Math 3385: Exam #1 review problems

Topologies, bases, closed sets

1. Let $X = \{a, b, c\}$, and let
$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}\}.$$
Is $\mathcal{T}$ a topology on X? Say why or why not.

2. Describe what a basis for $\mathbb{R}_l$ (lower-limit topology) looks like, and prove that this is a basis.

3. Let $\mathcal{B} = \{[a, b] \subset \mathbb{R} \mid a < b\}$. Show that $\mathcal{B}$ is not a basis for a topology on $\mathbb{R}$.

4. Prove: in any Hausdorff topological space, a set of a single point is closed.

5. Show that $\mathbb{R}_{fc}$ is not Hausdorff. Is a single point set closed in $\mathbb{R}_{fc}$?

6. In each of $\mathbb{R}_l$, $\mathbb{R}_{fc}$, and $\mathbb{R}$ with the discrete topology, give an example of a closed set (don't use $\emptyset$ or $\mathbb{R}$), and explain why it's closed.

7. A topological space X is called T_1 when: for any points $x, y \in X$, there are open sets U and V with $x \in U$ and $y \in V$ and $x \notin V$ and $y \notin U$. (Draw a picture- this is like Hausdorff, but not quite the same.) Show that any Hausdorff space is T_1, but that $\mathbb{R}_{fc}$ is T_1 but not Hausdorff.

Closure, interior, boundary

8. For these sets in $\mathbb{R}$ with the standard topology, find the interior, closure, and boundary.
$$(3, 5], \mathbb{R}, \mathbb{Q}, A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}, B = \mathbb{R} - A$$

9. Consider the sequence $x_n = (-1/n)$. Show that x_n does not converge to 0 in $\mathbb{R}_l$.

10. Let X be any set with the discrete topology, and let $A \subset X$ and $a \in A$. Show that a is not a limit point of A.

11. Show that if A is open, then $\text{Int}(A) = A$.

12. Show for any set A: if A has empty interior, then $\partial A = \bar{A}$.

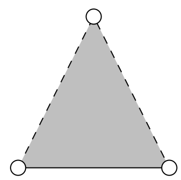
Subspace and product topologies

13. Explain why: in $\mathbb{Z}$ with the subspace topology from $\mathbb{R}$, any single point set is open. But in $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$, any single point set is not open.

14. Let $X = [0, 1]$, with the subspace topology from $\mathbb{R}$. Give an example of a set in X which is open in X with the subspace topology, but not open in $\mathbb{R}$.

15. Draw a picture and give some words explaining why the sphere S^2 with the subspace topology from $\mathbb{R}^3$ is Hausdorff.

16. Consider the subset $A \subset \mathbb{R}^2$ in the picture below. Explain why A is not open in $\mathbb{R}^2$ with the standard topology, but it is open in the product topology $\mathbb{R} \times \mathbb{R}_l$. Is it open in $\mathbb{R}_l \times \mathbb{R}_l$? Is it open in the vertical line topology in $\mathbb{R}^2$?



17. Let $\mathbb{R}_d$ be $\mathbb{R}$ with the discrete topology. Show that any horizontal interval of the form: $(a, b) \times \{c\} \subset \mathbb{R}^2$ is open in the product $\mathbb{R} \times \mathbb{R}_d$, but not open in the product $\mathbb{R}_d \times \mathbb{R}$.

Answers!

1. No it is not! It's not closed under unions, since $\{a\} \cup \{b\} = \{a, b\}$ and $\{a, b\} \notin \mathcal{T}$.

2. A basis for $\mathbb{R}_l$ looks like all intervals of the form $[a, b)$. To show it's a basis we have to prove two properties:

   #1 (covering property): Given any $x \in \mathbb{R}$, we have $x \in [x-1, x+1)$. So x is contained in a basis element.

   #2 (shrinking property): Let $[a_1, b_1)$ and $[a_2, b_2)$ be basis sets with $x \in [a_1, b_1) \cap [a_2, b_2)$. We must show that the intersection contains a basis set. Since that intersection is nonempty, we can assume that $a_2 < b_1$ (draw a picture to see what I mean). Then $[a_2, b_1) \subset [a_1, b_1) \cap [a_2, b_2)$, so there is a basis set in the intersection.

3. This does not satisfy the shrinking property: Consider sets $[0, 1]$ and $[1, 2]$. Then the intersection $[0, 1] \cap [1, 2]$ is not empty, but this intersection is only a single point, so it does not contain a smaller basis set (which would need to be an interval).

4. Let X be Hausdorff, and consider a single point set $\{x\}$. We'll show that the complement $X - \{x\}$ is open. Choose any point $y \in X - \{x\}$, and we'll show that some neighborhood of y is a subset of $X - \{x\}$. Since X is Hausdorff, there are neighborhoods U and V of x and y with $U \cap V =$. We don't care about U, but V is a neighborhood of y which omits x. Thus V is a subset of $X - \{x\}$ as desired.

5. To show $\mathbb{R}_{fc}$ is not Hausdorff: Take $0, 1 \in \mathbb{R}_{fc}$, and we'll show that any neighborhood of 0 must intersect with any neighborhood of 1. (There is nothing special about the numbers 0 and 1, but I just want two specific points.) A neighborhood of 0 in $\mathbb{R}_{fc}$ has finite complement, so it consists of all real numbers except for a few exceptions. Similarly any neighborhood of 1 consists of all real numbers except for a few exceptions. Thus these two neighborhoods must have many (infinitely many) points in common. So their intersection is not empty.

   A single point set in $\mathbb{R}_{fc}$ is closed. Take the single point set $\{x\}$: then its complement will be $\mathbb{R} - \{x\}$, which is open because $\mathbb{R} - \{x\}$ has finite complement. Since the complement of $\{x\}$ is open, this means $\{x\}$ is closed.

