Math 1172

Catchup exam #2

#23 $\displaystyle\sum_{n=0}^{\infty} \frac{4^n}{n^2} x^n$

Ratio: $\displaystyle L = \lim \left| \frac{4^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{4^n x^n} \right| = \lim \left| 4 \cdot x \cdot \left(\frac{n}{n+1}\right)^2 \right|$

$= 4|x| \cdot 1$

$L = 4|x|$

Conv. when $4|x| < 1$

$|x| < 1/4$

$-1/4 < x < 1/4$

Endpts

$x = 1/4$: $\displaystyle\sum \frac{4^n \cdot (1/4)^n}{n^2} = \sum \frac{1}{n^2}$ converges (p-series, $p=2$)

$x = -1/4$: $\displaystyle\sum \frac{4^n (-1/4)^n}{n^2} = \sum \frac{(-1)^n}{n^2}$ converges (abs series test)

So IOC is $[-1/4, 1/4]$

Old ones

#1 a) $f(x) = x\sin(x^2)$

$f'(x) = x \cdot \cos(x^2) \cdot 2x + \sin(x^2) \cdot 1$

$f'\left(\frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{2} \cos\left(\frac{\pi}{4}\right) \cdot 2\frac{\sqrt{\pi}}{2} + \sin\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{2}} \cdot 2\frac{\sqrt{\pi}}{2} + \frac{1}{\sqrt{2}}$

b) $\int x(x+3)^2 dx = \int x(x^2+6x+9)\,dx$

$= \int x^3+6x^2+9x \; dx = \frac{1}{4}x^4 + \frac{6}{3}x^3 + \frac{9}{2}x^2 + C$

#2 $\int_1^3 x^2\sqrt{5-x^3}\,dx$

$u = 5-x^3$

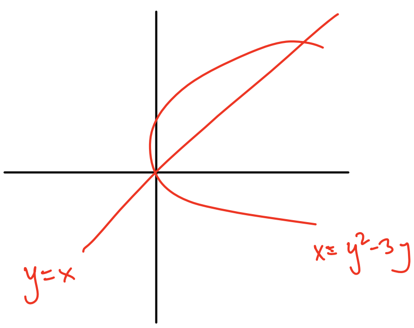
$du = -3x^2 dx$

$-\frac{1}{3}du = x^2 dx$

$-\frac{1}{3}\int \sqrt{u}\,du = \frac{-1}{3} \cdot \frac{2}{3} u^{3/2} = \frac{-2}{9}(5-x^3)^{3/2} \Big|_1^3$

$= \frac{-2}{9}(5-3^3)^{3/2} - \frac{-2}{9}(5-1)^{3/2}$

#3



Intersections

$y = y^2 - 3y$

$0 = y^2 - 4y$

$0 = y(y-4)$

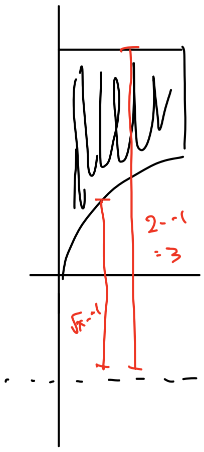
$y=0,\; y=4$

$\int_0^4 y - (y^2-3y)\,dy = \int_0^4 -y^2+4y \; dy$

$$= -\frac{1}{3}y^3 + 2y \Big|_0^4$$

$$= -\frac{1}{3}\cdot 4^3 + 2\cdot 4 - (0)$$

#4



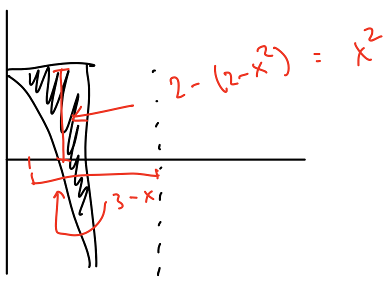
$$\pi \int_0^1 3^2 - (\sqrt{x}+1)^2 \, dx = \pi \int_0^1 9 - (x + 2\sqrt{x} + 1) \, dx$$

$$= \pi \int_0^1 8 - x + 2x^{1/2} \, dx$$

$$= \pi \left(8x - \frac{1}{2}x^2 + 2\cdot\frac{2}{3}x^{3/2} \right) \Big|_0^1$$

$$= \pi \left(8 - \frac{1}{2} + \frac{4}{3} \right) - \pi \cdot 0$$

#5



$$2\pi \int_0^2 (3-x)(x^2)\, dx = 2\pi \int_0^2 3x^2 - x^3 \, dx = 2\pi \left(x^3 - \frac{1}{4}x^4 \right) \Big|_0^2$$

