Spectral Analysis of the Transfer Operator for the Lorentz Gas

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Abstract

We study the billiard map associated with both the finite and infinite horizon Lorentz gases having smooth scatterers with strictly positive curvature. We introduce generalized function spaces (Banach spaces of distributions) on which the transfer operator is quasi-compact. The mixing properties of the billiard map then imply the existence of a spectral gap and related statistical properties such as exponential decay of correlations and the central limit theorem. Finer statistical properties of the map such as the identification of Ruelle resonances, large deviation estimates and an almost-sure invariance principle follow immediately once the spectral picture is established.

1 Introduction

Much attention has been given in recent years to developing a framework to study directly the transfer operator associated with hyperbolic maps on an appropriate Banach space. The goal of such a functional analytic approach is first to use the smoothing properties of the transfer operator to prove its quasi-compactness and then to derive statistical information about the map from the peripheral spectrum. For expositions of this subject, see [B1, HH, L1].

The link between the transfer operator and the statistical properties of the map traces back to classical results regarding Markov chains [DF, IM, N]. In the context of deterministic systems, this approach was first adapted to overcome the problem of discontinuities for expanding maps by using the smoothing effect of the transfer operator on functions of bounded variation [LY, K, S, Bu, T1, T2, BK]. Its extension to hyperbolic maps followed, using simultaneously the smoothing properties of the transfer operator in unstable directions and the contraction present in the stable directions: first to Anosov diffeomorphisms [R1, R2, R3, BKL, B2, BT, GL] and more recently to piecewise hyperbolic maps [DL, BG1, BG2]. Two crucial assumptions in the treatment of the piecewise hyperbolic case in two dimensions have been: (1) the map has a finite number of singularity curves and (2) the map admits a smooth extension up to the closure of each of its domains of definition. These assumptions and other technical difficulties have thus far prevented this approach from being successfully carried out for dispersing billiards.

In this paper, we apply the functional analytic approach to the billiard map associated with both a finite and infinite horizon Lorentz gas having smooth scatterers with strictly positive curvature.

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We introduce generalized function spaces (Banach spaces of distributions) on which the transfer operator is quasi-compact. The mixing properties of the billiard map then imply the existence of a spectral gap, the exponential decay of correlations and finer statistical properties such as Ruelle resonances. Many limit theorems such as local large deviation estimates, a central limit theorem, and an almost sure invariance principle for both invariant and non-invariant measures also follow immediately once the spectral picture is established.

Although the exponential decay of correlations and many limit theorems are already known for such classes of billiards [Y, Ch1, RY, MN2], the present approach provides a unified and greatly simplified framework in which to achieve these results and completely bypasses previous methods which relied on constructing countable Markov partitions [BSC1, BSC2], Markov extensions [Y, Ch1, CY], or magnets for coupling arguments [Ch2], all of which require a deep understanding of the regularity properties of the foliations. Indeed, we avoid entirely the need to work with the holonomy map matching unstable curves along real stable manifolds, which is a major technical difficulty present in each of the previous approaches.

In addition, the current functional analytic framework allows immediate extensions of well-known limit theorems to non-invariant measures. For example, we prove in Theorem 2.6 that our large deviation rate function is independent of the probability measure in our Banach space with which we measure the asymptotic deviations (see Section 2.4). Although the limit theorems with respect to invariant measures presented in Theorem 2.6 are already known for this class of billiards, the extensions to non-invariant measures constitute new results, with the partial exception of [D], which dealt with large deviations only. Finally, the spectral picture obtained via the method in this paper has been shown to be robust under a wide variety of perturbations in a number of settings [BY, KL] (see also the treatment of perturbations using norms similar to those in this paper in [DL]), and it is expected that the present framework will allow the unified treatment of large classes of perturbations in a way previously unattainable for billiards.

The paper is organized as follows. In Section 2, we define the Banach spaces on which we will study the transfer operator and state our main results. The norms we define follow closely those introduced in [DL], with the addition of an extra weighting factor to counteract the blow-up of the Jacobian of the map near singularities. In order to control distortion, we introduce additional cuts at the boundaries of homogeneity strips which implies that our expanded singularity sets comprise a countably infinite number of curves in both the finite and infinite horizon cases. In Section 3, we prove the necessary growth lemmas to control the cutting generated by the expanded singularity sets and prove preliminary properties of our Banach spaces including embeddings and compactness. Section 4 contains the required Lasota-Yorke inequalities and in Section 5 we characterize the peripheral spectrum and prove some related statistical properties. Section 6 contains the proofs of the limit theorems mentioned above.

2 Setting, Definitions and Results

2.1 Billiard maps associated with a Lorentz gas

We define here the class of maps to which our results apply and take the opportunity to establish some notation. Let $\{\Gamma_i\}_{i=1}^d$ be pairwise disjoint, simply connected convex regions in $\mathbb{T}^2$ having $\mathcal{C}^3$ boundary curves $\partial\Gamma_i$ with strictly positive curvature. We consider the billiard flow on the table $Q = \mathbb{T}^2 \setminus \cup_i\{\text{interior } \Gamma_i\}$ induced by a particle traveling at unit speed and undergoing elastic collisions at the boundaries. The phase space for the billiard flow is $\mathcal{M} = Q \times \mathbb{S}^1/\sim$ with the conventional identifications at the boundaries. Define $M = \cup_i \partial\Gamma_i \times [-\pi/2, \pi/2]$. The billiard map $T : M \to M$ is the Poincaré map corresponding to collisions with the scatterers. We will denote coordinates on M by (r,φ), where $r \in \cup_i \partial\Gamma_i$ is parametrized by arclength and φ is the angle that the unit tangent vector at r makes with the normal pointing into the domain Q. T preserves a probability measure μ defined by $d\mu = c\cos\varphi\, dr\, d\varphi$ on M, where c is the normalizing constant.

For any $x = (r,\varphi) \in M$, we denote by $\tau(x)$ the time of the first (non-tangential) collision of the trajectory starting at x under the billiard flow. The billiard map T is defined whenever $\tau(x) < \infty$ and is known to be uniformly hyperbolic, although its derivative DT becomes infinite near singularities (see for example [CM, Chapter 4]). We say T has *finite horizon* if there is an upper bound on the function τ. Otherwise, we say T has *infinite horizon*.

2.2 Transfer Operator

We define scales of spaces using a set of *admissible curves* $\mathcal{W}^s$ (defined in Section 3.1) on which we define the action of the *transfer operator* $\mathcal{L}$ associated with T. Such curves are homogeneous stable curves whose length is smaller than some fixed δ_0. Define $T^{-n}\mathcal{W}^s$ to be the set of homogeneous stable curves W such that T^n is smooth on W and $T^iW \in \mathcal{W}^s$ for $0 \le i \le n$. It follows from the definition that $T^{-n}\mathcal{W}^s \subset \mathcal{W}^s$.

We denote (normalized) Lebesgue measure on M by m, i.e., $dm = cdrd\varphi$. For $W \in T^{-n}\mathcal{W}^s$, a complex-valued test function $\psi : M \to \mathbb{C}$ and $0 < p \le 1$, define $H^p_W(\psi)$ to be the Hölder constant of ψ on W with exponent p measured in the Euclidean metric. Define $H^p_n(\psi) = \sup_{W \in T^{-n}\mathcal{W}^s} H^p_W(\psi)$ and let $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s) = \{\psi : M \to \mathbb{C} \mid |\psi|_\infty + H^p_n(\psi) < \infty\}$, denote the set of complex-valued functions which are Hölder continuous on elements of $T^{-n}\mathcal{W}^s$. The set $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$ equipped with the norm $|\psi|_{\mathcal{C}^p(T^{-n}\mathcal{W}^s)} = |\psi|_\infty + H^p_n(\psi)$ is a Banach space. We define $\mathcal{C}^p(T^{-n}\mathcal{W}^s)$ to be the closure of $\tilde{\mathcal{C}}^1(T^{-n}\mathcal{W}^s)$ in $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$. Similarly, we define $\tilde{\mathcal{C}}^p(T^{n}\mathcal{W}^u)$ and $\mathcal{C}^p(T^{n}\mathcal{W}^u)$ for each $n \ge 0$, the set of functions which are Hölder continuous with exponent p on unstable curves in $T^n\mathcal{W}^u$, defined in Section 3.1.

It follows from (4.3) that if $\psi \in \tilde{\mathcal{C}}^p(T^{-(n-1)}\mathcal{W}^s)$, then $\psi \circ T \in \tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$. Similarly, if $\xi \in \tilde{\mathcal{C}}^1(T^{-(n-1)}\mathcal{W}^s)$, then $\xi \circ T \in \tilde{\mathcal{C}}^1(T^{-n}\mathcal{W}^s)$. These two facts together imply that if $\psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s)$, then $\psi \circ T \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$.

If $h \in (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$, is an element of the dual of $\mathcal{C}^p(T^{-n}\mathcal{W}^s)$, then $\mathcal{L} : (\mathcal{C}^p(T^{-n}\mathcal{W}^s))' \to (\mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s))'$ acts on h by

$$\mathcal{L}h(\psi) = h(\psi \circ T) \quad \forall \psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s).$$

If $h \in L^1(M,m)$, then h is canonically identified with a signed measure absolutely continuous with respect to Lebesgue, which we shall also call h, i.e.,

$$h(\psi) = \int_M \psi h\, dm.$$

With the above identification, we write $L^1(M,m) \subset (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$ for each $n \in \mathbb{N}$. Then restricted to $L^1(M,m)$, $\mathcal{L}$ acts according to the familiar expression

$$\mathcal{L}^n h = h \circ T^{-n}\, |DT^n(T^{-n})|^{-1}$$

for any $n \ge 0$ and any $h \in L^1(M,m)$, where $|DT^n|$ denotes $|\det DT^n|$ to simplify notation.

2.3 Definition of the Norms

The norms we introduce below are defined via integration on the set of admissible stable curves $\mathcal{W}^s$ referred to in Section 2.2. In Section 3.1 we define precisely the notion of a distance $d_{\mathcal{W}^s}(\cdot,\cdot)$ between such curves as well as a distance $d_q(\cdot,\cdot)$ defined among functions supported on these curves.

The motivation for these norms is the following: We expect the action of the transfer operator to increase regularity in the unstable direction and to decrease regularity in the stable direction, so we integrate along stable curves in order to average the action of the transfer operator in the stable direction. The unstable norm $\|\cdot\|_u$ morally measures a Hölder constant in the unstable direction by comparing the norms of an element of $\mathcal{B}$ on two stable curves lying close together. The weights $\cos W$ assigned to the test functions are introduced to counteract the blowup of the Jacobian near singularities; they also help us sum over homogeneity strips as in the proof of Lemma 3.9. The weight α in $\|\cdot\|_s$ is important for the proof of compactness (see Lemma 3.10) as well as the Lasota-Yorke estimate for $\|\cdot\|_u$ (see Section 4.3).

Given a curve $W \in \mathcal{W}^s$, we denote by m_W the unnormalized Lebesgue (arclength) measure on W. We set $|W| = m_W(W)$. We also denote the Euclidean metric on W by $d_W(\cdot,\cdot)$. With a slight abuse of notation, we define $\cos W$ to be the average value of $\cos\varphi$ on $W \in \mathcal{W}^s$, i.e. $\cos W = |W|^{-1}\int_W \cos\varphi \, dm_W$.

For $0 \le p \le 1$, denote by $\tilde{\mathcal{C}}^p(W)$ the set of continuous complex-valued functions on W with Hölder exponent p, measured in the Euclidean metric. We then denote by $\mathcal{C}^p(W)$ the closure of $\tilde{\mathcal{C}}^1(W)$ in the $\tilde{\mathcal{C}}^p$-norm[1]: $|\psi|_{\mathcal{C}^p(W)} = |\psi|_{\mathcal{C}^0(W)} + H^p_W(\psi)$, where $H^p_W(\psi)$ is the Hölder constant of ψ along W. Notice that with this definition, $|\psi_1\psi_2|_{\mathcal{C}^p(W)} \le |\psi_1|_{\mathcal{C}^p(W)}|\psi_2|_{\mathcal{C}^p(W)}$. We define $\tilde{\mathcal{C}}^p(M)$ and $\mathcal{C}^p(M)$ similarly.

