Math 1172

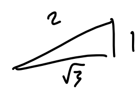
Midterm Answers

#1. a) $f(x) = 3x^2 \sin 2x$

$f'(x) = 3x^2 \cdot \cos 2x \cdot 2 + \sin 2x \cdot 6x$

b) $\int \sqrt{x} - \cos x \, dx = \int x^{1/2} - \cos x \, dx$

$= \frac{2}{3} x^{3/2} - \sin x + C$



#2 $\int_0^{\pi/6} 3\sin x - \cos x \, dx = -3\cos x - \sin x \Big|_0^{\pi/6}$

$= -3\cos \pi/6 - \sin \pi/6 - \left(-3\cos 0 - \sin 0\right)$

$= -3 \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} - \left(-3 \cdot 1 - 0\right)$

#3 $\int_1^4 5x(2+x^2)^{10} \, dx$

$u = 2 + x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

$= \int 5u^{10} \cdot \frac{1}{2} du = \frac{5}{2} \int u^{10} \, du = \frac{5}{2} \cdot \frac{1}{11} u^{11} \Big|_{x=1}^{x=4}$

$= \frac{5}{22} (2 + x^2)^{11} \Big|_1^4 = \frac{5}{22} (2+4^2)^{11} - \frac{5}{22} (2+1^2)^{11}$

#4

intersections: $x^3 - 2x = 4x - x^2$

$$x^3 + x^2 - 6x = 0$$

$$x(x^2 + x - 6) = 0$$

$$x(x+3)(x-2) = 0$$

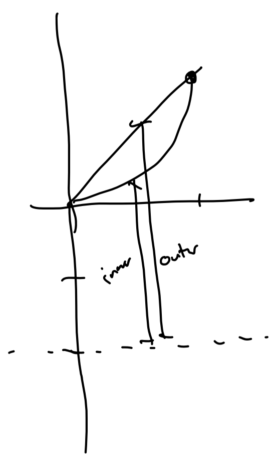
$$\boxed{x=0,\ x=2} \quad x=-3$$

$$\int_0^2 (4x - x^2) - (x^3 - 2x)\,dx = \int_0^2 4x - x^2 - x^3 + 2x\,dx$$

$$= \int_0^2 6x - x^2 - x^3\,dx = \frac{6}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^2$$

$$= 3 \cdot 2^2 - \frac{1}{3} \cdot 2^3 - \frac{1}{4} \cdot 4^2 - (0)$$

#5



$$\pi \int_0^1 (x+2)^2 - (x^2+2)^2\,dx$$

$$\pi \int_0^1 x^2 + 4x + 4 - (x^4 + 4x^2 + 4)\,dx$$

$$= \pi \int_0^1 -x^4 - 3x^2 + 4x\,dx = \pi\left(-\frac{1}{5}x^5 - \frac{3}{3}x^3 + \frac{4}{2}x^2\right)\Big|_0^1$$

$$= \pi\left(-\frac{1}{5} - 1 + 2\right) - \pi(0)$$

#6

$$2\pi \int_2^4 x \cdot \left(\frac{x}{2}+2\right)\, dx = 2\pi \int_2^4 \frac{1}{2}x^2 + 2x \, dx$$

$$= 2\pi \left(\frac{1}{6}x^3 + x^2 \right)\Big|_2^4$$

$$= 2\pi \left(\frac{1}{6} \cdot 4^3 + 4^2\right) - 2\pi\left(\frac{1}{6} \cdot 2^3 + 2^2\right)$$

#7

$y = 4 + 2\ln x$

$y - 4 = 2\ln x$

$\frac{1}{2}(y-4) = \ln x$

$x = e^{\frac{1}{2}(y-4)}$

$f^{-1}(x) = e^{\frac{1}{2}(x-4)}$

#8 a)

$$\int_1^4 x + \frac{4}{x}\, dx = \frac{1}{2}x^2 + 4\ln x \Big|_1^4 = \frac{1}{2} \cdot 16 + 4\ln 4 - \frac{1}{2} \cdot 1^2 + 4\ln 1$$

b)

$f(x) = x\ln x$

$f'(x) = x \cdot \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x$

$f'(e^2) = 1 + \ln(e^2) = 1 + 2 = 3$

#9 a)

$f(x) = x^2 e^x$

$f'(x) = x^2 \cdot e^x + e^x \cdot 2x$

$f'(\ln 3) = (\ln 3)^2 \cdot e^{\ln 3} + e^{\ln 3} \cdot 2\ln 3$

$= (\ln 3)^2 \cdot 3 + 3 \cdot 2\ln 3$

$= 3(\ln 3)^2 + 6\ln 3$

#9b

$\int e^x \sqrt{4+e^x}\, dx \qquad u = 4+e^x$

$du = e^x\, dx$

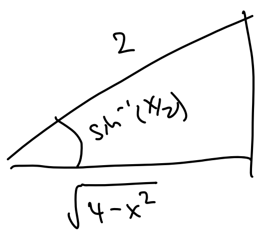
$= \int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(4+e^x)^{3/2} + C$

