The Ratio Test
Applied Calculus II, MA 120
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The Ratio Test is a test to determine if a given series converges (has a finite sum) or diverges. It determines the convergence or divergence of a series by comparing it to a geometric series.

Ratio Test. *Given a series $\sum_n a_n$, consider the limit*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = R.$$

If $R < 1$, then the series $\sum_n a_n$ converges.
If $R > 1$, then the series $\sum_n a_n$ diverges.
If $R = 1$, then the test is inconclusive: we cannot tell if the series converges or diverges using this test.

Proof. By the definition of limit, $\frac{|a_{n+1}|}{|a_n|} \approx R$ for very large n. If $R < 1$, then we can pick a number r, $R < r < 1$ so that $\frac{|a_{n+1}|}{|a_n|} < r$ for all very large n. This means,

$$|a_{n+1}| < r|a_n|, \text{ and also } |a_{n+2}| < r^2|a_n|, \text{ and also } |a_{n+3}| < r^3|a_n|,$$

and in general, $|a_{n+k}| < r^k|a_n|$. So the series after the nth term looks geometric!

$$\begin{aligned} |a_n| + |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+k}| + \cdots &< |a_n| + r|a_n| + r^2|a_n| + \cdots + r^k|a_n| + \cdots \\ &= |a_n|(1 + r + r^2 + \cdots + r^k + \cdots). \end{aligned}$$

This converges since $r < 1$.

If $R > 1$, then in a similar way, we can pick a number r so that $R > r > 1$ and $\frac{|a_{n+1}|}{|a_n|} > r$ for all very large n. This means,

$$|a_{n+1}| > r|a_n|, \text{ and also } |a_{n+2}| > r^2|a_n|, \text{ and also } |a_{n+3}| > r^3|a_n|,$$

and in general, $|a_{n+k}| > r^k|a_n|$. So again the series after the nth term looks geometric.

$$\begin{aligned} |a_n| + |a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+k}| + \cdots &> |a_n| + r|a_n| + r^2|a_n| + \cdots + r^k|a_n| + \cdots \\ &= |a_n|(1 + r + r^2 + \cdots + r^k + \cdots). \end{aligned}$$

This diverges since $r > 1$. $\square$

Example 1. Determine if the series $\displaystyle\sum_{n=1}^{\infty} n3^{-n}$ converges or diverges.

Solution. Notice that the nth term a_n is $a_n = n3^{-n} = \frac{n}{3^n}$. So,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \frac{n+1}{n} \cdot \frac{1}{3}.$$

Now,

$$\lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{1} = \frac{1+0}{1} = 1,$$

so that,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{3} = 1 \cdot \frac{1}{3} = \frac{1}{3} = R.$$

Since $R < 1$, this series converges. $\square$

Example 2. Determine if the series $\displaystyle\sum_{n=5}^{\infty} \frac{n^2}{e^n}$ converges or diverges.

Solution. The nth term a_n is $a_n = \frac{n^2}{e^n}$. So,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2} = \frac{n^2+2n+1}{n^2} \cdot \frac{1}{e}.$$

Now,

$$\lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1} = \frac{1+0+0}{1} = 1,$$

so that,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} \cdot \frac{1}{e} = 1 \cdot \frac{1}{e} = \frac{1}{e} = R.$$

Since $R < 1$, this series converges. $\square$

Example 3. Determine if the series $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{n!}$ converges or diverges.

Solution. Recall that $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. The nth term of the series a_n is $a_n = \frac{2^n}{n!}$. So,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} \cdot \frac{2}{1} = \frac{2}{n+1}.$$

Now,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 = R.$$

Since $R < 1$, this series converges. $\square$

Example 4. Determine if the series $\displaystyle\sum_{n=1}^{\infty} 4^n n^{-6}$ converges or diverges.

Solution. The nth term a_n is $a_n = 4^n n^{-6} = \frac{4^n}{n^6}$. So,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{(n+1)^6}}{\frac{4^n}{n^6}} = \frac{4^{n+1}}{(n+1)^6} \cdot \frac{n^6}{4^n} = \frac{n^6}{(n+1)^6} \cdot \frac{4}{1}.$$

Now, we know from Example 1 that $\lim_{n\to\infty} \frac{n+1}{n} = 1$ and therefore its reciprocal is also 1, $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Using this,

$$\lim_{n\to\infty} \frac{n^6}{(n+1)^6} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^6 = (1)^6 = 1,$$

so that,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^6}{(n+1)^6} \cdot \frac{4}{1} = 1 \cdot 4 = 4 = R.$$

Since $R > 1$, this series diverges. $\square$