For $\alpha \ge 0$, define the following norms for test functions,

$$|\psi|_{W,\alpha,p} := |W|^\alpha \cdot \cos W \cdot |\psi|_{\mathcal{C}^p(W)}.$$

Now fix $0 < p \le 1/3$. Given a function $h \in \mathcal{C}^1(M)$, define the *weak norm* of h by

$$|h|_w := \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in\mathcal{C}^p(W)\\ |\psi|_{W,0,p}\le 1}} \int_W h\psi \, dm_W. \tag{2.1}$$

Choose[2] α, β, $q > 0$ such that $\alpha < 1/6$, $q < p$ and $\beta \le \min\{\alpha, p-q\}$. We define the *strong stable norm* of h as

$$\|h\|_s := \sup_{W\in\mathcal{W}^s} \sup_{\substack{\psi\in\mathcal{C}^q(W)\\ |\psi|_{W,\alpha,q}\le 1}} \int_W h\psi \, dm_W, \tag{2.2}$$

and the *strong unstable norm* as

$$\|h\|_u := \sup_{\varepsilon\le\varepsilon_0} \sup_{\substack{W_1,W_2\in\mathcal{W}^s\\ d_{\mathcal{W}^s}(W_1,W_2)\le\varepsilon}} \sup_{\substack{\psi_i\in\mathcal{C}^p(W_i)\\ |\psi_i|_{W_i,0,p}\le 1\\ d_q(\psi_1,\psi_2)\le\varepsilon}} \frac{1}{\varepsilon^\beta}\left|\int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W\right|, \tag{2.3}$$

where $\varepsilon_0 > 0$ is chosen less than δ_0, the maximum length of $W \in \mathcal{W}^s$ which is determined after (3.2). We then define the *strong norm* of h by

$$\|h\|_{\mathcal{B}} = \|h\|_s + b\|h\|_u,$$

where b is a small constant chosen in Section 2.4.

We define $\mathcal{B}$ to be the completion of $\mathcal{C}^1(M)$ in the strong norm and $\mathcal{B}_w$ to be the completion of $\mathcal{C}^1(M)$ in the weak norm. In Section 4, we will actually apply these norms to functions of the form $\mathcal{L}h$ where $h \in \mathcal{C}^1(M)$. The fact that $\mathcal{L}h \in \mathcal{B}$ when $h \in \mathcal{C}^1(M)$ is established in Lemma 3.8.

[1]Note that for $p < 1$, while $\mathcal{C}^p(W)$ may not contain all of $\tilde{\mathcal{C}}^p(W)$, it does contain $\mathcal{C}^{p'}(W)$ for all $p' > p$.

[2]The restrictions on the constants are placed according to the dynamical properties of T. For example, $p \le 1/3$ due to the distortion estimate (3.1) while $\alpha < 1/6$ so that Lemma 3.4 can be applied with $\varsigma = 1 - \alpha > 5/6$. In the finite horizon case, $\alpha < 1/2$ suffices.

2.4 Statement of Results

We assume throughout that T is the billiard map associated to a finite or infinite horizon Lorentz gas as described in Section 2.1.

The first result gives a more concrete description of the above abstract spaces.

Lemma 2.1. *For $\gamma > 2\beta$ and each $n \geq 0$, $\mathcal{C}^\gamma(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$, each of the embeddings is continuous and the first two are injective. Moreover, $\mathcal{L}$ is well defined as an operator on both $\mathcal{B}$ and $\mathcal{B}_w$.*

Proof. The continuity of the embeddings follows from the fact that $\|h\|_{\mathcal{B}} \leq C|h|_{\mathcal{C}^\gamma(M)}$ by (3.24) in the proof of Lemma 3.7, that $|\cdot|_w \leq \|\cdot\|_{\mathcal{B}}$ by definition, and Lemma 3.9 which implies that $|h(\psi)| \leq C|h|_w|\psi|_{\mathcal{C}^p(T^{-n}\mathcal{W}^s)}$ for all $h \in \mathcal{B}_w$ and any $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$.

The injectivity of the first embedding is immediate while that of the second follows from the fact that our test functions for $\|\cdot\|_s$ are in $\mathcal{C}^q(M)$ rather than $\tilde{\mathcal{C}}^q(M)$. Finally, the fact that $\mathcal{L}$ is well defined on $\mathcal{B}$ follows from Lemma 3.8. The proof that $\mathcal{L}$ is well defined on $\mathcal{B}_w$ is similar and is omitted. □

Remark 2.2. *In fact, one could make the embedding $\mathcal{B}_w \hookrightarrow (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$ injective by using test functions ψ in the weak norm satisfying $|\psi|_{\mathcal{C}^p(W)}|W|^a \cos W \leq 1$, with the requirements that $p < a < \alpha$ and $\beta \leq \alpha - a$. We do not do this since we do not need the injectivity of this embedding for any of the results of our paper. Also the modification would complicate the Lasota-Yorke inequalities slightly and would reduce our best estimate on the essential spectral radius to be $\Lambda^{-1/12}$ (see Remark 2.4).*

The following inequalities are proven in Section 4.

Proposition 2.3. *Let $\Lambda > 1$ be the minimum expansion from (2.8) and let $\delta_1 > 0$, $\theta_1 < 1$ be constants defined by (3.3). There exists $C > 0$ such that for all $h \in \mathcal{B}$ and $n \geq 0$,*

$$
\begin{aligned}
|\mathcal{L}^n h|_w &\leq C|h|_w\,, && (2.4)\\
\|\mathcal{L}^n h\|_s &\leq C(\theta_1^{(1-\alpha)n} + \Lambda^{-qn})\|h\|_s + C\delta_1^{-\alpha}|h|_w\,, && (2.5)\\
\|\mathcal{L}^n h\|_u &\leq Cn^\beta\Lambda^{-\beta n}\|h\|_u + CC_1^n\|h\|_s\,, && (2.6)
\end{aligned}
$$

where $C_1 > 0$ is from Lemma 3.4

If we choose $1 > \sigma > \max\{\Lambda^{-\beta}, \theta_1^{1-\alpha}, \Lambda^{-q}\}$, then there exists $N \geq 0$ such that

$$
\begin{aligned}
\|\mathcal{L}^N h\|_{\mathcal{B}} = \|\mathcal{L}^N h\|_s + b\|\mathcal{L}^N h\|_u &\leq \frac{\sigma^N}{2}\|h\|_s + C\delta_1^{-\alpha}|h|_w + b\sigma^N\|h\|_u + bCC_1^N\|h\|_s\\
&\leq \sigma^N\|h\|_{\mathcal{B}} + C_{\delta_1}|h|_w,
\end{aligned}
\tag{2.7}
$$

provided b is chosen small enough with respect to N. The above represents the traditional Lasota-Yorke inequality.

The final ingredient in the strategy to prove the quasi-compactness of the operator $\mathcal{L}$ is the relative compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$. This is proven in Lemma 3.10. It thus follows by standard arguments (see [B1, HH]) that the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is bounded by σ, while the estimate for the spectral radius is one.

Remark 2.4. *Since by (3.3) we choose $\theta_1 \leq \Lambda^{-1/2}$, and given the constraints among β, α and q, our best estimate on the essential spectral radius is $\Lambda^{-1/6}$.*

With these estimates on the spectral radius and essential spectral radius of $\mathcal{L}$, we next prove the spectral decomposition of the transfer operator in Section 5. Those results and the resulting information about the statistical properties of T are summarized in the following theorem. We denote by Π_0 the projection onto the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue 1.

Theorem 2.5. *The peripheral spectrum of $\mathcal{L}$ on $\mathcal{B}$ consists of a simple eigenvalue at 1. The unique (normalized) eigenvector corresponding to 1 is the smooth invariant measure $d\mu = \rho\, dm$, where $\rho = c \cos\varphi$ and c is a normalizing constant. In addition:*

1. *For any probability measure $\nu \in \mathcal{B}$, we have $\lim_{n\to\infty} \|\mathcal{L}^n \nu - \mu\|_{\mathcal{B}} = 0$ and this convergence occurs at an exponential rate given by $\sigma' :=$ the spectral radius of $\mathcal{L} - \Pi_0$ on $\mathcal{B}$, $\sigma' < 1$.*
2. *(T, μ) exhibits exponential decay of correlations for Hölder observables. More precisely, for $\phi \in \mathcal{C}^\gamma(M)$, $\gamma > 2\beta$, and $\psi \in \mathcal{C}^p(\mathcal{W}^s)$, we have*
$$\left| \int_M \phi\psi \circ T^n \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right| \le C(\sigma')^n |\phi|_{\mathcal{C}^\gamma(M)} |\psi|_{\mathcal{C}^p(\mathcal{W}^s)}.$$
3. *More generally, the Fourier transform of the correlation function (sometimes called the* power spectrum*) admits a meromorphic extension in the annulus $\{z \in \mathbb{C}\ ;\ \sigma < |z| < \sigma^{-1}\}$ and the poles (Ruelle resonances) correspond exactly to the eigenvalues of $\mathcal{L}$, where σ is from (2.7).*

Item (1) is proved in Section 5.1 while items (2) and (3) are proved in Section 5.2.

2.4.1 Limits theorems for billiards

Once the spectral picture described above has been established, a variety of limit theorems become immediately accessible, testifying to the concise nature of the present approach. Such limit theorems have been the subject of many recent studies and we refer the interested reader to the following partial list [HH, MN1, CG, RY, G].

We state several limit theorems here and show how they follow from our functional analytic framework in Section 6. Although these limit theorems with respect to invariant measures are known for this class of billiards, their extension to non-invariant probability measures is a new result, with the exception of [D]. Throughout this section, g denotes a real-valued function in $\mathcal{C}^\gamma(M)$, where $\gamma = \max\{p, 2\beta + \varepsilon\}$ for some $\varepsilon > 0$, and $S_n g = \sum_{j=0}^{n-1} g \circ T^j$.

Large deviation estimates. Large deviation estimates provide exponential bounds on the rate of convergence of $\frac{1}{n} S_n g$ to $\mu(g)$. They typically take the form

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log \mu \left(x \in M : \frac{1}{n} S_n g(x) \in [t - \varepsilon, t + \varepsilon] \right) = -I(t),$$

where $I(t) \ge 0$ is called the rate function. More generally, one can ask about the above limit when μ is replaced by a non-invariant measure, for example Lebesgue measure. In the present context, we prove a large deviation estimate for all probability measures in $\mathcal{B}$ with the same rate function I.

Central Limit Theorem. Assume $\mu(g) = 0$ and let $(g \circ T^j)_{j\in\mathbb{N}}$ be a sequence of random variables on the probability space (M, ν), where ν is a (not necessarily invariant) probability measure on the Boreal σ-algebra. We say that the triple (g, T, ν) satisfies a Central Limit Theorem if there exists a constant $\varsigma^2 \ge 0$ such that

$$\frac{S_n g}{\sqrt{n}} \overset{\text{dist.}}{\longrightarrow} \mathcal{N}(0, \varsigma^2),$$

where $\mathcal{N}(0, \varsigma^2)$ denotes the normal distribution with mean 0 and variance ς^2.

Almost-sure Invariance Principle. Assume again that $\mu(g) = 0$ and as above distribute $(g \circ T^j)_{j\in\mathbb{N}}$ according to a probability measure ν. Suppose there exists $\varepsilon > 0$, a probability space Ω with a sequence of random variables $\{X_n\}$ satisfying $S_n g \overset{\text{dist.}}{=} X_n$ and a Brownian motion W with variance $\varsigma^2 \geq 0$ such that

$$X_n = W(n) + \mathcal{O}(n^{1/2-\varepsilon}) \quad \text{as } n \to \infty \text{ almost-surely in } \Omega.$$

Then we say that the process $(g \circ T^j)_{j\in\mathbb{N}}$ satisfies an almost-sure invariance principle.

Theorem 2.6. *Let $\gamma = \max\{p, 2\beta + \varepsilon\}$, for some $\varepsilon > 0$. If $g \in \mathcal{C}^\gamma(M)$, then*

(a) g satisfies a large deviation estimate with uniform rate function I for all (not necessarily invariant) probability measures $\nu \in \mathcal{B}$.

Assume that $\mu(g) = 0$, let $\nu \in \mathcal{B}$ be a probability measure and distribute $(g \circ T^j)_{j\in\mathbb{N}}$ according to ν. Then,

(b) (g, T, ν) satisfies the Central Limit Theorem;

(c) the process $(g \circ T^j)_{j\in\mathbb{N}}$ satisfies an almost-sure invariance principle.

The proof of this theorem appears in Section 6.

2.5 Known Facts about the Lorentz gas

Before exploring the properties of the Banach spaces we have defined, we recall some of the important properties of dispersing billiards that we shall need and refer the reader to [BSC1, BSC2, CM] for details.