$$= 2\pi \left(2^3 - \frac{1}{4}\cdot 2^4 \right) - 0$$

#6 a

$$f(x) = \ln(x^2 + x\sin x)$$

$$f'(x) = \frac{1}{x^2 + x\sin x}(2x + x\cdot\cos x + \sin x)$$

b

$$5 + 4^{2x} = 10$$

$$4^{2x} = 5$$

$$\ln 4^{2x} = \ln 5$$

$$2x \ln 4 = \ln 5$$

$$X = \frac{\ln 5}{2\ln 4}$$

#7 a

$$f(x) = e^{x^7 \sin x}$$

$$f'(x) = e^{x^7 \sin x}\left(x^7 \cdot \cos x + \sin x \cdot 7x^6 \right)$$

b

$$\int \frac{e^x}{(3-e^x)^5}\, dx \qquad u = 3-e^x \qquad du = -e^x\, dx$$

$$= -\int \frac{1}{u^5}\, du = -\int u^{-5}\, du = \frac{1}{4} u^{-4} + C$$

$$= \frac{1}{4}(3-e^x)^{-4} + C$$

#8 a

$$f(x) = 5x^2 \cdot 4^{2x}$$

$$f'(x) = 5x^2 \cdot 4^{2x} \ln 4 \cdot 2 + 4^{2x} \cdot 10x$$

b

$$\int x\, 5^{4-x^2}\, dx \qquad u = 4-x^2 \qquad du = -2x\, dx$$

$$-\frac{1}{2}\int 5^u\, du = -\frac{1}{2} \cdot \frac{1}{\ln 5} \cdot 5^u + C$$

$$= -\frac{1}{2\ln 5}\, 5^{4-x^2} + C$$

#9

$$\int_0^{2/3} \frac{2}{\sqrt{4-9x^2}}\,dx = \cancel{2}\int \frac{1}{\cancel{2}\sqrt{1-\frac{9}{4}x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-(\frac{3}{2}x)^2}}\,dx$$

$u = 3/2\,x$

$du = \frac{3}{2}dx$

$\frac{2}{3}du = dx$

$$= \frac{2}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{2}{3}\sin^{-1}u \qquad = \frac{2}{3}\sin^{-1}\left(\frac{3}{2}x\right)\Big|_0^{2/3}$$

$$= \frac{2}{3}\sin^{-1}(1) - \frac{2}{3}\sin^{-1}0$$

$$= \frac{2}{3}\cdot \pi/2 - 0 = \pi/3$$

#10

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{1/x} \overset{H}{=} \lim_{x\to 0} \frac{1/x}{-x^{-2}} = \lim_{x\to 0} -\frac{x^2}{x}$$

$$= \lim_{x\to 0} -x = 0$$

#11

$$\int_1^3 \frac{\ln x}{x^5}\,dx = \int_1^3 x^{-5}\ln x\,dx$$

$u = \ln x$ $\qquad dv = x^{-5}\,dx$

$du = \frac{1}{x}dx$ $\qquad v = -\frac{1}{4}x^{-4}$

$$= \ln x \cdot \frac{-1}{4}x^{-4} - \int -\frac{1}{4}x^{-4}\cdot\frac{1}{x}\,dx$$

$$= -\frac{1}{4}\ln x\cdot x^{-4} + \frac{1}{4}\int x^{-5}\,dx = -\frac{1}{4}\ln x\cdot x^{-4} + \frac{1}{4}\cdot\frac{-1}{4}x^{-4}\Big|_1^3$$

$$= -\frac{1}{4}\ln 3\cdot 3^{-4} - \frac{1}{16}\cdot 3^{-4} - \left(\frac{-1}{4}\ln 1\cdot 1^{-4} - \frac{1}{16}\cdot 1^{-4}\right)$$

#12

$$\int \sin^2 x \cos^2 x \, dx$$

$$= \int \frac{1}{2}(1+\cos 2x) \cdot \frac{1}{2}(1-\cos 2x)\, dx$$

$$= \frac{1}{4}\int 1 - \cos^2 2x \, dx = \frac{1}{4}\int 1 - \left(\frac{1}{2}(1-\cos 4x)\right) dx$$

$$= \frac{1}{4}\int \frac{1}{2} + \frac{1}{2}\cos 4x \, dx = \frac{1}{4}\left(\frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{4}\sin 4x\right) + C$$

