-1/n) \times (-1+1/n, 1-1/n)$ $(3+1/n, 5-1/n)$





$B(\vec{0}, 1-1/n)$

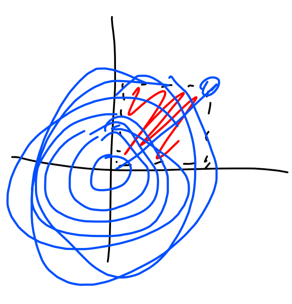


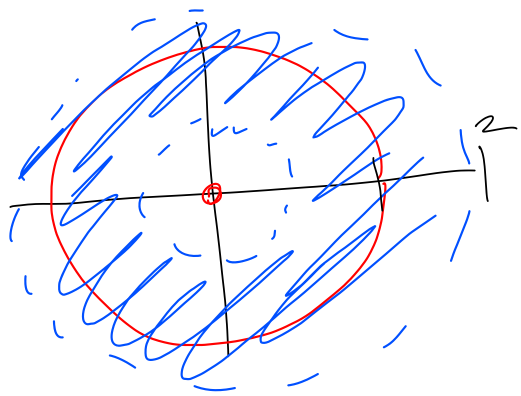


the open unit square

$(0,1) \times (0,1)$

$\mathcal{O} = \{ (0, 1-1/n) \times (0, 1-1/n) \}$





$B(\vec{0},2) - \overline{B(\vec{0},1/n)}$