2.5.1 Hyperbolicity

Since we have assumed that the scatterers have strictly positive curvature $\mathcal{K}(x)$ at each $x \in M$, there exist constants $\mathcal{K}_{\min}, \mathcal{K}_{\max}, \tau_{\min}$ such that

$$0 < \mathcal{K}_{\min} \leq \mathcal{K}(x) \leq \mathcal{K}_{\max}, \quad \tau_{\min} \leq \tau(x), \quad \forall x \in M.$$

This allows us to define global stable and unstable cones as follows. Let $(dr, d\varphi)$ be an element of the tangent space. Then

$$C^u(x) := \{(dr, d\varphi) \in \mathcal{T}_x M : \mathcal{K}_{\min} \leq \frac{d\varphi}{dr} \leq \mathcal{K}_{\max} + \frac{1}{\tau_{\min}}\} \quad \text{and}$$
$$C^s(x) := \{(dr, d\varphi) \in \mathcal{T}_x M : -\mathcal{K}_{\max} - \frac{1}{\tau_{\min}} \leq \frac{d\varphi}{dr} \leq -\mathcal{K}_{\min}\}.$$

Note that the angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero. The cones also enjoy the following two properties.

(i) *Strict invariance.* $DT_x(C^u(x)) \subset C^u(Tx)$ and $DT_x^{-1}(C^s(x)) \subset C^s(T^{-1}x)$ whenever DT and DT^{-1} exist.

(ii) *Uniform expansion.* Let $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$. There exists $\hat{c} > 0$ such that

$$\|DT_x^n(v)\| \geq \hat{c}\Lambda^n\|v\|, \forall v \in C^u(x), \qquad \text{and} \qquad \|DT_x^{-n}(v)\| \geq \hat{c}\Lambda^n\|v\|, \forall v \in C^s(x), \tag{2.8}$$

where $\|\cdot\|$ is the Euclidian norm. In addition, letting $T^{-1}(r,\varphi) = (r_{-1}, \varphi_{-1})$, the expansion factor for T^{-1} in the stable cone satisfies for any $x = (r, \varphi)$,

$$C^{-1}\frac{\tau(T^{-1}x)}{\cos\varphi_{-1}} \leq \frac{\|DT_x^{-1}v\|}{\|v\|} \leq C\frac{\tau(T^{-1}x)}{\cos\varphi_{-1}}, \quad \forall v \in C^s(x), v \neq 0, \tag{2.9}$$

for some $C > 1$ independent of x.

Note that the expansion may not be larger than 1 at the first iteration. We can always define a norm $\|\cdot\|_*$, uniformly equivalent to $\|\cdot\|$, as an adapted norm on the tangent bundle such that (see [CM, Section 5.10])

$$\|DT_x^n(v)\|_* \geq \Lambda^n\|v\|_*, \forall v \in C^u(x), \qquad \text{and} \qquad \|DT_x^{-n}(v)\|_* \geq \Lambda^n\|v\|_*, \forall v \in C^s(x). \tag{2.10}$$

We say that a smooth curve $W \subset M$ is a *stable curve* if at every point $x \in W$, the tangent line $\mathcal{T}_xW$ belongs to the stable cone $C^s(x)$. We define unstable curves in the same way.

2.5.2 Singularities

The singularity curves of the billiard map T comprise two types of curves: discontinuity curves and the boundaries of homogeneity strips.

We denote by $\mathcal{S}_0 := \{\varphi = \pm\pi/2\}$ the boundary of the collision space, which consists of all grazing collisions. Then the map T lacks smoothness on the set $\mathcal{S}_1 := \mathcal{S}_0 \cup T^{-1}\mathcal{S}_0$. In general, denote

$$\mathcal{S}_{\pm n} = \cup_{i=0}^n T^{\mp i}\mathcal{S}_0.$$

For each $n = 1, 2, 3, \ldots$, the map $T^n : M \setminus \mathcal{S}_n \to M \setminus \mathcal{S}_{-n}$ is a C^2 diffeomorphism on each connected component. The time-reversibility of T implies that $\mathcal{S}_{-n}$ and $\mathcal{S}_n$ are symmetric about $\varphi = 0$ in M. Moreover the set $\mathcal{S}_n \setminus \mathcal{S}_0$ is a union of compact smooth stable curves for $n \geq 1$ and unstable curves for $n \leq -1$. The number of such curves is countable for billiards with infinite horizon and finite otherwise.

Each smooth curve $S \subset \mathcal{S}_n \setminus \mathcal{S}_0$ terminates on a smooth curve in $\mathcal{S}_n$. Furthermore, every curve $S \subset \mathcal{S}_n \setminus \mathcal{S}_0$ is contained in one monotonically decreasing (or increasing for $n < 0$) continuous curve which stretches all the way from $\varphi = -\pi/2$ to $\varphi = \pi/2$. This property is often referred to as *continuation of singularity lines.*

Next we describe briefly $\mathcal{S}_{-1}$ for the infinite horizon case and refer to [BSC1, BSC2] for more details. A point $x \in M$ is called *an infinite-horizon point* if the free path along its forward trajectory is infinite, or there are infinitely many consecutive grazing collisions along the trajectory of x. There are only finitely many infinite-horizon points in M, denoted by $IH := \{x_1, \cdots, x_\ell\}$. By symmetry, it suffices to consider only singular curves in the upper part of M, $\varphi \geq 0$. In the vicinity of any $x_i \in IH$, the set $\mathcal{S}_{-1}$ contains a long increasing curve s' having x_i as an endpoint. In addition $\mathcal{S}_{-1}$ also contains a sequence of short increasing curves $\{s_n\}$, connecting s' and $\mathcal{S}_0$, approaching x_i at the speed of order $\mathcal{O}(1/n)$ along $\mathcal{S}_0$ and of order $\mathcal{O}(1/\sqrt{n})$ along s'. More precisely, for any n large, let D_n be the cell that is bounded by $s_n, s_{n+1}, s', \mathcal{S}_0$. Then $|s_n| = \mathcal{O}(1/\sqrt{n})$, as it is almost parallel to s'. There exists a constant $C > 1$ such that for any $n \geq 1$ and any point $x \in D_n$, we have $C^{-1}n \leq \tau(x) \leq Cn$.

In order to control distortion along stable curves, we define homogeneity strips, $\mathbb{H}_k$, following [BSC1]. We fix $k_0 \in \mathbb{N}$, where k_0 is chosen so that (3.2) holds, and define for $k \geq k_0$,

$$\mathbb{H}_k = \{(r,\varphi) \, : \, \pi/2 - k^{-2} < \varphi < \pi/2 - (k+1)^{-2}\} \quad \text{and}$$
$$\mathbb{H}_{-k} = \{(r,\varphi) \, : \, -\pi/2 + (k+1)^{-2} < \varphi < -\pi/2 + k^{-2}\}.$$

We also put

$$\mathbb{H}_0 = \{(r,\varphi) \, : \, -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\}.$$

Denote by

$$S^H_{\pm k} = \{(r,\varphi) \, : \, |\varphi| = \pm\pi/2 \mp k^{-2}\} \text{ and } \mathcal{S}_{0,H} = \mathcal{S}_0 \cup (\cup_{k \geq k_0}^{\infty} S^H_{\pm k}).$$

In general, we set $\mathcal{S}^{\mathbb{H}}_{\pm n} = \cup_{i=0}^{n} T^{\mp i}\mathcal{S}_{0,H}$ and call this the expanded singularity set for $T^{\pm n}$. We call a stable or unstable curve *homogeneous* if it lies entirely in one of the homogeneity strips $\mathbb{H}_k$.

3 Preliminary Estimates and Properties of the Banach Spaces

3.1 Family of Admissible Stable Curves

Due to our definition of the stable cones $C^s(x)$, each stable curve W can be viewed as the graph of a function $\varphi_W(r)$ of the arc length parameter r. For each stable curve W, let I_W denote the interval on which φ_W is defined and set $G_W(r) = (r, \varphi_W(r))$ to be its graph so that $W = \{G_W(r) : r \in I_W\}$.

We fix constants $B > 0$ and $\delta_0 > 0$, where δ_0 is chosen small enough to satisfy the one-step expansion (3.2). We call a homogeneous stable curve *admissible* if $|W| \leq \delta_0$ and $\left|\frac{d^2\varphi_W}{dr^2}\right| \leq B$. We define $\mathcal{W}^s$ to be the set of admissible stable curves in M. It follows directly from the uniform contraction of $C^s(x)$ under the action of T^{-1} that if $W \in \mathcal{W}^s$, then each (sufficiently short) component of $T^{-1}W$ on which T is smooth is a homogeneous stable curve. It then follows from [CM, Proposition 4.29] that each such smooth component is in $\mathcal{W}^s$ if B is chosen sufficiently large.

We define an analogous family of homogeneous unstable curves $\mathcal{W}^u$ which lie in the unstable cone C^u.

Let $W_1, W_2 \in \mathcal{W}^s$ and identify them with the graphs G_{W_i} of their functions φ_{W_i}, $i = 1, 2$. Let $I_i := I_{W_i}$ be the r-interval on which each curve is defined and denote by $\ell(I_1 \triangle I_2)$ the length of the symmetric difference between I_1 and I_2. Let $\mathbb{H}_{k_i}$ be the homogeneity strip containing W_i. We define the distance between W_1 and W_2 to be,

$$d_{\mathcal{W}^s}(W_1, W_2) = \eta(k_1, k_2) + \ell(I_1 \triangle I_2) + |\varphi_{W_1} - \varphi_{W_2}|_{\mathcal{C}^1(I_1 \cap I_2)},$$

where $\eta(k_1, k_2) = 0$ if $k_1 = k_2$ and $\eta(k_1, k_2) = \infty$ otherwise, i.e., we only compare curves which lie in the same homogeneity strip.

Given two functions $\psi_i \in \mathcal{C}^q(W_i, \mathbb{C})$, we define the distance between ψ_1, ψ_2 as

$$d_q(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathcal{C}^q(I_1 \cap I_2)}.$$

We recall one final fact regarding distortion bounds for stable curves (see [CM, Lemma 5.27]). Suppose that $W \in \mathcal{W}^s$ and that $T^iW \in \mathcal{W}^s$ for $i = 0, 1, \ldots, n$ (i.e., each T^iW is a homogeneous stable curve with uniformly bounded curvature). Then there exists $C_d > 0$, independent of n and W, such that for any $x, y \in W$,

$$|\ln J_W T^n(x) - \ln J_W T^n(y)| \leq C_d d_W(x,y)^{1/3}, \tag{3.1}$$

where $J_W T^n(x) = |\det(DT^n_x | \mathcal{T}_x W)|$ denotes the Jacobian of T^n along W and $d_W(\cdot,\cdot)$ is the arc-length distance on W.

3.2 Growth Lemmas

In order to prove the characterization of our Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ given by Lemma 2.1 as well as the estimates of Proposition 2.3, we need some understanding of the properties of $T^{-n}W$ for $W \in \mathcal{W}^s$. In this section we prove some growth lemmas that we shall need in Section 4.

One Step Expansion. Let W be a homogeneous stable curve. We partition the connected components of $T^{-1}W$ into maximal pieces V_i such that each V_i is a homogeneous stable curve in some $\mathbb{H}_k$. We choose k_0 large enough that the following estimate holds for both classes of billiards we consider (see [CM, Lemma 5.56]):

$$\limsup_{\delta \to 0} \sup_{|W|<\delta} \sum_i \frac{|TV_i|_*}{|V_i|_*} < 1, \tag{3.2}$$

where $|V_i|_*$ is the length of V_i in the metric induced by the adapted norm $\| \cdot \|_*$. Now we choose δ_0 sufficiently small that for any homogeneous stable curve W with $|W| \le \delta_0$, the sum in (3.2) is $\le \theta_*$ for a fixed $\theta_* < 1$. In fact, by choosing δ_0 sufficiently small and k_0 sufficiently large, one can choose θ_* arbitrarily close to Λ^{-1} [CM, eq. (5.39)]. From this point forward, we will consider δ_0 and k_0 to be fixed by these relations. Note that this also fixes the distortion constant C_d from (3.1). Next we choose $\delta_1 < \delta_0/2$ sufficiently small that

$$\theta_1 := \theta_* e^{C_d|\delta_1|^{1/3}} < \Lambda^{-1/2}. \tag{3.3}$$

To ensure that each component of $T^{-1}W$ is in $\mathcal{W}^s$, we subdivide any of the long pieces V_i whose length is $> \delta_0$. This process is then iterated so that given $W \in \mathcal{W}^s$, we construct the components of $T^{-n}W$, which we call the n^{th} generation $\mathcal{G}_n(W)$, inductively as follows. Let $\mathcal{G}_0(W) = \{W\}$ and suppose we have defined $\mathcal{G}_{n-1}(W) \subset \mathcal{W}^s$. First, for any $W' \in \mathcal{G}_{n-1}(W)$, we partition $T^{-1}W'$ into at most countably many pieces W'_i so that T is smooth on each W'_i and each W'_i is a homogeneous stable curve. If any W'_i have length greater than δ_0, we subdivide those pieces into pieces of length between $\delta_0/2$ and δ_0. We define $\mathcal{G}_n(W)$ to be the collection of all pieces $W^n_i \subset T^{-n}W$ obtained in this way. Note that each W^n_i is in $\mathcal{W}^s$ since we chose B sufficiently large in the definition of $\mathcal{W}^s$.