6. In $\mathbb{R}_l$, the set $(-\infty, 1)$ is closed because the complement is $[1, \infty)$, which is open in $\mathbb{R}_l$.

   In $\mathbb{R}_{fc}$, the set $\{0\}$ is closed, because the complement is $\mathbb{R} - \{0\}$, which is open in $\mathbb{R}_{fc}$.

   In $\mathbb{R}$ with the discrete topology, all sets are closed, because their complement will always be open (since any set in the discrete topology is open).

7. Any Hausdorff space is T_1: Let X be Hausdorff, and we'll show it's T_1. Take $x, y \in X$, and we must find neighborhoods U and V of x and y with $x \notin V$ and $y \notin U$. Since X is Hausdorff we have neighborhoods U, V of x, y with $U \cap V = \emptyset$. Since this intersection is empty, we must have $x \notin V$ and $y \notin Y$, which is what we needed to show.

   $\mathbb{R}_{fc}$ is T_1 but not Hausdorff: we showed above that $\mathbb{R}_{fc}$ is not Hausdorff. To show it's T_1, take any $x, y \in \mathbb{R}_{fc}$. Let $U = \mathbb{R} - \{y\}$ and $V = \mathbb{R} - \{x\}$. Then these are neighborhoods of x and y with $x \notin V$ and $y \notin U$, so $\mathbb{R}_{fc}$ is T_1.

8. $(3, 5]$: the interior is $(3, 5)$, closure is $[3, 5]$, boundary is $\{3, 5\}$.

   $\mathbb{R}$: the interior and closure are both $\mathbb{R}$, and boundary is empty.

   $\mathbb{Q}$: the interior is empty, the closure is $\mathbb{R}$, and boundary is $\mathbb{R}$.

   A: The interior is empty, the closure is $A \cup \{0\}$, and boundary is $A \cup \{0\}$.

   B: The interior is $\mathbb{R} - (A \cup \{0\})$, the closure is $\mathbb{R}$, and the boundary is $A \cup \{0\}$.

9. The set $[0, 1)$ is a neighborhood of 0 in $\mathbb{R}_l$ which excludes all terms of the sequence. Thus the sequence does not converge to 0.

10. "Limit point of A" means that any neighborhood of a intersects A at a point other than a itself. In the discrete topology, $\{a\}$ is a neighborhood of a. But $\{a\}$ does not intersect A at any point other than a itself. Thus a is not a limit point of A.

11. $\text{Int}(A)$ is the union of all open sets containing A. Since A is open, the whole set A is an open set containing A, and so $A = \text{Int}(A)$.

12. The definition of boundary is $\partial A = \bar{A} - \text{Int}(A)$. So if $\text{Int}(A)$ is empty, then $\partial A = \bar{A}$.

13. For $\mathbb{Z}$: Let $n \in \mathbb{Z}$. Then $\{n\} = (n - 1/2, n + 1/2) \cap \mathbb{Z}$, and so $\{n\}$ is an open set in the subspace topology.

    For $\mathbb{Q}$: If we take $x \in \mathbb{Q}$, we can never write $\{x\} = U \cap \mathbb{Q}$ where U is an open set in $\mathbb{R}$, since any open set in $\mathbb{R}$ containing x will also contain other rationals close to x.

14. $[0, 1/2)$ is open in X but not open in $\mathbb{R}$.

15. Open sets in S^2 with the subspace topology from $\mathbb{R}^3$ look like disc-shaped patches on the surface of a sphere. To show it's Hausdorff: take $x, y \in S^2$. Draw a picture and you'll see that you can put small patches around them to get neighborhoods U and V with $U \cap V = \emptyset$.

16. In each case, think in terms of each point having a neighborhood inside A.

    In the standard topology: If we choose a point on the line at the bottom, any basis neighborhood of that point looks like an open ball, and so will stretch outside of the triangle. So these points do not have neighborhoods inside of A, so A is not open in $\mathbb{R}^2$.

    In $\mathbb{R} \times \mathbb{R}_l$: This time, a basis neighborhood looks like a rectangle including its bottom side but not the other 3 (and not any corners). This time, if we choose any point in A (even ones on the bottom), we can draw a basis box around it inside of A. So A is open in $\mathbb{R} \times \mathbb{R}_l$.

    In $\mathbb{R}_l \times \mathbb{R}$: This time, a basis neighborhood looks like a rectangle including its left side but not the other 3 (and not any corners). This time if we choose a point of A on the line at the bottom, any such neighborhood must leave A. So A is not open in $\mathbb{R}_l \times \mathbb{R}$.

    In $\mathbb{R}_l \times \mathbb{R}_l$: This time, a basis neighborhood looks like a rectangle including its left and bottom sides (and bottom-left corner). This time, if we choose any point in A (even ones on the bottom), we can draw a basis box around it inside of A. So A is open in $\mathbb{R}_l \times \mathbb{R}_l$.

    In vertical line topology: This time, a basis neighborhood looks like a small vertical open interval. Again, if we choose a point of A on the line at the bottom, any such neighborhood must leave A. So A is not open in the vertical line topology.

17. $(a, b) \times \{c\}$ is the product of an open set from $\mathbb{R}$ together with an open set from $\mathbb{R}_d$ (since single points are open in $\mathbb{R}_d$). Thus $(a, b) \times \{c\}$ is open in $\mathbb{R} \times \mathbb{R}_d$.

    The product $\mathbb{R}_d \times \mathbb{R}$ is the same as the vertical line topology: basis open sets look like vertical open intervals. A horizontal interval is not open in this topology, since any neighborhood of a point will stretch outside of the horizontal interval.