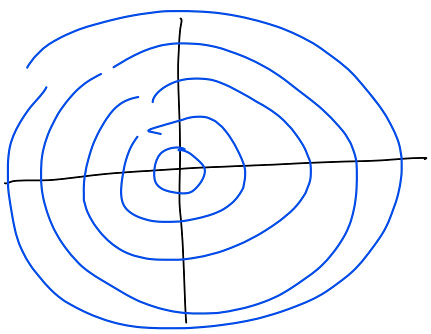
Heine-Borel: In $\mathbb{R}^n$,

<u>A is compact</u> iff A is closed & bounded.

any open cover has a finite subcover.

Warmup: $A \subseteq \mathbb{R}^n$ is compact $\Rightarrow$ A is bounded



$B(\vec{0}, n)$

Assume A is compact,

use $\mathcal{O} = \{ B(\vec{0}, n) \mid n \in \mathbb{N} \}$ as our open cover.

Then $\mathcal{O}$ has a finite subcover,

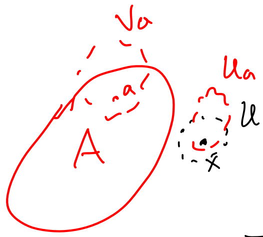
so actually $A \subseteq B(\vec{0}, N)$ for some N. So A is bounded.

So compact $\Rightarrow$ bounded in $\mathbb{R}^n$

Thm If X is Hausdorff, then A is compact $\Rightarrow$ A is closed

Pf Assume A is compact, WTS A is closed, i.e. $X - A$ is open.

Choose a point $x \in X - A$, we'll find a nbhd $\mathcal{U} \subseteq X - A$ with $x \in \mathcal{U}$.



For each $a \in A$, since X is Hausdorff, $\exists$ nbhds $\mathcal{U}_a$, V_a with $x \in \mathcal{U}_a$, $a \in V_a$, and $\mathcal{U}_a \cap V_a = \phi$.

Do this for all $a \in A$

V_a's are a open cover of a.

$\mathcal{U}_a$'s all are nbhds of x.

We'll want to take $\mathcal{U} = \bigcap_{a \in A} \mathcal{U}_a$ as our nbhd. of x.

But its not a finite intersection so this $\mathcal{U}$ may not be open.

Since A is compact, there is a finite subcover of the V_a's.

So $A \subseteq V_1 \cup V_2 \cup \ldots \cup V_n$.

Then $x \subseteq \mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n$,

so let $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_i$, then

$x \in \mathcal{U}$ and $\mathcal{U} \cap A = \phi$.

so $\mathcal{U} \subseteq X - A$ as desired.

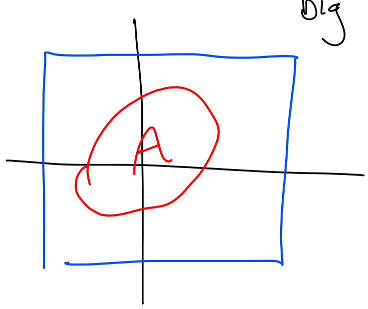
So compact $\Rightarrow$ bounded in $\mathbb{R}^n$
compact $\Rightarrow$ closed in Hausdorff.

Thm (Heine-Borel Thm) If $A \subseteq \mathbb{R}^n$, $\boxed{\text{DAFPOTR}^N}$ then A is compact iff A is closed & bounded.

$\Longrightarrow$ is done already done.

$\Longleftarrow$ Assume $A \subseteq \mathbb{R}^n$ is closed & bounded.

Since A is bounded, A fits inside some big box



$$B = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$$

B is a product of compact things, so B is compact.

So A is a subset of a compact set (B), and A is closed, so $\boxed{A \text{ is compact.}}$

Old Thm:
If $A \subseteq B$ & A is closed & B is compact,
then A is compact

functions & compact ~~sets~~:

Thm If $f : X \to Y$ is continuous
and $A \subseteq X$ is compact,
then $f(A)$ is compact.

This is a familiar calc. theorem:
the Extreme Value Theorem

In calc: If $[a,b]$ is closed intrval
& $f : [a,b] \to \mathbb{R}$ is continuous,
then f attains its max & min value
on $[a,b]$