At each iterate of T^{-1}, typical short curves in $\mathcal{G}_n(W)$ grow in size, but there exist a portion of curves which are trapped in tiny Homogeneity strips and in the infinite horizon case, stay too close to the infinite horizon points. Our first lemma shows that the proportion of curves (in a sense made precise below) that never grow to a fixed length in $\mathcal{G}_n(W)$ decays exponentially fast.

For $W \in \mathcal{W}^s$, $n \ge 0$, and $0 \le k \le n$, let $\mathcal{G}_k(W) = \{W^k_i\}_i$ denote the k^{th} generation pieces in $T^{-k}W$. Let $B_k = \{i : |W^k_i| < \delta_1\}$ and $L_k = \{i : |W^k_i| \ge \delta_1\}$ denote the index of the short and long elements of $\mathcal{G}_k(W)$, respectively. We consider $\{\mathcal{G}_k\}_{k=0}^n$ as a tree with W as its root and $\mathcal{G}_k$ as the k^{th} level.

At level n, we group the pieces as follows. Let $W^n_{i_0} \in \mathcal{G}_n(W)$ and let $W^k_j \in L_k$ denote the most recent long "ancestor" of $W^n_{i_0}$, i.e. $k = \max\{0 \le \ell \le n : T^{n-\ell}(W^n_{i_0}) \subset W^\ell_j \text{ and } j \in L_\ell\}$. If no such ancestor exists, set $k = 0$ and $W^k_j = W$. Note that if $W^n_{i_0}$ is long, then $W^k_j = W^n_{i_0}$. Let

$$\mathcal{I}_n(W^k_j) = \{i : W^k_j \in L_k \text{ is the most recent long ancestor of } W^n_i\}.$$

When $k = 0$, the set $\mathcal{I}_n(W)$ represents those curves $W^n_i \in \mathcal{G}_n(W)$ such that $T^\ell W^n_i$ belongs to a short curve in $\mathcal{G}_{n-\ell}(W)$ for each $0 \le \ell \le n-1$.

Lemma 3.1. *Let $W \in \mathcal{W}^s$ and for $n \ge 0$, let $\mathcal{I}_n(W)$ be defined as above. There exists $C > 0$, independent of W, such that for any $n \ge 0$,*

$$\sum_{i \in \mathcal{I}_n(W)} \frac{|T^n W^n_i|}{|W^n_i|} \le C\theta_1^n.$$

Proof. We define a function

$$\mathcal{Z}_n(W) = \sum_{i \in \mathcal{I}_n(W)} \frac{|T^n W_i^n|_*}{|W_i^n|_*}. \tag{3.4}$$

We will show that for any admissible curve W, the function $\mathcal{Z}_n(W)$ decays exponentially as n goes to infinity. Then, since $\|\cdot\|_*$ is uniformly equivalent to $\|\cdot\|$, the lemma follows.

We prove by induction on $n \in \mathbb{N}$ that for any $W \in \mathcal{W}^s$, the following formula holds:

$$\mathcal{Z}_{n+1}(W) \le \theta_1^n \theta_*. \tag{3.5}$$

Note that at each iterate between 1 and n, every piece W_i^n, $i \in \mathcal{I}_n(W)$, is created by genuine cuts due to singularities and homogeneity strips and not by any artificial subdivisions, since those are only made when a piece has grown to length greater than δ_0 and δ_1 was chosen $< \delta_0/2$. Thus we may apply the one-step expansion (3.2) to conclude,

$$\mathcal{Z}_1(W) \le \theta_*. \tag{3.6}$$

Assume that (3.5) is proved for some $n \ge 1$ and all $W \in \mathcal{W}^s$. We apply it to each component $W_i^1 \in \mathcal{G}_1(W)$ such that $i \in \mathcal{I}_1(W)$. Then by assumption, $\mathcal{Z}_n(W_i^1) \le \theta_1^{n-1}\theta_*$, since $W_i^1 \in \mathcal{W}^s$.

We group the components of $W_i^{n+1} \in \mathcal{G}_{n+1}(W)$ with $i \in \mathcal{I}_{n+1}(W)$ according to elements with index in $\mathcal{I}_1(W)$. More precisely, let A_k^n denote those indices of W_i^{n+1} such that $T^n W_i^{n+1} \subset W_k^1$, $k \in \mathcal{I}_1(W)$. It follows from (3.1) that for any W_k^1, the maximum distortion of T is bounded by $e^{C_d |W_k^1|^{\frac{1}{3}}}$. Thus

$$\frac{|T^{n+1} W_i^{n+1}|}{|T^n W_i^{n+1}|} \le e^{C_d |W_k^1|^{\frac{1}{3}}} \frac{|T W_k^1|}{|W_k^1|}.$$

Combining this and (3.6) with the inductive hypothesis, we get

$$\begin{aligned} \mathcal{Z}_{n+1}(W) &= \sum_{k \in \mathcal{I}_1(W)} \sum_{i \in A_k^n} \frac{|T^{n+1} W_i^{n+1}|_*}{|W_i^{n+1}|_*} \le \sum_{k \in \mathcal{I}_1(W)} e^{C_d |W_k^1|^{\frac{1}{3}}} \left(\sum_{i \in A_k^n} \frac{|T^n W_i^{n+1}|_*}{|W_i^{n+1}|_*} \right) \frac{|T W_k^1|_*}{|W_k^1|_*} \\ &= \sum_{k \in \mathcal{I}_1(W)} e^{C_d |W_k^1|^{\frac{1}{3}}} \mathcal{Z}_n(W_k^1) \cdot \frac{|T W_k^1|_*}{|W_k^1|_*} \le \theta_1^{n-1} \theta_* e^{C_d \delta_1^{1/3}} \cdot \mathcal{Z}_1(W) \le \theta_1^n \theta_*. \end{aligned}$$

□

Our next lemma allows us to iterate the control given by the one-step expansion (3.2) over pieces in $\mathcal{G}_n(W)$.

Lemma 3.2. *There exists $C_s > 0$, depending only on θ_1, such that for any $W \in \mathcal{W}^s$ and any $n \ge 0$,*

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|} \le C_s.$$

Proof. Fix $W \in \mathcal{W}^s$ and $n > 0$. For any $1 \le k \le n$, since T^k is smooth on each $W_j^k \in \mathcal{G}_k(W)$, the bounded distortion (3.1) implies that if $T^{n-k} W_i^n \subset W_j^k$, then

$$\frac{|T^n W_i^n|}{|T^{n-k} W_i^n|} \le e^{C_d \delta_0^{1/3}} \frac{|T^k W_j^k|}{|W_j^k|}. \tag{3.7}$$

Now grouping $W_i^n \in \mathcal{G}_n(W)$ by most recent long ancestor as described before the statement of Lemma 3.1 and using (3.7), we have

$$\begin{aligned}
\sum_i \frac{|T^n W_i^n|}{|W_i^n|} &= \sum_{k=0}^{n} \sum_{W_j^k \in \mathcal{G}_k(W)} \sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|T^n W_i^n|}{|W_i^n|} \\
&\le \sum_{k=1}^{n-1} \sum_{W_j^k \in L_k(W)} \left(\sum_{i \in \mathcal{I}_n(W_j^k)} \frac{|T^{n-k} W_i^n|}{|W_i^n|} \right) e^{C_d \delta_0^{1/3}} \frac{|T^k W_j^k|}{|W_j^k|} \; + \sum_{i \in \mathcal{I}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|},
\end{aligned}$$

where we have split off the sum for $k = 0$. Note that $\mathcal{I}_n(W_j^k)$ and $\mathcal{I}_{n-k}(W_j^k)$ correspond to the same set of short pieces in the $(n-k)^{\text{th}}$ generation of W_j^k, so we can apply Lemma 3.1 to each of these sums separately. Thus,

$$\begin{aligned}
\sum_i \frac{|T^n W_i^n|}{|W_i^n|} &\le \sum_{k=1}^{n-1} \sum_{W_j^k \in L_k(W)} C\theta_1^{n-k} e^{C_d \delta_0^{1/3}} \frac{|T^k W_j^k|}{|W_j^k|} \; + \; C\theta_1^n \\
&\le C\delta_1^{-1} \sum_{k=1}^{n-1} \sum_{W_j^k \in L_k(W)} \theta_1^{n-k} |T^k W_j^k| + C\theta_1^n \; \le \; C\delta_1^{-1} |W| \sum_{k=1}^{n-1} \theta_1^{n-k} + C\theta_1^n,
\end{aligned}$$

which is uniformly bounded in n. □

The following lemma is a straight-forward consequence of Lemma 3.2.

Lemma 3.3. *Let $W \in \mathcal{W}^s$ and $0 \le \varsigma \le 1$. Then for any $n \ge 0$,*

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|W_i^n|^\varsigma}{|W|^\varsigma} \cdot \frac{|T^n W_i^n|}{|W_i^n|} \le C_s^{1-\varsigma}.$$

Proof. Multiplying by $|W|/|W|$, we write,

$$\sum_i \frac{|W_i^n|^\varsigma}{|W|^\varsigma} \cdot \frac{|T^n W_i^n|}{|W_i^n|} = \sum_i \frac{|W|^{1-\varsigma}}{|W_i^n|^{1-\varsigma}} \cdot \frac{|T^n W_i^n|}{|W|} \le C_s^{1-\varsigma},$$

by Jensen's inequality since $\sum_i |T^n W_i^n| |W|^{-1} = 1$. □

Our final result of this section concerns an extension of these results when the expansion on each piece is weakened by an exponent < 1.

Lemma 3.4. *Let $\varsigma > 5/6$. There exists a constant $C_1 = C_1(\delta_0, \varsigma) > 0$ such that for any $W \in \mathcal{W}^s$ and $n \ge 0$,*

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|T^n W_i^n|^\varsigma}{|W_i^n|^\varsigma} \le C_1^n.$$

In the case of the finite horizon Lorentz gas, it suffices to take $\varsigma > 1/2$.

Proof. The proof relies on the following version of the one step expansion (3.2) for the exponent ς.

Sublemma 3.5. *Let $\varsigma > 5/6$. Then there exists $C = C(\delta_0, \varsigma) > 0$ such that for any $W \in \mathcal{W}^s$,*

$$\sum_i \frac{|TV_i|^\varsigma}{|V_i|^\varsigma} < C. \tag{3.8}$$

where the V_i's are the maximal homogeneous components of $T^{-1}W$. In the case of the finite horizon Lorentz gas, it suffices to take $\varsigma > 1/2$.

Before proving the sublemma, we use it to prove the following estimate by induction on n:

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|T^n W_i^n|^\varsigma}{|W_i^n|^\varsigma} \le \delta_0^{-n} C^n c^{2n}.$$

where $c = e^{\varsigma C_d}$ and $C = C(\delta_0, \varsigma) > 0$ is from (3.8).

For $n = 1$: Recall that the $W_i^1 \in \mathcal{G}_1(W)$ are obtained by subdividing the maximal homogeneous components V_j of $T^{-1}W$ of length $> \delta_0$. Since T is smooth with bounded distortion on each V_j and the number of W_i^1 in each V_j is at most $1/\delta_0$, we have by Sublemma 3.5,

$$\sum_i \frac{|TW_i^1|^\varsigma}{|W_i^1|^\varsigma} \le \sum_j e^{\varsigma C_d} \delta_0^{-1} \frac{|TV_j|^\varsigma}{|V_j|^\varsigma} \le \delta_0^{-1} C c.$$

Assume at the n-th iteration, for all $W \in \mathcal{W}^s$, we have

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|T^n W_i^n|^\varsigma}{|W_i^n|^\varsigma} \le \delta_0^{-n} C^n c^{2n}. \tag{3.9}$$

We group the elements $W_i^{n+1} \in \mathcal{G}_{n+1}(W)$ according to their ancestors (long or short) $W_k^1 \in \mathcal{G}_1(W)$. More precisely, define $A_k = \{i : T^n W_i^{n+1} \subset W_k^1\}$. Then $\mathcal{G}_{n+1}(W) = \cup_{k \ge 1} \mathcal{G}_n(W_k^1) := \cup_{k \ge 1} \{W_i^{n+1} : i \in A_k\}$. Applying (3.9) to each family $\mathcal{G}_n(W_k^1)$, we obtain

$$\begin{aligned}
\sum_{W_i^{n+1} \in \mathcal{G}_{n+1}(W)} \frac{|T^{n+1} W_i^{n+1}|^\varsigma}{|W_i^{n+1}|^\varsigma} &= \sum_{W_k^1 \in \mathcal{G}_1(W)} \sum_{i \in A_k} \frac{|T^{n+1} W_i^{n+1}|^\varsigma}{|W_i^{n+1}|^\varsigma} \\
&\le \sum_{W_k^1 \in \mathcal{G}_1(W)} \sum_{i \in A_k} e^{\varsigma C_d} \frac{|T^n W_i^{n+1}|^\varsigma}{|W_i^{n+1}|^\varsigma} \cdot \frac{|TW_k^1|^\varsigma}{|W_k^1|^\varsigma} \\
&\le \delta_0^{-n} C^n c^{2n+1} \sum_{W_k^1 \in \mathcal{G}_1(W)} \frac{|TW_k^1|^\varsigma}{|W_k^1|^\varsigma} \le \delta_0^{-(n+1)} C^{n+1} c^{2(n+1)}.
\end{aligned}$$

□

Proof of Sublemma 3.5. We first prove (3.8) in the finite horizon case and then indicate the necessary modifications in the infinite horizon case.

