Algebraic Properties of $\mathbb{R}$ (Field Axioms)

The set $\mathbb{R}$ has two binary operations, addition, $+$, and multiplication, $\cdot$, which satisfy the following axioms.

(A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$ (commutative property of addition).

(A2) $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathbb{R}$ (associative property of addition).

(A3) There exists an element 0 in $\mathbb{R}$ such that $0+a = a$ for all $a \in \mathbb{R}$ (existence of a zero element).

(A4) For each $a \in \mathbb{R}$ there exists an element $-a \in \mathbb{R}$ such that $a + (-a) = 0$ (existence of negative elements).

(M1) $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{R}$ (commutative property of multiplication).

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in \mathbb{R}$ (associative property of multiplication).

(M3) There exists an element 1 in $\mathbb{R}$, distinct from 0, such that $1 \cdot a = a$ for all $a \in \mathbb{R}$ (existence of a unit element).

(M4) For each $a \in \mathbb{R}$, $a \neq 0$, there exists an element $1/a \in \mathbb{R}$ such that $a \cdot (1/a) = 1$ (existence of reciprocals).

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in \mathbb{R}$ (distributive property of multiplication over addition).

Example of a Finite Field

The set of four elements $F_4 = \{0, 1, A, B\}$ with the following operations is a field, i.e. it satisfies the algebraic properties listed above.

$+$	**0**	**1**	**A**	**B**
0	0	1	A	B
1	1	0	B	A
A	A	B	0	1
B	B	A	1	0

$\cdot$	**0**	**1**	**A**	**B**
0	0	0	0	0
1	0	1	A	B
A	0	A	B	1
B	0	B	1	A