A is <u>compact</u> when any open cover of A has a finite subcover.

Generally, sets which omit some boundary pts or stretch to ∞ are not compact.

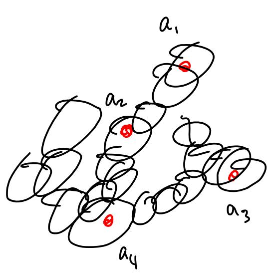
A compact set should include all boundary pts, and not stretch to ∞

closed | bounded.

This all works great in $\mathbb{R}^n$ & similar

Weird stuff can happen in weird spaces.

<u>Thm</u> In any top space, if A is finite, then A is compact.



<u>Pf</u> let $\mathcal{O}$ be an open cover.

WTS we can find a finite subcover.

let $A = \{a_1, \ldots, a_n\}$, Since $\mathcal{O}$ is an open cover, there are $U_1, \ldots, U_n$ from $\mathcal{O}$ with $a_i \in U_i \ \forall i$.

Then $\{U_1, \ldots, U_n\}$ is a finite subcover.

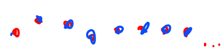
For infinite sets, some are compact, some not.

<u>Thm</u> If X has the discrete topology, Any infinite set is not compact.

<u>Pf</u> Let $A \subseteq X$ be infinite, WTS A is not compact.

i.e. WTS there's an open cover of A with no finite subcover.

Use sets like: $\{a\}$ for $a \in A$.



Let $\mathcal{O} = \{ \{a\} \mid a \in A \}$

this is an open cover, but has no finite subcover since A is infinite.

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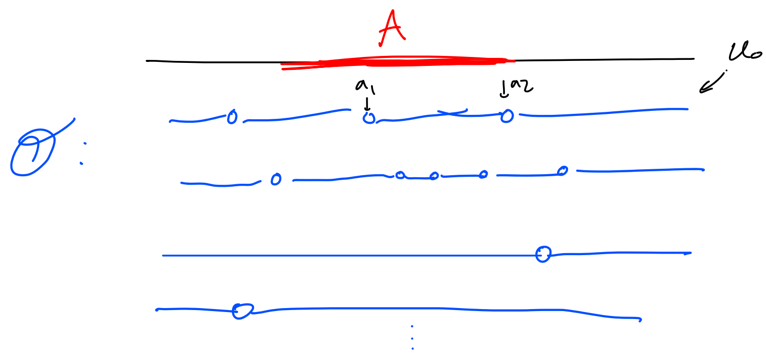
in $\mathbb{R}_{fc}$ an open set is like

<u>Thm</u> In $\mathbb{R}_{fc}$, all sets are compact.

<u>Pf</u> Let $A \subseteq \mathbb{R}_{fc}$, let $\mathcal{O}$ be an open cover, we'll find a finite subcover.



Choose any $U_0 \in \mathcal{O}$,

U_0 already covers all of A, except finitely many points.

Say U_0 covers A, except misses $\{a_1, a_2, \dots, a_n\}$

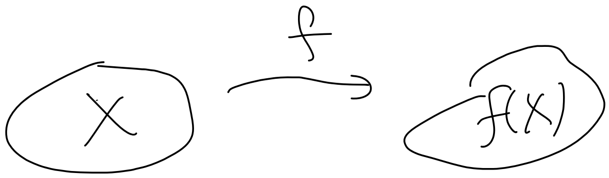
We can find $U_1, \dots, U_n \in \mathcal{O}$

with $a_1 \in U_1, \dots, a_n \in U_n$

So $U_0, U_1, \dots, U_n$ is a finite subcover.

So any set $A \subseteq \mathbb{R}_{fc}$ is compact.

(even unbounded, even not closed)

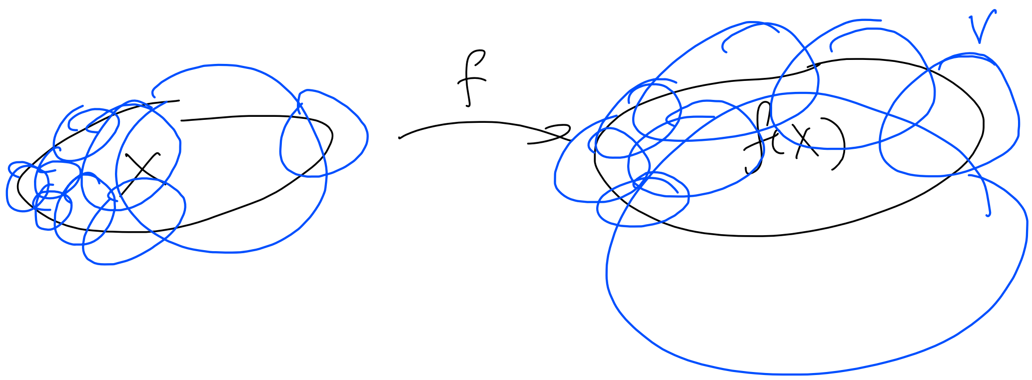


Thm If X is compact and $f: X \to Y$ is continuous, then $f(X)$ is compact.

Pf Let X be compact, WTS $f(X)$ is compact.

Let $\mathcal{O}$ be any open cover of $f(X)$

WTS there's a finite subcover.



For each $V \in \mathcal{O}$,

$f^{-1}(V)$ will be open in X

Then these sets $\{f^{-1}(V)\}$ make an open cover of X.

Since X is compact, $\exists$ a finite subcover

$$f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n) \quad \text{covers } X.$$

Apply f:

$$V_1, \ldots, V_n \quad \text{covers } f(X).$$