Notice that a stable curve of length $\le \delta_0$ can be cut by at most $N \le \tau_{\max}/\tau_{\min}$ singularity curves in $\mathcal{S}_{-1}$ (see [CM, §5.10]). For each $s \in \mathcal{S}_{-1}$ intersecting W, W is cut further by images of the boundaries of homogeneity strips S_k^H, $k \ge k_0$. For one such s, we relabel the components V_i of $T^{-1}W$ on which T is smooth by V_k, k corresponding to the homogeneity strip $\mathbb{H}_k$ containing V_k. By (2.9), there exists $c_1 > 0$ such that on TV_k, the expansion under T^{-1} is $\ge c_1 k^2$. So for all $\varsigma > 1/2$,

$$\sum_{k \ge k_0} \frac{|TV_k|^\varsigma}{|V_k|^\varsigma} \le c_1^{-\varsigma} \sum_{k \ge k_0} \frac{1}{k^{2\varsigma}} \le \frac{c_1^{-\varsigma}}{k_0^{2\varsigma - 1}}. \tag{3.10}$$

An upper bound for (3.8) in this case is given by N times the bound in (3.10).

It may happen that W does not intersect any $s \in \mathcal{S}_{-1}$, but may intersect one or more preimages of the S_k^H. In this case, the uniform expansion in $\mathbb{H}_0$ combined with the sum in (3.10) provides an upper bound for (3.8).

In the infinite horizon case, in addition to the scenarios above, it may be that a stable curve W intersects a countable number of singularity curves. In the notation of Section 2.5.2, assume that W intersects at least two adjacent singular curves s_n and s_{n+1} in a neighborhood of one of the infinite horizon points and denote the least index of the intersected $s_n \in \mathcal{S}_{-1}$ to be n_1. Since $|W| < \delta_0$ and the distance between the s_n along a stable curve is of order $\mathcal{O}(n^{-2})$,

$$n_1 = \mathcal{O}(|W|^{-1/2}). \tag{3.11}$$

According to [CM, Remark 5.59], there exists $c > 0$, such that for any singular curve s_n belonging to $\mathcal{S}_{-1}$, there is a sequence $\{s_{n,k} \subset TS_k^H : |k| \geq cn^{1/4}\}$ that accumulates on s_n as k goes to ∞ (or $-\infty$). We call the set bounded by $s_{n,k}, s_{n,k+1}, s'$ and $\mathcal{S}_0$, a $D_{n,k}$-cell and note that by (2.9) the expansion along stable curves under T^{-1} is $\mathcal{O}(nk^2)$ in each $D_{n,k}$. Note that $D_{n,k} \subset D_n$ where D_n was defined in Section 2.5.2. We relabel the components of $T^{-1}W$ as $\{W_{n,k}\}$ corresponding to the cell $D_{n,k}$ in which $TW_{n,k}$ lies. Then for any $\varsigma > 5/6$, we have

$$\begin{aligned}\sum_i \frac{|TV_i|^\varsigma}{|V_i|^\varsigma} &\leq \sum_{n \geq n_1} \sum_{|k| \geq cn^{1/4}} \frac{|TW_{n,k}|^\varsigma}{|W_{n,k}|^\varsigma} \\ &\leq \sum_{n \geq n_1} \sum_{k \geq cn^{1/4}} \frac{c_1}{(nk^2)^\varsigma} \leq C n_1^{-\frac{3}{2}\varsigma + \frac{5}{4}} \leq C|W|^{\frac{3}{4}\varsigma - \frac{5}{8}} \leq C|\delta_0|^{\frac{3}{4}\varsigma - \frac{5}{8}},\end{aligned}$$

where we have used the relation (3.11) between n_1 and $|W|$.

This estimate together with the considerations in the finite horizon case proves (3.8) in the infinite horizon case. □

3.3 Properties of the Banach spaces

We begin by verifying that our Banach spaces contain an interesting class of measures. We first record the following simple observation.

Lemma 3.6. *There exists a constant $C_0 > 1$ such that for any homogeneous stable curve W and any $x \in W$,*

$$C_0^{-1} \leq \frac{\cos\varphi(x)}{\cos W} \leq C_0,$$

where $\varphi(x)$ is the angle at x and $\cos W$ is as defined in Section 2.3. Similar bounds hold for $\cos W / \cos W'$ whenever W and W' lie in the same homogeneity strip.

Proof. The proof is straightforward and uses the fact that $\cos(\pi/2 - 1/(k+1)^2) \leq \cos\varphi(x) \leq \cos(\pi/2 - 1/k^2)$ for $x \in \mathbb{H}_k$. □

Our first main lemma shows that $\mathcal{B}$ contains functions with discontinuities that are transverse to the stable cone. The approximation argument rests on the fact that the contribution to the norm of a given function from homogeneity strips with high index is small.

Lemma 3.7. *Let $\mathcal{P}$ be a (mod 0) countable partition of M into open, simply connected sets such that (1) there is a constant $K > 0$ such that for each $P \in \mathcal{P}$, ∂P comprises at most K smooth* *curves, each of which is transverse to $C^s(x)$, with a minimum angle uniform for all $P \in \mathcal{P}$; (2) each strip $\mathbb{H}_k$ intersects at most finitely many $P \in \mathcal{P}$.*

Let $\gamma > 2\beta$. Suppose h is a function on M such that $\sup_{P\in\mathcal{P}} |h|_{\mathcal{C}^\gamma(P)} < \infty$. Then $h \in \mathcal{B}$. In particular, $\mathcal{C}^\gamma(M) \subset \mathcal{B}$ for each $\gamma > 2\beta$ and Lebesgue measure is in $\mathcal{B}$.

Proof. Since $\mathcal{B}$ is defined as the completion of $\mathcal{C}^1(M)$, we must show that h as above can be approximated by functions in $\mathcal{C}^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

For $P \in \mathcal{P}$ we define P_k to be a single simply connected component of $P \cap \mathbb{H}_k$. The labeling may not be unique, but there are only finitely many elements of $\mathcal{P}$ labelled P_k for each $k \geq k_0$ by assumption (2) on $\mathcal{P}$.

Let h be as in the statement of the lemma. Since $\|h\|_{\mathcal{B}} = \sup_k \|h|_{\mathbb{H}_k}\|_{\mathcal{B}}$ by definition of $\mathcal{W}^s$, we may fix k and approximate h one $\mathbb{H}_k$ at a time. We fix P_k and for simplicity first consider $h \equiv 0$ off of P_k.

Choose $\eta > 0$ such that $\tilde{P}_k := B_{\eta/k^3}(P_k)$, the η/k^3 neighborhood of P_k, satisfies $\tilde{P}_k \subset \mathbb{H}_{k-1} \cup \mathbb{H}_k \cup \mathbb{H}_{k+1}$ (for $\mathbb{H}_0$, we use $k = k_0$). Choose a smooth foliation of stable curves on $\tilde{P}_k$ and extend h to the smaller neighborhood $B_{\eta/(2k^3)}(P_k)$ by extending h as a constant function along each stable curve in the foliation. Denote this extended function by $\tilde{h}_k$ and set it equal to 0 elsewhere.

Let $\rho_\eta(x,y)$ be a nonnegative $\mathcal{C}^\infty$ bump function such (1) $\int_{\tilde{P}_k} \rho_\eta(x,y)dm(y) = 1$ for each $x \in \tilde{P}_k$, and (2) $\rho_\eta(x,y) = 0$ whenever $d(x,y) > \eta/(2k^3)$. Define

$$f_\eta(x) = \int_{\tilde{P}_k} \tilde{h}_k(y)\rho_\eta(x,y)\, dm(y), \quad \text{for } x \in M.$$

Note that $f_\eta \in \mathcal{C}^\infty(M)$ and that $f_\eta(x) \equiv 0$ for $x \notin \tilde{P}_k$. We may also arrange it so that $|f_\eta|_{\mathcal{C}^\gamma(P_k)} \leq |h|_{\mathcal{C}^\gamma(P_k)}$, while $|f_\eta|_{\mathcal{C}^\gamma(M)} \leq C|h|_{\mathcal{C}^\gamma(P_k)} k^{3\gamma}/\eta^\gamma$ for some $C > 0$ independent of k and η.

Now let $W \in \mathcal{W}^s$, $W \subset \mathbb{H}_k$, and take $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \leq 1$. Notice that $|\psi|_\infty \leq |W|^{-\alpha}(\cos W)^{-1}$. Thus,

$$\begin{aligned}\int_W (h - f_\eta)\psi\, dm_W &= \int_{W\cap P_k} (h - f_\eta)\psi\, dm_W + \int_{W\setminus P_k} (h - f_\eta)\psi\, dm_W \\ &\leq |h - f_\eta|_{\mathcal{C}^0(W\cap P_k)}|W|^{1-\alpha}(\cos W)^{-1} + |f_\eta|_\infty |(\mathrm{supp} f_\eta) \cap (W \setminus P_k)||W|^{-\alpha}(\cos W)^{-1}.\end{aligned} \tag{3.12}$$

For the first term above, we estimate the difference in functions for $x \in W \cap P_k$ by,

$$|h(x) - f_\eta(x)| \leq \int_{\tilde{P}_k} |h(x) - \tilde{h}_k(y)|\rho_\eta(x,y)\, dm(y)$$

and notice that we only need consider y such that $d(x,y) \leq \eta/(2k^3)$ by definition of ρ_η, i.e. y such that $\tilde{h}_k(y) = h(z)$ for some $z \in P_k$ by definition of $\tilde{h}_k$. Also, since ∂P_k is transverse to $\mathcal{C}^s(x)$ and h was extended along stable curves, we have $d(y,z) \leq C\eta/(2k^3)$. Thus $d(x,z) \leq C\eta/k^3$ and so

$$|h(x) - f_\eta(x)| \leq C|h|_{\mathcal{C}^\gamma(P_k)}\eta^\gamma k^{-3\gamma}.$$

Now consider the expression $|W|^{1-\alpha}(\cos W)^{-1}$. Since W is a homogeneous curve, it lies either in $\mathbb{H}_0$ or in a homogeneity strip indexed by $k \geq k_0$. In the former case, $\cos W \geq 1/k_0^2$ so that the above expression is bounded. In the latter case, $\cos W \geq 1/k^2$ and $|W| \leq Ck^{-3}$ since the stable cone is uniformly transverse to the boundaries of the homogeneity strips. Thus

$$|W|^{1-\alpha}(\cos W)^{-1} \leq Ck^{3(\alpha-1)}k^2 < Ck^{-1/2}, \tag{3.13}$$

since $\alpha < 1/6$. Putting these estimates together, we obtain for the first term of (3.12),

$$|h - f_\eta|_{\mathcal{C}^0(W\cap P_k)}|W|^{1-\alpha}(\cos W)^{-1} \le C|h|_{\mathcal{C}^\gamma(P)}\eta^\gamma k^{-1/2}.$$

For the second term of (3.12), we consider two cases.