#10

$f(x) = \log_4 x + \log_4(\sin x) - \log_4(x+2)$

$f'(x) = \frac{1}{x \ln 4} + \frac{1}{\sin x \ln 4} \cdot \cos x - \frac{1}{(x+2)\ln 4}$

#11 a)



$x^2 + a^2 = 4$

$a = \sqrt{4-x^2}$

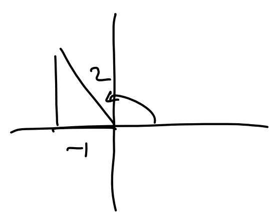
So $\tan\left(\sin^{-1}\left(\frac{x}{2}\right)\right) = \frac{x}{\sqrt{4-x^2}}$

b)

$\sin^{-1}(1) = \pi/2 \qquad$ since $\sin \pi/2 = 1$

$\tan^{-1}(1) = \pi/4 \qquad$ since $\tan \pi/4 = 1$

$\cos^{-1}(-1/2) = 2\pi/3 \qquad$ since $\cos(2\pi/3) = -1/2$



#12

$\lim_{x \to 2} \frac{\ln(x-1)}{x^2-4} \to \frac{\ln(2-1)}{2^2-4} = \frac{\ln 1}{0} = \frac{0}{0}$

L'Hop: $\lim_{x \to 2} \frac{\frac{1}{x-1}}{2x} \to \frac{1/1}{4} = \boxed{\frac{1}{4}}$

#13

$$\int_1^2 x^4 \ln x \, dx \qquad u = \ln x \quad du = \tfrac{1}{x}\,dx \qquad dv = x^4\,dx \quad v = \tfrac{1}{5}x^5$$

$$= \ln x \cdot \frac{1}{5}x^5 - \int_1^2 \frac{1}{5}x^5 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int_1^2 x^4\,dx = \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 \Big|_1^2$$

$$= \frac{1}{5}2^5 \ln 2 - \frac{1}{25}2^5 - \left(\frac{1}{5}1^5 \cdot \ln 1 - \frac{1}{25}\cdot 1^5\right)$$

#14

$$\int \sin^5 x\,dx = \int \sin^4 x \cdot \sin x\,dx$$

$$= \int (\sin^2 x)^2 \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx \qquad u = \cos x \quad du = -\sin x\,dx$$

$$= -\int (1-u^2)^2\,du = -\int 1 - 2u^2 + u^4\,du$$

$$= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C$$

$$= -\left(\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right) + C$$

#15

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx \qquad \begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}$$

$$\int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta = 4\int \frac{\sin^2\theta}{\cancel{2}\cancel{\cos\theta}}\cdot \cancel{2}\cancel{\cos\theta}\,d\theta$$

$$= 4\int \sin^2\theta\,d\theta = 4\int \frac{1}{2}\left(1-\cos 2\theta\right)d\theta$$

$$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = 2\left(\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\sin\left(2\sin^{-1}\frac{x}{2}\right)\right) + C$$

#16

$$\begin{array}{r|l} & x^2 + 2x \\ \hline x^2+x & x^3 + 3x^2 + 2x + 1 \\ & \underline{x^3 + x^2} \\ & 2x^2 + 2x \\ & \underline{2x^2 + 2x} \\ & 1 \end{array}$$

$$\int x^2 + 2x + \frac{1}{x^2+x}\,dx$$

partial fractions

$$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$

$$A(x+1) + Bx = 1$$

$$Ax + A + Bx = 1$$

$$(A+B)x + A = 1$$

$$A = 1 \qquad A + B = 0$$

$$B = -1$$

$$\int x^2 + 2x + \frac{1}{x} - \frac{1}{x+1}\,dx = \frac{1}{3}x^3 + x^2 - \ln|x| - \ln|x+1| + C$$

#17

$$\int_3^\infty \frac{x^2}{(1+x^3)^4}\,dx$$

$u = 1+x^3$
$du = 3x^2 dx$
$\frac{1}{3} du = x^2 dx$

$$= \int \frac{1}{u^4} \cdot \frac{1}{3} du = \frac{1}{3} \int u^{-4} = \frac{1}{3} \cdot \frac{1}{-3} u^{-3}$$

$$= -\frac{1}{9} (1+x^3)^{-3} \Big|_3^t$$

$$\lim_{t\to\infty} \frac{-1}{9} (1+x^3)^{-3} \Big|_3^t$$

$$= \lim_{t\to\infty} \frac{-1}{9} \underbrace{(1+t^3)^{-3}}_{\downarrow\, 0} - \frac{-1}{9} (1+3^3)^{-3}$$

$$= \frac{1}{9} (1+3^3)^{-3}$$

#18

$$\int_1^3 \sqrt{1+(f'(x))^2}\,dx$$

$f'(x) = 3x^{1/2}$
$(f'(x))^2 = 9x$

$$= \int_1^3 \sqrt{1+9x}\,dx$$

$u = 9x$
$du = 9dx$

$$= \frac{1}{9} \int \sqrt{u}\,du = \frac{1}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{x=1}^{x=3}$$

$$= \frac{2}{27} (1+9x)^{3/2} \Big|_1^3 = \frac{2}{27} (1+27)^{3/2} - \frac{2}{27} (1+9)^{3/2}$$