Math 3385: Exam #2 review problems

Quotients

1. For each of these quotients, draw a picture or describe in words what it would look like. The quotient of $\mathbb{R}^2$ after identifying a circle to a point. The quotient of $\mathbb{R}$ after identifying all of $\mathbb{Z}$ to a point. The quotient of a disc after identifying its boundary circle to a point. The quotient of a disc in $\mathbb{R}^2$ after identifying any points on its boundary circle which have equal y-coordinate.
2. Let $X = \mathbb{R}$ with the standard topology, and let X^* be the quotient space obtained by identifying $[0, 1]$ to a point. Draw or describe what X^* looks like. Describe what a small neighborhood of -1 looks like in X^*. Describe what a small neighborhood of 0 looks like in X^*. Describe what a small neighborhood of $1/2$ looks like in X^*.
3. As above let $X = \mathbb{R}$ but this time let X^* be the quotient space obtained by identifying $(0, 1)$ to a single point. Show that X^* is not Hausdorff. (That is, find two points $x, y \in X^*$ such that any two neighborhoods of x and y must intersect each other.)

Continuous functions

4. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X with $\mathcal{T}_2$ finer than $\mathcal{T}_1$ (so $\mathcal{T}_1 \subset \mathcal{T}_2$). Show that if $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_1)$ is continuous, then $f : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous (this is the same function, but the continuity is considered with respect to different topologies).
5. Let X and Y be topological spaces. Fill in the blanks in two different ways to make two different theorems, and prove them: "If ??? has the ??? topology, then every map from X to Y is continuous." where the first ??? can be either X or Y, and the second ??? can be either "discrete" or "trivial".
6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by:
$$f(x) = \begin{cases} x^2 & \text{if } x \neq 3, \\ 0 & \text{if } x = 3. \end{cases}$$
Show using the open-set definition of continuity that f is not continuous.
7. Let X and Y be any topological spaces with some value $c \in Y$, and let f be a constant function $f(x) = c$. Show using the open-set definition of continuity that f is continuous.

Homeomorphisms

8. Let $f(x) = x + 1$. Is this a homeomorphism of $\mathbb{R}$ to $\mathbb{R}$? Please explain why or why not.
9. Explain why $f(x) = x^2$ is not a homeomorphism of $\mathbb{R}$ to $\mathbb{R}$.
10. Let $f : [0, 1) \to S^1$ be the function which wraps the interval around and makes it into a circle by identifying the points 0 and 1. In a parametric formula, this is $f(t) = (\cos(2\pi t), \sin(2\pi t)$, so that $f(0) = (0, 0)$. Explain why f is not a homeomorphism.

Metric spaces

11. Let d be a metric on some set X, and let $e(x, y) = 2 \cdot d(x, y)$. Show that e is also a metric on X.

12. For $x, y \in S^1$, let $d(x, y)$ be the distance from x to y as measured along the circumpherence, going the shortest way around the circle. Is d a metric? Explain generally why or why not. (You can use words since writing formulas for d is a pain.)

13. Let $d(x, y)$ be the distance from x to y as measured along the circumpherence, going counter-clockwise. Is d a metric? Explain generally why or why not. (You can use words since writing formulas for d is a pain.)

14. Let $d(x, y) = |x^2 - y^2|$. Is this a metric on $\mathbb{R}$? Explain why or why not.

15. Let $d(x, y) = |xy|$. Show that this is not a metric. (*Hint*: take 3 numbers with one less than 1, and the other two greater than 1.)

Answers

1. The quotient of $\mathbb{R}^2$ after identifying a circle to a point: This is something that looks like $\mathbb{R}^2$ again, but with a balloon sitting on the floor.

   The quotient of $\mathbb{R}$ after identifying all of $\mathbb{Z}$ to a point: This looks like infinitely many loops all joined at a single point.

   The quotient of a disc after identifying its boundary circle to a point: This looks like a sphere.

   The quotient of a disc in $\mathbb{R}^2$ after identifying any points on its boundary circle which have equal y-coordinate: This also looks like a sphere: like an old-timey hinged coin purse. When it is open lying flat, it looks like a topological disc, but then you close the two circular sides up and you get a topological sphere. (I mean like one of these: `https://www.amazon.com/Marshal-Womens-Leather-Kiss-Purse/dp/B07BJ7CFV6`)

2. X^* is like the entire line, but we crunched the interval $[0, 1]$ into a point. Topologically this doesn't really change anything, so it still looks like a line. A small neighborhood of -1 in X^* will still look like an ordinary neighborhood of -1, like an open interval $(-1 - \epsilon, -1 + \epsilon)$.

   For a small neighborhood of 0, remember that 0 has been identified with all other points of $[0, 1]$, so a neighborhood of 0 in X^* must additionally include all of $[0, 1]$. So it will look something like $(-\epsilon, 1+\epsilon)$.

   In X^*, $1/2$ and 0 are the same point. So a small neighborhood of $1/2$ will also look something like $(-\epsilon, 1 + \epsilon)$.