Case 1: $|W| < \eta/k^3$. Then $|(\text{supp} f_\eta) \cap (W \setminus P_k)| < |W|$ so that using (3.13),

$$|f_\eta|_\infty |(\text{supp} f_\eta) \cap (W \setminus P_k)||W|^{-\alpha}(\cos W)^{-1} \le C|h|_\infty \eta^{1-\alpha}k^{-1/2}.$$

Case 2: $|W| > \eta/k^3$. Then since $|(\text{supp} f_\eta) \cap (W \setminus P_k)| < \eta/(2k^3)$, we have

$$|f_\eta|_\infty |(\text{supp} f_\eta) \cap (W \setminus P_k)||W|^{-\alpha}(\cos W)^{-1} \le C|h|_\infty \eta(2k^3)^{-1}(\eta/k^3)^{-\alpha}k^2 \le C|h|_\infty \eta^{1-\alpha}k^{-1/2}.$$

Putting together these estimates and taking the suprema over $W \subset \mathbb{H}_k$ and $\psi \in \mathcal{C}^q(W)$, we have by (3.12),

$$\|(h - f_\eta^{P_k})|_{\mathbb{H}_k}\|_s \le C|h|_{\mathcal{C}^\gamma(P)}(\eta^\gamma + \eta^{1-\alpha})k^{-1/2}.$$

Notice that if we were not concerned with approximation, (3.12) and (3.13) would imply,

$$\|h|_{\mathbb{H}_k}\|_s \le C|h|_\infty k^{-1/2} \quad \text{for all bounded functions } h. \tag{3.14}$$

Since f_η is supported on $\mathbb{H}_{k-1} \cup \mathbb{H}_k \cup \mathbb{H}_{k+1}$, we must estimate the norm of $h - f_\eta$ on $\mathbb{H}_{k\pm1}$ as well. Recalling that $h \equiv 0$ on $M \setminus P_k$ and $f_\eta \equiv 0$ on $M \setminus \tilde{P}_k$, for $W \subset \mathbb{H}_{k\pm1}$ and $|\psi|_{W,\alpha,q} \le 1$ we estimate,

$$\int_W (h - f_\eta)\psi \, dm_W \le |f_\eta|_\infty |W \cap \tilde{P}_k||W|^{-\alpha}(\cos W)^{-1} \le C|h|_\infty \eta^{1-\alpha}k^{-1/2},$$

again using (3.13) and cases 1 and 2 above since $|W \cap \tilde{P}_k| \le C\eta/k^3$. Putting this together with our estimate on $\mathbb{H}_k$, we have $\|h - f_\eta\|_s \le C|h|_{\mathcal{C}^\gamma(P)}(\eta^\gamma + \eta^{1-\alpha})k^{-1/2}$.

To estimate $\|(h - f_\eta)|_{\mathbb{H}_k}\|_u$, fix $0 < \varepsilon \le \varepsilon_0$, where ε_0 is from (2.3), and let $W_1, W_2 \subset \mathbb{H}_k$ be two admissible stable curves such that $d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon$. In the notation of Section 3.1, we identify W_i with the graph G_{W_i} of its defining function $\varphi_{W_i}(r)$, $r \in I_i$. Let ψ_1, ψ_2 be two test functions satisfying $|\psi_i|_{W_i,0,p} \le 1$, $i = 1, 2$, and $|\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathcal{C}^q(I_1 \cap I_2)} \le \varepsilon$. Without loss of generality, assume $\gamma = 2\beta + \delta \le 1/2$, for some $\delta > 0$. This is always possible since $\beta < 1/6$ by definition of the norms.

First assume that $\varepsilon \ge \eta^2 k^{-\frac{1}{2\beta}}$. Then by the estimate on the stable norm, we have

$$\varepsilon^{-\beta}\left|\int_{W_1}(h - f_\eta)\psi_1 \, dm_W - \int_{W_2}(h - f_\eta)\psi_2 \, dm_W\right| \le C\varepsilon^{-\beta}|h|_{\mathcal{C}^\gamma}(P)\eta^\gamma k^{-1/2} \le C\eta^\delta |h|_{\mathcal{C}^\gamma(P)}.$$

It remains to estimate the case $\varepsilon < \eta^2 k^{-\frac{1}{2\beta}}$. For this estimate, we split up the terms involving h and f_η,

$$\begin{aligned}\int_{W_1}(h - f_\eta)\psi_1 \, dm_W - \int_{W_2}(h - f_\eta)\psi_2 \, dm_W &= \int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W \\ &\quad + \int_{W_2} f_\eta\psi_2 \, dm_W - \int_{W_1} f_\eta\psi_1 \, dm_W.\end{aligned} \tag{3.15}$$

We first estimate the difference involving h.

We match W_1 and W_2 using a foliation of vertical line segments of length at most ε wherever possible. This partitions W_1 in the following way: curves $U_1^i \subset W_1$ for which the vertical segment connecting U_1^i to W_2 lies entirely in P_k; curves $V_1^j \subset W_1$ which either are not matched to W_2 (near the endpoints of W_1) or for which the vertical segment connecting V_1^i to W_2 does not lie entirely in P_k. This induces a corresponding partition on W_2 into curves U_2^i and V_2^j. We call U_k^i the matched pieces and V_k^j the unmatched pieces and note that by assumption on $\mathcal{P}$, there can be no more than K matched pieces and $K+2$ unmatched pieces.

We split up the integrals on W_1 and W_2 on matched and unmatched pieces,

$$\int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W = \sum_i \int_{U_1^i} h\psi_1 \, dm_W - \int_{U_2^i} h\psi_2 \, dm_W + \sum_{j,k} \int_{V_k^j} h\psi_k \, dm_W. \quad (3.16)$$

We estimate the integrals on the unmatched pieces first. Since $h \equiv 0$ off of P_k and ∂P_k and the vertical lines are both uniformly transverse to the stable cone (see property (1) of $\mathcal{P}$ in the statement of the lemma), we have $|\text{supp}(h) \cap V_k^j| \le C\varepsilon$ for each V_j^k. Then using (3.14), we estimate

$$\int_{V_k^j} h\psi_i \, dm_W \le \|h|_{\mathbb{H}_k}\|_s \, |\text{supp}(h) \cap V_k^j|^\alpha \cos V_k^j |\psi_k|_{\mathcal{C}^q(W_k)} \le C|h|_\infty \varepsilon^\alpha k^{-1/2}, \quad (3.17)$$

where in the last inequality, $|\psi_k|_{\mathcal{C}^q(W_k)} \le (\cos W_k)^{-1}$ and we have used Lemma 3.6 to bound $\cos V_k^j / \cos W_k$.

Next we estimate the difference on matched pieces in (3.16). To do this, we change variables to the r intervals I_i common to U_1^i and U_2^i.

$$\int_{I_i} (h\psi_1) \circ G_{U_1^i} \, JG_{U_1^i} - (h\psi_2) \circ G_{U_2^i} \, JG_{U_2^i} \, dr \le \ell(I_i) |(h\psi_1) \circ G_{U_1^i} \, JG_{U_1^i} - (h\psi_2) \circ G_{U_2^i} \, JG_{U_2^i}|_\infty$$

where $JG_{U_k^i}$ denotes the Jacobian of $G_{U_k^i}$. Notice that

$$JG_{U_k^i}(r) = \sqrt{1 + \Big(\frac{d\varphi_{U_k^i}}{dr}\Big)^2} \le \sqrt{1 + \big(\mathcal{K}_{\max} + \tau_{\min}^{-1}\big)^2} := C_g. \quad (3.18)$$

We split the difference on matched pieces into the sum of three terms. The first term is,

$$A := |h \circ G_{U_1^i} - h \circ G_{U_2^i}|_\infty |\psi_1 \circ G_{U_1^i} JG_{U_1^i}|_\infty \le H^\gamma(h) \sup_{r \in I_i} d(G_{U_1^i}(r), G_{U_2^i}(r))^\gamma \frac{C_g}{\cos W_1},$$

where $H^\gamma(h)$ denotes the Hölder constant of h with exponent γ on P_k. Now $d(G_{U_1^i}(r), G_{U_2^i}(r)) = |\varphi_{U_1^i}(r) - \varphi_{U_2^i}(r)| \le \varepsilon$ by definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$. Thus,

$$A \le C_g H^\gamma(h) \varepsilon^\gamma / \cos W_1. \quad (3.19)$$

The second term of the difference is,

$$B := |\psi_1 \circ G_{U_1^i} - \psi_2 \circ G_{U_2^i}|_\infty |h \circ G_{U_2^i} JG_{U_1^i}|_\infty \le \varepsilon |h|_\infty C_g, \quad (3.20)$$

by assumption on ψ_1 and ψ_2. Finally, the last difference we must estimate is,

$$E := |h \circ G_{U_2^i} \psi_2 \circ G_{U_2^i}|_\infty |JG_{U_1^i} - JG_{U_2^i}|_\infty \le |h|_\infty |\psi_2|_\infty |\varphi'_{U_1^i} - \varphi'_{U_2^i}|_\infty \le |h|_\infty \varepsilon / \cos W_2, \quad (3.21)$$

again by definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$, where $\varphi'_{U_k^i} = \frac{d\varphi_{U_k^i}}{dr}$.

Putting together the estimates for A, B and E, as well as (3.17), into (3.16), we have

$$\varepsilon^{-\beta}\left|\int_{W_1} h\psi_1\,dm_W - \int_{W_2} h\psi_2\,dm_W\right| \le C|h|_{\mathcal{C}^\gamma(P_k)}|W_1|\Big(\frac{\varepsilon^{\gamma-\beta}}{\cos W_1} + \frac{\varepsilon^{1-\beta}}{\cos W_2}\Big) + C|h|_\infty \varepsilon^{\alpha-\beta}k^{-1/2}. \tag{3.22}$$

Notice that the estimate (3.22) holds without the assumption $\varepsilon < \eta^2 k^{-\frac{1}{2\beta}}$ which is what makes (3.24) below possible.

A similar estimate holds for f_η, although now we use the assumption $\varepsilon < \eta^2 k^{-\frac{1}{2\beta}}$. Indeed the estimate is simpler since f_η is Hölder continuous on all of M with $H^\gamma(f_\eta) \le C|h|_{\mathcal{C}^\gamma(P_k)}k^{3\gamma}/\eta^\gamma$. Thus we may partition W_1 and W_2 into one matched piece and at most two unmatched pieces near their endpoints. The unmatched pieces have length at most $C\varepsilon$ so that an estimate similar to (3.17) holds for f_η. Then since f_η is Hölder continuous everywhere, estimates A, B and E hold on the single matched piece and so,

$$\varepsilon^{-\beta}\left|\int_{W_1} f_\eta\psi_1\,dm_W - \int_{W_2} f_\eta\psi_2\,dm_W\right| \le C|W_1|\Big(\frac{H^\gamma(h)\varepsilon^{\gamma-\beta}k^{3\gamma}}{\eta^\gamma\cos W_1} + \frac{|h|_\infty\varepsilon^{1-\beta}}{\cos W_2}\Big) + C|h|_\infty\varepsilon^{\alpha-\beta}. \tag{3.23}$$

Since $|W_1|/\cos W_1$ is bounded by C/k by (3.13) and $\cos W_1/\cos W_2 \le C_0$ by Lemma 3.6 because W_1 and W_2 lie in the same homogeneity strip, it is clear that the only term that can cause a problem is the first one in (3.23). We estimate,

$$\frac{|W_1|}{\cos W_1}\frac{\varepsilon^{\gamma-\beta}k^{3\gamma}}{\eta^\gamma} \le C\frac{1}{k}\frac{\eta^{2(\beta+\delta)}k^{3\gamma}}{\eta^\gamma k^{(\gamma-\beta)/(2\beta)}} \le C\eta^\delta k^{(6\gamma\beta-\beta-\gamma)/(2\beta)}.$$

Notice that the exponent of k is negative since $6\beta\gamma < \gamma < \gamma+\beta$ for any $\gamma > 0$ and $\beta < 1/6$.

We have shown that $\|(h-f_\eta)|_{\mathbb{H}_k}\|_u \le C|h|_{\mathcal{C}^\gamma(P_k)}\eta^{\delta'}$, where $\delta' = \min\{\delta, 2(\alpha-\beta)\}$. Since $h \equiv 0$ outside P_k, we have $\|(h-f_\eta)|_{\mathbb{H}_{k\pm 1}}\|_u = \|f_\eta|_{\mathbb{H}_{k\pm1}}\|_u$ and this expression is similarly bounded by (3.23) since the bound on $H^\gamma(f_\eta)$ used there holds on all of M.

