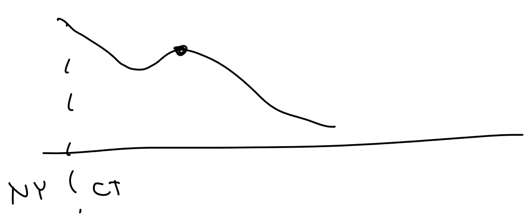
Thm If $f: X \to Y$ is continuous and $A \subseteq X$ is compact then $f(A)$ is compact.
$\subseteq Y$

A compact
$\Rightarrow f(A)$ compact

Extreme Value Thm If $f: [a,b] \to \mathbb{R}$ is continuous, then f has a min & max value.



In $\mathbb{R}$, some sets have min & max values, some don't:

$[0, 1]$ ← has a min & max.

$(0, \infty)$ ← has no max value, no min value ← must be in the set.

$(0, 1)$ ← has no min or max.

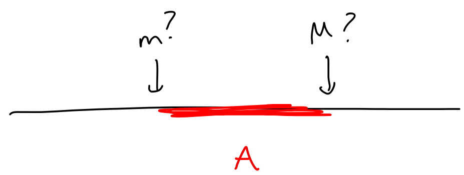
For a set in $\mathbb{R}$ to have a min/max, it ought to be: bounded & closed.

Thm If $A \subseteq \mathbb{R}$ is compact, it has a min & max.

i.e. $\exists\, m, M \in A$ s.t. $m \le a \le M$ $\forall a \in A$.

Pf Let A be compact.

So A is closed & bounded.

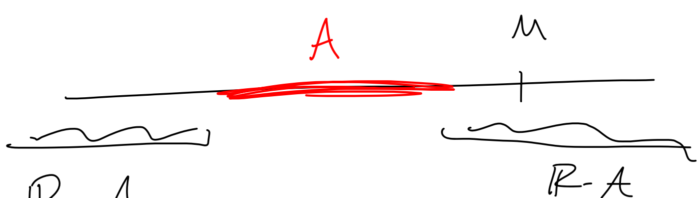


Let $M = \sup A$
$m = \inf A$
(these exist since A is bounded)

We have $m \le a \le M$, we only need to show $m, M \in A$.

Let's show $M \in A$.

For a contradiction, Assume $M \notin A$.



A is closed, so $\mathbb{R} - A$ is open,

$M \in \mathbb{R} - A$, so $\exists$ a nbhd

$(M-\varepsilon, M+\varepsilon) \subseteq \mathbb{R} - A$.

so M is not the sup
since $M-\varepsilon$ is a smaller
upper bound for A.

Given $f: X \to \mathbb{R}$, $A \subseteq X$,

A compact $\Rightarrow$ $f(A)$ is compact in $\mathbb{R}$
$\Rightarrow$ $f(A)$ has a min & max.

<u>General EVT</u> If A is compact &
$f: X \to \mathbb{R}$ is continuous,
then $f(A)$ has a min & max.

Note: Domain need not be $\mathbb{R}$.

i.e. any $f: S^2 \to \mathbb{R}$ has a min & max,
since S^2 is compact.