Math 3385: Exam #3 review problems

The test will cover everything, but will focus on connectedness and compactness.

Connectedness

1. Give examples showing that if A and B are connected, then $A \cup B$ may be connected or it may be disconnected. Also do the same for $A \cap B$.
2. If $A \subset X$ is disconnected and $f : X \to Y$ is continuous, must $f(A)$ be disconnected? Either prove it, or give a counterexample.
3. Show that $\mathbb{R}_l$ and $\mathbb{R}_d$ are disconnected.
4. Show that if X has a proper nonempty clopen subset, then X is disconnected.
5. Show that: if X has some point p such that p is in every nonempty open set, then X is connected.
6. Let X and Y be path connected. Show that $X \times Y$ is path connected. (Hint: Start with two points $(a, b) \in X \times Y$ and $(c, d) \in X \times Y$, and you need a path from (a, b) to (c, d). But a and c are in X, so since X is path connected...)

Compactness

7. For each of these non-compact sets, give an open cover which has no finite subcover: $\mathbb{Z}$, $(0, 1)$, $\{\frac{1}{n} \mid n \in \mathbb{N}\}$, $\mathbb{Q} \cap [0, 10]$, the open ball $B(\vec{0}, 1) \subset \mathbb{R}^2$.
8. If A and B are compact, then must $A - B$ be compact? Say why or why not.
9. Show that any compact set in $\mathbb{R}^n$ is bounded.
10. Show that any finite set is compact.
11. Call a set $A \subset X$ *isolated* if every point of A has a neighborhood in X which includes no other points of A. (For example $\mathbb{Z}$ in $\mathbb{R}$ is isolated.) Show that any compact isolated set is finite.

Answers

1. If $A = [0, 1]$ and $B = [1, 2]$, then A and B are connected, and $A \cup B$ is also connected. If $A = [0, 1]$ and $B = [2, 3]$, then A and B are connected, and $A \cup B$ is disconnected.

   If $A = [0, 1]$ and $B = [1, 2]$, then A and B are connected, and $A \cap B$ is also connected. For an example where A and B are connected, but $A \cap B$ is not connected, this was on one of our homeworks— also discussed in class.

2. No! It is possible for A to be disconnected, but $f(A)$ is connected. For example the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then $f(\{-1, 1\}) = \{1\}$.

3. In $\mathbb{R}_l$, the interval $(-\infty, 0)$ is open, and so is its complement $[0, \infty)$. Then these two sets form a separation of $\mathbb{R}_l$, so $\mathbb{R}_l$ is disconnected.

   In $\mathbb{R}_d$, all sets are open. So we can take any two nonempty sets U and V, and they are a separation of $\mathbb{R}_d$.

4. Let U be a proper nonempty clopen set in X. Then let $V = X - U$, and V is also a proper nonempty clopen set, since the complement of a clopen set is clopen. (Because complement of open is closed, and complement of closed is open.) So both U and V are nonempty and open with empty intersection, so they make a separation of X. Thus X is disconnected.

5. We'll prove this by contradiction: Assume that X is disconnected. Then there is a separation $X = U \cup V$ where U and V are disjoint open sets. Since U and V are open, they both contain the special point p, which contradicts the fact that they are disjoint.

6. Start with two points $(a, b) \in X \times Y$ and $(c, d) \in X \times Y$, and we will make a path from (a, b) to (c, d). Since $a, c \in X$, and X is path connected, there is a path $p : [0, 1] \to X$ with $p(0) = a$ and $p(1) = c$. Since $b, d \in Y$ and Y is path connected, there is a path $q : [0, 1] \to Y$ with $q(0) = c$ and $q(1) = d$. Now to build a path $r : [0, 1] \to X \times Y$, we do:

   $$r(t) = (p(t), q(t)).$$

   Then r is continuous since p and q are continuous, and $r(0) = (p(0), q(0)) = (a, b)$ and $r(1) = (p(1), q(1)) = (c, d)$, so r is a path from (a, b) to (c, d) as desired.

7. For $\mathbb{Z}$, use $\mathcal{O} = \{(-n, n) \mid n \in \mathbb{N}\}$. Or you could use $\mathcal{O} = \{(x - 1/2, x + 1/2) \mid x \in \mathbb{Z}\}$.

   For $(0, 1)$, use $\mathcal{O} = \{(1/n, 1) \mid n \in \mathbb{N}\}$.

   For $\{\frac{1}{n} \mid n \in \mathbb{N}\}$, use $\mathcal{O} = \{(1/n, 2) \mid n \in \mathbb{N}\}$.

   For $\mathbb{Q} \cap [0, 10]$, use something like $\mathcal{O} = \{(\pi, \infty)\} \cup \{(-1, \pi - \frac{1}{n}) \mid n \in \mathbb{N}\}$

   For $B(\vec{0}, 1) \subset \mathbb{R}^2$, use $\mathcal{O} = \{B(\vec{0}, 1/n) \mid n \in \mathbb{N}\}$.

8. No! Generally it will not be compact. For example in $\mathbb{R}$, where compact means closed and bounded, let $A = [0, 2]$ and $B = [1, 3]$. Then A and B are compact, but $A - B = [0, 1)$, which is not compact.

9. Let $A \subset \mathbb{R}^n$ be compact, and we'll show it's bounded. Let $\mathcal{O} = \{B(\vec{0}, n) \mid n \in \mathbb{N}\}$. Since X is compact, there is a finite subcover, so only finitely many of these balls are necessary to cover A. But since the balls are all nested, this means that there is some largest ball that covers A all by itself. Thus we have $A \subset B(\vec{0}, N)$ for some $N \in \mathbb{N}$, and so A is bounded.

10. Let A be finite, and we'll show it's compact. Let $\mathcal{O}$ be any open cover of A, and we must find a finite subcover. Since A is finite we can write $A = \{a_1, \ldots, a_n\}$. Since $\mathcal{O}$ covers A, we can find $U_1, \ldots, U_n \in \mathcal{O}$ with $a_i \in U_i$ for each i. In this case, these sets $\{U_1, \ldots, U_n\}$ form a finite subcover.

11. Let A be isolated and compact. Since A is isolated, for every $a \in A$, there is a neighborhood $U_a \subset X$ with $U_a \cap A = \{a\}$. These U_a form an open cover of A, and so since A is compact, there is some finite subcover $U_{a_1}, \ldots, U_{a_n}$.

    Since this finite subcover still covers A, we have:

$$A \subseteq U_{a_1} \cup \cdots \cup U_{a_n}$$

    and the sets on the right side contain only 1 point each from A. Thus A has at most n points, so A is finite.