This together with the estimate on the strong stable norm implies that $\|h-f_\eta\|_{\mathcal{B}} \le C|h|_{\mathcal{C}^\gamma(P_k)}\eta^{\delta'}$. Notice that if we are not concerned with approximation, then (3.13), (3.14) and (3.22) together imply that

$$\|h|_{\mathbb{H}_k}\|_{\mathcal{B}} \le C\sup_{P\in\mathcal{P}}|h|_{\mathcal{C}^\gamma(P)}k^{-1/2}. \tag{3.24}$$

In making this approximation argument, we have assumed that $h \equiv 0$ outside P_k. More general h can be expressed as $h = \sum_k\sum_{P_k} h\mathbf{1}_{P_k}$ where $h\mathbf{1}_{P_k} \equiv 0$ outside of P_k and so can be approximated by a $\mathcal{C}^1$ function $f_\eta^{P_k}$ as above. Due to (3.24), given $\epsilon > 0$, we first choose K'_ϵ so that $\|h|_{\mathbb{H}_k}\|_{\mathcal{B}} < \epsilon$ for all $k > K'_\epsilon$. By property (2) of $\mathcal{P}$ in the statement of the lemma, there exists a constant N_ϵ such that each strip $\mathbb{H}_k$ for $k_0 \le k \le K'_\varepsilon$ intersects at most N_ϵ elements $P \in \mathcal{P}$. We thus form the finite sum $\sum_{k_0\le k\le K'_\epsilon}\sum_{P_k} f_\eta^{P_k}$ and approximate h by 0 on $\cup_{k>K'_\epsilon}\mathbb{H}_k$. Note that there are at most N_ε elements P_k for each $k \le K'_\varepsilon$. Thus,

$$\Big\|\big(h - \sum_{k_0\le k\le K'_\epsilon}\sum_{P_k} f_\eta^{P_k}\big)\Big\|_{\mathcal{B}} \le \epsilon + \sup_{k_0\le k\le K'_\epsilon}\Big\|\sum_{P_k}\big(h\mathbf{1}_{P_k} - f_\eta^{P_k}\big)\Big\|_{\mathcal{B}} \le \epsilon + CN_\epsilon\eta^{\delta'}\sup_{P\in\mathcal{P}}|h|_{\mathcal{C}^\gamma(P)},$$

and finally we choose η sufficiently small so that $\eta^{\delta'}N_\varepsilon < \varepsilon$. □

Our next lemma shows that $\mathcal{L}$ is well-defined as an operator from $\mathcal{B}$ to $\mathcal{B}$. Its proof uses the fact that $\|\mathcal{L}h\|_{\mathcal{B}} < \infty$ from Section 4. This is the only point in this section where we use results from Section 4.

Lemma 3.8. *If $h \in \mathcal{C}^1(M)$, then $\mathcal{L}h \in \mathcal{B}$.*

Proof. Let $h \in \mathcal{C}^1(M)$. As in the proof of Lemma 3.7, we must approximate $\mathcal{L}h$ by $\mathcal{C}^1$ functions in the norm $\| \cdot \|_{\mathcal{B}}$. Note that $\mathcal{L}h$ has a countable number of smooth discontinuity curves given by $T(\mathcal{S}_{0,H})$ (we include the images of boundaries of the homogeneity strips). These curves define a countable partition $\mathcal{P}$ of M into open simply connected sets which does not satisfy the assumption (2) of Lemma 3.7 since each $\mathbb{H}_k$ can intersect countably many $P \in \mathcal{P}$. In addition, the $\mathcal{C}^1$ norm of $\mathcal{L}h$ blows up near the curves $T\mathcal{S}_0$.

For $j \geq k_0$ let P^j denote an element of $\mathcal{P}$ such that $T^{-1}P^j \subseteq \mathbb{H}_j$. Again, the labeling is not unique, but for each j, the number of elements in $\mathcal{P}$ which are assigned the label j is finite (even in the infinite horizon case). Let $P^J = \cup_{j>J} P^j$. We claim that $\|\mathcal{L}h|_{P^J}\|_{\mathcal{B}}$ is arbitrarily small for J sufficiently large. On the finite set of P^j with $j \leq J$, the $\mathcal{C}^1$ norm of $\mathcal{L}h$ is finite and the modified partition $\mathcal{P}^* = \{P^j\}_{j \leq J} \cup \{P^J\}$ satisfies the requirements of Lemma 3.7. So we may approximate $\mathcal{L}h$ as in Lemma 3.7 on $M \setminus P^J$ and approximate $\mathcal{L}h$ by 0 on P^J. Thus the lemma follows once we establish our claim.

Indeed, the claim is trivial using the estimates of Section 4. For example, we must estimate $\|\mathcal{L}h|_{P^J}\|_s = \|1_{P^J}\mathcal{L}h\|_s$. Taking $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \leq 1$, we write

$$\int_W 1_{P^J} \mathcal{L}h \, \psi \, dm_W = \int_{T^{-1}(W \cap P^J)} h |DT|^{-1} J_{T^{-1}W} T \, \psi \circ T \, dm_W,$$

and the homogeneous stable components of $T^{-1}(W \cap P^J)$ correspond precisely to the tail of the series considered in (4.2) and following and so can be made arbitrarily small by choosing J large (notice that we do not need contraction here so that we may use the simpler estimate similar to Section 4.1 applied to the strong stable norm rather than the estimate of Section 4.2).

Similarly, in estimating $\|\mathcal{L}h\|_u$, one can see that the contribution from P^J corresponds to the tail of the series from the estimates of Section 4.3, and so this too can be made arbitrarily small by choosing J large. □

The next lemma allows us to establish a connection between our Banach spaces and the space of distributions introduced in Section 2.2. Recall that $H^p_n(\psi) = \sup_{W \in T^{-n}\mathcal{W}^s} H^p_W(\psi)$.

Lemma 3.9. *For each $h \in \mathcal{C}^1(M)$, $n \geq 0$, and $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$ we have*

$$|h(\psi)| = \left| \int_M h\psi \, dm \right| \leq C|h|_w(|\psi|_\infty + H^p_n(\psi)).$$

Proof. On each $M_\ell = \partial\Gamma_\ell \times [-\pi/2, \pi/2]$, we partition the set $\mathbb{H}_0 \cap M_\ell$ into finitely many boxes B_j whose boundary curves are straight line segments in $\mathcal{W}^s$ and $\mathcal{W}^u$ as well as the horizontal lines $\pm\pi/2 \mp 1/k_0^2$. We construct the boxes so that each B_j has diameter $\leq \delta_0$ and is foliated by curves $W \in \mathcal{W}^s$. On each B_j, we choose a smooth foliation $\{W_\xi\}_{\xi \in E_j} \subset \mathcal{W}^s$ of parallel straight line segments, each of whose elements completely crosses B_j in the approximate stable direction (this is always possible if we originally choose the stable and unstable boundaries of B_j to be parallel).

We decompose Lebesgue measure on B_j into $dm = \lambda(d\xi)dm_{W_\xi}$, where m_{W_ξ} is the conditional measure of m on W_ξ and λ is the transverse measure on E_j. We normalize the measures so that $m_\xi(W_\xi) = |W_\xi|$ and note that the conditional measure m_{W_ξ} is the arclength measure on W_ξ since the foliation is comprised of straight line segments. Note also that $\lambda(E_j) \leq C\delta_0$ due to the transversality of curves in $\mathcal{W}^s$ and $\mathcal{W}^u$.

Next we foliate each homogeneity strip $\mathbb{H}_k \cap M_\ell$, $k \geq k_0$, using a smooth family of parallel line segments $\{W_\xi\}_{\xi \in E_k} \subset \mathcal{W}^s$ whose elements all have endpoints lying in the two boundary curves of $\mathbb{H}_k$. We again decompose m on $\mathbb{H}_k$ into $dm = \lambda(d\xi)dm_{W_\xi}$, $\xi \in E_k$, and $m_\xi(W_\xi) = |W_\xi|$ is normalized as above. By construction, $\lambda(E_k) = \mathcal{O}(1)$.

Now given $h \in \mathcal{C}^1(M)$ and $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$, note that since $M = T^{-n}M$ (mod 0), $\int_M h\psi\, dm = \int_M \mathcal{L}^n h\, \psi \circ T^{-n}\, dm$. We estimate the second integral one ℓ at a time,

$$\begin{aligned}\int_{M_\ell} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm &= \sum_j \int_{B_j} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm + \sum_{|k|\geq k_0} \int_{\mathbb{H}_k \cap M_\ell} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm \\ &= \sum_j \int_{E_j} \int_{W_\xi} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm_{W_\xi} d\lambda(\xi) + \sum_{|k|\geq k_0} \int_{E_k} \int_{W_\xi} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm_{W_\xi} d\lambda(\xi).\end{aligned}$$

We change variables and estimate the integrals on one W_ξ at a time. Letting $W^n_{\xi,i}$ denote the components of $\mathcal{G}_n(W_\xi)$ defined in Section 3.2, we define $J_{W^n_{\xi,i}} T^n$ to be the stable Jacobian of T^n along the curve $W^n_{\xi,i}$, and write

$$\begin{aligned}\int_{W_\xi} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm_W &= \sum_i \int_{W^n_{\xi,i}} h\psi |DT^n|^{-1} J_{W^n_{\xi,i}} T^n\, dm_W \\ &\leq \sum_i |h|_w \cos(W^n_{\xi,i}) |\psi|_{\mathcal{C}^p(W^n_{\xi,i})} ||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^p(W^n_{\xi,i})}.\end{aligned}$$

From the distortion bounds (A.1) we have $||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^p(W^n_{\xi,i})} \leq C_d^2 ||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})}$. Since by [CM, (2.29)], the Jacobian $|DT^n|^{-1}(x) = \cos(T^n x)/\cos x$ for $x \in W^n_{\xi,i}$, we have by Lemma 3.6,

$$\cos(W^n_{\xi,i})\, ||DT^n|^{-1}|_{C^0(W^n_{\xi,i})} \leq C_0^2 \cos W_\xi.$$

Also by (3.1), $|J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})} \leq e^{C_d \delta_0^{1/3}} \frac{|T^n W^n_{\xi,i}|}{|W^n_{\xi,i}|}$. Putting these estimates together yields,

$$\int_{W_\xi} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm_W \leq C|h|_w (|\psi|_\infty + H^p_n(\psi)) \cos W_\xi \sum_i \frac{|T^n W^n_{\xi,i}|}{|W^n_{\xi,i}|},$$

where the sum is $\leq C_s$ by Lemma 3.2. Thus

$$\begin{aligned}\left| \int_{M_\ell} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm \right| &\leq C|h|_w (|\psi|_\infty + H^p_n(\psi)) \Big(\sum_j \int_{E_j} \cos W_\xi\, d\lambda(\xi) + \sum_{|k|\geq k_0} \int_{E_k} \cos W_\xi\, d\lambda(\xi) \Big) \\ &\leq C|h|_w (|\psi|_\infty + H^p_n(\psi)) \Big(\sum_j \lambda(E_j) + \sum_{|k|\geq k_0} k^{-2} \lambda(E_k) \Big),\end{aligned}$$

where in the last line we have used the fact that $\cos W \leq Ck^{-2}$ for $W \subset \mathbb{H}_k$. Both sums are finite since there are only finitely many E_j and $\lambda(E_k)$ is of order 1 for each k. Since there are only finitely many M_ℓ, we may sum over ℓ and the lemma is proved. □

We conclude this section by proving the following important fact.

Lemma 3.10. *The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.*

Proof. First notice that on a fixed $W \in \mathcal{W}^s$, $|\cdot|_{W,0,p}$ is equivalent to $|\cdot|_{\mathcal{C}^p(W)}$ and $|\cdot|_{W,\alpha,q}$ is equivalent to $|\cdot|_{\mathcal{C}^q(W)}$ so that $p > q$ implies that the unit ball of $|\cdot|_{W,0,p}$ is compactly embedded in $|\cdot|_{W,\alpha,q}$. Since $\|\cdot\|_s$ is the dual of $|\cdot|_{W,\alpha,q}$ and $|\cdot|_w$ is the dual of $|\cdot|_{W,0,p}$ on each stable curve

$W \in \mathcal{W}^s$, the unit ball of $\|\cdot\|_s$ is compactly embedded in $|\cdot|_w$ on W. It remains to compare the weak norm on different stable curves.

We argue one component $M_\ell = \Gamma_\ell \times [-\pi/2, \pi/2]$ at a time. Let $0 < \varepsilon \le \varepsilon_0$ be fixed. Let $k_\varepsilon \in \mathbb{N}$ be the first integer k such that $1/k^2 < \varepsilon$. We split M_ℓ into two parts, $A = \{-\pi/2 + 1/k_\varepsilon^2 \le \varphi \le \pi/2 - 1/k_\varepsilon^2\}$ and $B = M_\ell \setminus A$. Since curves in $\mathcal{W}^s$ are graphs of functions φ_W whose slopes are greater than $\mathcal{K}_{\min} > 0$ and have uniformly bounded second derivative, there exists $C = C(Q) > 0$ such that any admissible curve $W \subset B$ must have length no longer than $C\epsilon$.

Let $h \in \mathcal{C}^1(M)$ with $\|h\|_{\mathcal{B}} \le 1$. First we estimate the weak norm of h on curves W in B. If $W \subset \mathbb{H}_k$ for $|k| \ge k_\varepsilon$, and $|\psi|_{W,0,p} \le 1$, then

$$\int_W h\psi \, dm_W \le \|h\|_s |W|^\alpha \cos W |\psi|_{\mathcal{C}^q(W)} \le C\|h\|_s \varepsilon^\alpha.$$

Now for $W \subset A$, notice that there exists a constant $K_\varepsilon > 1$ such that $1/\cos W \le K_\varepsilon$. On a fixed interval I, the set of functions $\{\varphi_W\}_{W \in \mathcal{W}^s}$ defined on I and lying in one homogeneity strip is compact in the $\mathcal{C}^1$-norm. Since A contains only finitely many homogeneity strips, we may choose finitely many stable curves $W_i \in \mathcal{W}^s$ such that $\{W_i\}_{i=1}^{N_\varepsilon}$ forms an ε-covering of $\mathcal{W}^s|_A$ in the distance $d_{\mathcal{W}^s}$.