#13

$$\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\, dx$$

$x = \sin\theta$ $\quad x=0: \theta = 0$

$dx = \cos\theta\, d\theta$ $\quad x=1: \theta = \pi/2$

$$= \int_0^{\pi/2} \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^3\theta}{\cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta = \int_0^{\pi/2} \sin^5\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \quad \sin\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= \int_0^{\pi/2} (1-\cos^2\theta) \quad \sin\theta\, d\theta$$

$$= -\int (1-u^2)\, du = -\left(u - \frac{1}{3}u^3\right)$$

$$= -\cos\theta + \frac{1}{3}\cos^3\theta \Big|_0^{\pi/2}$$

$$= -\cos\pi/2 + \frac{1}{3}(\cos\pi/2)^3 - \left(-\cos 0 + \frac{1}{3}(\cos 0)^3\right)$$

$$= 0 \qquad 0 - \left(-1 + \frac{1}{3}\right) = 1 - \frac{1}{3} = 2/3$$

#14

$$\begin{array}{r|l} & 3 \\ x^2+x-2 & 3x^2+2x+1 \\ & 3x^2+3x-6 \\ \hline & -x+7 \end{array}$$

$$\frac{3x^2+2x+1}{x^2+x-2} = 3 + \frac{-x+7}{x^2+x-2}$$

$$\frac{-x+7}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$-x+7 = A(x-1) + B(x+2)$$

$$-x+7 = Ax - A + Bx + 2B$$

$-1 = A + B$ $\qquad$ $6 = 3B$

$7 = -A + 2B$ $\qquad$ $B = 2$

$A = -1 - B = -3$

$$\int \frac{3x^2+2x+1}{x^2+x-2} = \int 3 + \frac{-3}{x+2} + \frac{2}{x-1}\, dx$$

$$= 3x - 3\ln|x+2| + 2\ln|x-1| + C$$

#15

$$\int_1^4 \frac{x}{\sqrt{x^2-1}}\,dx = \lim_{t\to 1} \int_t^4 \frac{x}{\sqrt{x^2-1}}\,dx$$

$u = x^2 - 1$

$du = 2x\,dx$

$$= \lim_{t\to 1} \frac{1}{2}\int u^{-1/2}\,du = \lim_{t\to 1} \frac{1}{2}\cdot 2u^{1/2}$$

$$= \lim_{t\to 1} (x^2-1)^{1/2}\Big|_t^4 = \lim_{t\to 1} (4^2-1)^{1/2} - (t^2-1)^{1/2}$$

$$= (4^2-1)^{1/2} - (1^2-1)^{1/2} = \sqrt{15}$$

\# 17 $a_n = n^2 + 2n - 5$

Try some: $n=1: \quad 1^2+2-5=-2$

$n=2: \quad 2^2+2\cdot 2-5 = 3$

looks inc.

WTS: $a_{n+1} \geq a_n$

WB $(n+1)^2+2(n+1)-5 \geq n^2+2n-5$

i.e. $n^2+2n+1+2n+2-5 \geq$ ..

WB $\cancel{n^2}+4n-2 \geq \cancel{n^2}+2n-5$

WB $2n \geq -3$

This is true since n is positive!

\# 18 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{4^n}{3^{2n}}$

$a_1 = -\frac{4}{3^2}$

$a_2 = +\frac{4^2}{3^4}$

$r = \frac{4^2}{3^4} \cdot \frac{-3^2}{4} = -\frac{4}{3^2} = -\frac{4}{9}$

$|r|<1$ so it converges to

$\frac{a}{1-r} = \frac{-4/9}{1+4/9}$

\# 19 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^4}$

$$\int_1^{\infty} \frac{1}{(2x)^4}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{16x^4}\,dx = \lim_{t\to\infty} \frac{1}{16}\int_1^t x^{-4}\,dt$$

$$= \lim_{t\to\infty} \frac{1}{16} \cdot \frac{-1}{3}x^{-3}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{1}{16}\cdot\frac{-1}{3}\underset{\downarrow \atop 0}{t^{-3}} - \frac{1}{16}\cdot\frac{-1}{3}\cdot 1$$

$$= \frac{1}{16}\cdot\frac{1}{3}$$

#20 $\quad \sum \frac{5+\sin n}{n^2} \leq \sum \frac{6}{n^2} = 6\sum\frac{1}{n^2}$

The bigger converges since it's a p-series, $p=2$.

So the smaller converges

#21 $\quad \sum (-1)^n \frac{n}{2^n}$

lim $\quad \lim \frac{n}{2^n} \overset{H}{=} \lim \frac{1}{2^n \cdot \ln 2} = \frac{1}{\infty} = 0 \quad \checkmark$

dec $\quad$ WB $\quad \frac{n+1}{2^{n+1}} \leq \frac{n}{2^n}$

WB $\quad (n+1)2^n \leq n\, 2^{n+1}$

WB $\quad n+1 \leq n\cdot 2$

WB $\quad 1 \leq n \quad$ which is true since n is positive. $\checkmark$

#22 $\quad \sum (-1)^n \frac{n}{e^n}$

ratio: $\quad L = \lim \left| \frac{(-1)^{n+1}(n+1)}{e^{n+1}} \cdot \frac{e^n}{(-1)^n\cdot n} \right|$

$$= \lim \left| -1 \cdot \frac{n+1}{n} \cdot \frac{e^n}{e^{n+1}} \right|$$

$$= 1 \cdot 1 \cdot \frac{1}{e}$$

$L = \frac{1}{e} < 1$ so it <u>converges</u>