3. This time 0 and 1 are *not* the same point, though there is only 1 point in between (the point obtained by collapsing $(0, 1)$), which is weird enough to make it non-Hausdorff. Specifically, any neighborhood of 0 will look like $(-\epsilon, 1 + \epsilon)$, and any neighborhood of 1 will also have the same form. So there are no neighborhoods U and V with $0 \in U$ and $1 \in V$ and $U \cap V = \emptyset$.

4. We need to show that if U is open in $\mathcal{T}_1$, then $f^{-1}(U)$ is open in $\mathcal{T}_2$. So let U be open in $\mathcal{T}_1$, and we already know f is continuous from $\mathcal{T}_1$ to $\mathcal{T}_1$, so $f^{-1}(U)$ is open in $\mathcal{T}_1$. Since $\mathcal{T}_1 \subset \mathcal{T}_2$ this means U is open in $\mathcal{T}_2$ as desired.

5. First: If X has the discrete topology, then every map from X to Y is continuous. Proof: We need to show that $f^{-1}(U)$ is open in X whenever U is open in Y. But if X has the discrete topology then $f^{-1}(U)$ is open no matter what, so f is continuous automatically.

   Second: If Y has the trivial topology, then every map from X to Y is continuous. Proof: We need to show that $f^{-1}(U)$ is open in X whenever U is open in Y. But there are only 2 open sets in Y, so we can just check them both. If $U = \emptyset$, then $f^{-1}(U) = \emptyset$ which is open. And if $U = Y$ then $f^{-1}(Y) = X$ which is open. So $f^{-1}(U)$ is open in either case, so f is continuous.

6. Let $U = (-1, 1)$, which is open. Then $f^{-1}(U) = (-1, 1) \cup \{3\}$, which is not open. So f is not continuous.

7. Let $U \subset Y$ be open. We consider two cases: if $c \in U$, then $f^{-1}(U) = X$, since all points of x map to c. Thus in this case $f^{-1}(U)$ is open. In the other case, say $c \notin U$, and then $f^{-1}(U) = \emptyset$ which is open. So in either case, $f^{-1}(U)$ is open as desired.

8. Yes it is a homeomorphism: it is a bijection because it has an inverse $f^{-1}(x) = x - 1$. And clearly both f and f^{-1} are continuous.

9. It is not a bijection. It's not 1-1 because $f(-1) = f(1)$, and also it's not onto because there is no x with $f(x) = -1$. (You really only need to mention one or the other of these.)

10. Intuitively, it cannot be a homeomorphism because $[0,1)$ is not topologically the same as a circle (but that's not a proof). Specifically, it is not a homeomorphism because f^{-1} is not continuous. To see this, we will find an open set $U \subseteq [0,1)$ with $f(U) \subset S^1$ not open. (Since we are showing f^{-1} is not continuous, we need to look at preimages under f^{-1}, that is, things like ${f^{-1}}^{-1}(U)$, which is $f(U)$.)

    Let $U = [0, 1/2)$, which is open in $[0,1)$ because we are using the subspace topology on $[0,1) \subset \mathbb{R}$. Then $f(U)$ is the top half of the circle, including the endpoint on the right, but not the endpoint on the left. This is not an open set in S^1 with the subspace topology from $\mathbb{R}^2$.

11. We need to verify the 3 properties. In all cases, the property for e is proved by using the same property for d.

    Positive definite: We have $e(x,y) = 2d(x,y) \geq 2 \cdot 0 = 0$ so $e(x,y) \geq 0$. And also $e(x,y) = 0$ means that $2d(x,y) = 0$ and so $d(x,y) = 0$, so $e(x,y) = 0$ if and only if $d(x,y) = 0$, which happens only when $x = y$.

    Symmetric: We have $e(x,y) = 2d(x,y) = 2d(y,x) = e(y,x)$ as desired.

    Triangle inequality: We have:

    $$e(x,y) + e(y,z) = 2d(x,y) + 2d(y,z) = 2(d(x,y) + d(y,z)) \geq 2d(x,z) = e(x,z)$$

    as desired.

12. Yes this is a metric. Let's discuss the three properties.

    Positive definite: measuring the distance along the circumpherence is positive definite because $d(x,y)$ is the distance along the circumpherence is always positive, and it can only be zero when the two points are the same.

    Symmetry: the distance from x to y along the circumpherence going the shortest way is the same as the distance from y to x going the shortest way.

    Triangle inequality: this is also true in this case. The distance from x to z going the shortest way is always less than or equal to the distance from x to y plus the distance from y to z going the shortest way.

13. This is not symmetric. For example if $x = (1,0)$ and $y = (0,1)$, then the distance from x to y along the circle going counter clockwise is not the same as the distance from y to x along the circle going counter clockwise.

14. This is not positive definite, since $d(1,-1) = 0$.

15. This does not satisfy the triangle inequality: Let $x = 2$, $y = 1/2$, and $z = 3$. Then:

    $$d(x,y) + d(y,z) = 1 + 3/2 = 5/2$$

    and $d(x,z) = 6$. So the triangle inequality is violated.