Let $|\Gamma_\ell|$ denote the arclength of Γ_ℓ and define $\mathbb{S}^1_\ell$ to be the circle of length $|\Gamma_\ell|$. Since any ball of finite radius in the $\mathcal{C}^p$-norm is compactly embedded in $\mathcal{C}^q$, we may choose finitely many functions $\overline{\psi}_j \in \mathcal{C}^p(\mathbb{S}^1_\ell)$ such that $\{\overline{\psi}_j\}_{j=1}^{L_\varepsilon}$ forms an ε-covering in the $\mathcal{C}^q(\mathbb{S}^1_\ell)$-norm of the ball of radius $C_g K_\varepsilon$ in $\mathcal{C}^p(\mathbb{S}^1_\ell)$, where C_g is from (3.18).

Now let $W = G_W(I_W) \in \mathcal{W}^s|_A$, and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,0,p} \le 1$. We view I_W as a subset of $\mathbb{S}^1_\ell$. Let $\overline{\psi} = \psi \circ G_W$ be the push down of ψ to I_W. Note that $|\overline{\psi}|_{\mathcal{C}^p(I_W)} \le C_g/\cos W \le C_g K_\varepsilon$.

Choose $W_i = G_{W_i}(I_{W_i})$ such that $d_{\mathcal{W}^s}(W, W_i) \le \varepsilon$ and choose $\overline{\psi}_j \in \mathcal{C}^p(\mathbb{S}^1_L)$ such that $|\overline{\psi} - \overline{\psi}_j|_{\mathcal{C}^q(I_W)} \le \varepsilon$. Define $\psi_j = \overline{\psi}_j \circ G_{W_i}^{-1}$ to be the lift of $\overline{\psi}_j$ to W_i. Note that $|\psi_j|_{W_i,0,p} \le \cos W_i (2C_g/\cos W) \le 2C_g C_0$ by Lemma 3.6 since W_i and W lie in the same homogeneity strip. Then normalizing ψ and ψ_j by $2C_0C_g$, we estimate

$$\left| \int_W h\psi \, dm_W - \int_{W_i} h\psi_j \, dm_W \right| \;\le\; \varepsilon^\beta \|h\|_u 2C_0C_g.$$

We have proved that for each $0 < \varepsilon \le \varepsilon_0$, there exist finitely many bounded linear functionals $\ell_{i,j}$, $\ell_{i,j}(h) = \int_{W_i} h\psi_j dm_W$, such that

$$|h|_w \le \max_{i \le N_\varepsilon, j \le L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C\|h\|_u + \varepsilon^\alpha C\|h\|_s \le \max_{i \le N_\varepsilon, j \le L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C b^{-1}\|h\|_{\mathcal{B}},$$

which implies the required compactness. $\square$

4 Lasota-Yorke Estimates

It suffices to prove Proposition 2.3 for $h \in \mathcal{C}^1(M)$ since then by density of $\mathcal{C}^1(M)$ in $\mathcal{B}$, $\mathcal{L}$ is continuous on $\mathcal{B}$. To see this, assume Proposition 2.3 has been proved for $h \in \mathcal{C}^1(M)$ and identify $h \in \mathcal{B}$ with a Cauchy sequence $\{g_n\}_{n \ge 0} \subset \mathcal{C}^1(M)$. Since $\mathcal{L}$ is bounded when applied to functions in $\mathcal{C}^1(M)$, by the assumption that Proposition 2.3 holds for $\mathcal{C}^1$ functions, it follows that $\{\mathcal{L}g_n\}$ is a Cauchy sequence in $\mathcal{B}$. By Lemma 3.9, we identify its limit with $\mathcal{L}h$ and so $\|\mathcal{L}h\|_{\mathcal{B}} = \lim_n \|\mathcal{L}g_n\|_{\mathcal{B}} \le \lim_n C\|g_n\|_{\mathcal{B}} = C\|h\|_{\mathcal{B}}$. Thus $\mathcal{L}$ is bounded and therefore continuous on $\mathcal{B}$. A similar argument holds for $\mathcal{B}_w$.

We use the distortion bounds of Appendix A throughout this section.

4.1 Estimating the Weak Norm

Let $h \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ such that $|\psi|_{W,0,p} \le 1$. For $n \ge 0$, we write,

$$\int_W \mathcal{L}^n h \, \psi \, dm_W = \sum_{W_i^n \in \mathcal{G}_n(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} \psi \circ T^n dm_W, \tag{4.1}$$

where $J_{W_i^n} T^n$ denotes the Jacobian of T^n along W_i^n.

Using the definition of the weak norm on each W_i^n, we estimate (4.1) by

$$\int_W \mathcal{L}^n h \, \psi \, dm_W \ \le \ \sum_{W_i^n \in \mathcal{G}_n} |h|_w ||DT^n|^{-1} J_{W_i^n} T^n|_{\mathcal{C}^p(W_i^n)} |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} \cos W_i^n. \tag{4.2}$$

The disortion bounds given by equation (A.1) imply that

$$||DT^n|^{-1} J_{W_i^n} T^n|_{\mathcal{C}^p(W_i^n)} \le C_d^2 ||DT^n|^{-1} J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)}.$$

For $x, y \in W_i^n$, we record for future use,

$$\frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(T^n x, T^n y)^p} \cdot \frac{d_W(T^n x, T^n y)^p}{d_W(x,y)^p} \le C|\psi|_{\mathcal{C}^p(W)} |J_{W_i^n} T^n|^p_{\mathcal{C}^0(W_i^n)}, \le C\Lambda^{-pn} |\psi|_{\mathcal{C}^p(W)} \tag{4.3}$$

by (2.8) so that $|\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} \le C|\psi|_{\mathcal{C}^p(W)} \le C/\cos W$. Using these estimates in equation (4.2), we obtain

$$\int_W \mathcal{L}^n h \, \psi \, dm_W \ \le \ C|h|_w \sum_{W_i^n \in \mathcal{G}_n} \frac{\cos W_i^n}{\cos W} ||DT^n|^{-1} J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)}.$$

Since $|DT^n(x)| = \cos\varphi(x)/\cos\varphi(T^n x)$ for $x \in W_i^n$, by Lemma 3.6, we have

$$||DT^n|^{-1}|_{C^0(W_i^n)} \frac{\cos W_i^n}{\cos W} \le C_0^2. \tag{4.4}$$

Notice also that by the bounded distortion estimate (3.1), $|J_{W_i^n} T^n|_{C_0(W_i^n)} \le e^{C_d \delta_0^{1/3}} |T^n W_i^n| |W_i^n|^{-1}$. Gathering these estimates together, we obtain

$$\int_W \mathcal{L}^n h \, \psi \, dm_W \le C|h|_w \sum_{W_i^n \in \mathcal{G}_n} \frac{|T^n W_i^n|}{|W_i^n|} \le C C_s |h|_w,$$

where in the last inequality we have used Lemma 3.2. Taking the supremum over all $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,0,p} \le 1$ yields (2.4).

4.2 Estimating the Strong Stable Norm

Let $W \in \mathcal{W}^s$ and let W_i^n denote the elements of $\mathcal{G}_n(W)$ as defined in Section 3.2. For $\psi \in \mathcal{C}^q(W)$, $|\psi|_{W,\alpha,q} \le 1$, define $\overline{\psi}_i = |W_i^n|^{-1} \int_{W_i^n} \psi \circ T^n \, dm_W$. Using equation (4.1), we write

$$\int_W \mathcal{L}^n h \, \psi \, dm_W = \sum_i \int_{W_i^n} h \, \frac{J_{W_i^n} T^n}{|DT^n|} \, (\psi \circ T^n - \overline{\psi}_i) \, dm_W + \overline{\psi}_i \int_{W_i^n} h \, \frac{J_{W_i^n} T^n}{|DT^n|} \, dm_W. \tag{4.5}$$

To estimate the first term of (4.5), we first estimate $|\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W_i^n)}$. If $H_W^q(\psi)$ denotes the Hölder constant of ψ along W, then equation (4.3) implies

$$\frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(x,y)^q} \le C\Lambda^{-nq} H_W^q(\psi), \tag{4.6}$$

for any $x, y \in W_i^n$. Since $\overline{\psi}_i$ is constant on W_i^n, we have $H^q_{W_i^n}(\psi \circ T^n - \overline{\psi}_i) \le C\Lambda^{-qn} H^q_W(\psi)$. To estimate the $\mathcal{C}^0$ norm, note that $\overline{\psi}_i = \psi \circ T^n(y_i)$ for some $y_i \in W_i^n$. Thus for each $x \in W_i^n$,

$$|\psi \circ T^n(x) - \overline{\psi}_i| = |\psi \circ T^n(x) - \psi \circ T^n(y_i)| \le H^q_{W_i^n}(\psi \circ T^n)|W_i^n|^q \le C H^q_W(\psi)\Lambda^{-nq}.$$

This estimate together with (4.6) and the fact that $|\varphi|_{W,\alpha,q} \le 1$, implies

$$|\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W_i^n)} \le C\Lambda^{-nq}|\psi|_{\mathcal{C}^q(W)} \le C\Lambda^{-qn}|W|^{-\alpha}(\cos W)^{-1}. \tag{4.7}$$

We apply (4.7), the distortion estimate (A.1) and the definition of the strong stable norm to the first term on the right hand side of (4.5),

$$\begin{aligned}\sum_i \int_{W_i^n} h\frac{J_{W_i^n}T^n}{|DT^n|}(\psi \circ T^n - \overline{\psi}_i)\, dm_W &\le C\sum_i \|h\|_s \frac{|W_i^n|^\alpha}{|W|^\alpha}\frac{\cos W_i^n}{\cos W}\left|\frac{J_{W_i^n}T^n}{|DT^n|}\right|_{\mathcal{C}^0(W_i^n)}\Lambda^{-qn} \\ &\le\ Ce^{C_d\delta_0^{1/3}}C_0^2\Lambda^{-qn}\|h\|_s\sum_i \frac{|W_i^n|^\alpha}{|W|^\alpha}\frac{|T^nW_i^n|}{|W_i^n|}\ \le\ C'\Lambda^{-qn}\|h\|_s,\end{aligned} \tag{4.8}$$

where in the second line we have used (4.4) and Lemma 3.3 with $\varsigma = \alpha$.

For the second term on the right hand side of (4.5), we use the fact that $|\overline{\psi}_i| \le |W|^{-\alpha}(\cos W)^{-1}$ since $|\psi|_{W,\alpha,q} \le 1$. Recall the notation introduced before the statement of Lemma 3.1. Grouping the pieces $W_i^n \in \mathcal{G}_n(W)$ according to most recent long ancestors, we have

$$\begin{aligned}\sum_i \frac{1}{|W|^\alpha\cos W}\int_{W_i^n} h\frac{J_{W_i^n}T^n}{|DT^n|}\, dm_W =& \sum_{k=1}^n\sum_{j\in L_k}\sum_{i\in\mathcal{I}_n(W_j^k)}\frac{1}{|W|^\alpha\cos W}\int_{W_i^n} h\frac{J_{W_i^n}T^n}{|DT^n|}\, dm_W \\ &+ \sum_{i\in\mathcal{I}_n(W)}\frac{1}{|W|^\alpha\cos W}\int_{W_i^n} h\frac{J_{W_i^n}T^n}{|DT^n|}\, dm_W,\end{aligned}$$

where we have split up the terms involving $k = 0$ and $k \ge 1$. We estimate the terms with $k \ge 1$ by the weak norm and the terms with $k = 0$ by the strong stable norm,

$$\begin{aligned}\sum_i \frac{1}{|W|^\alpha\cos W}\int_{W_i^n} h\frac{J_{W_i^n}T^n}{|DT^n|}\, dm_W \le& C\sum_{k=1}^n\sum_{j\in L_k}\sum_{i\in\mathcal{I}_n(W_j^k)}\frac{\cos W_i^n}{|W|^\alpha\cos W}|h|_w\left|\frac{J_{W_i^n}T^n}{|DT^n|}\right|_{\mathcal{C}^0(W_i^n)} \\ &+ C\sum_{i\in\mathcal{I}_n(W)}\frac{|W_i^n|^\alpha\cos W_i^n}{|W|^\alpha\cos W}\|h\|_s||DT^n|^{-1}J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)}.\end{aligned}$$

As usual, by (4.4), the ratio of cosines times $|DT^n|^{-1}$ is uniformly bounded.

In the first sum above corresponding to $k \ge 1$, we write

$$|J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)} \le |J_{W_i^n}T^{n-k}|_{\mathcal{C}^0(W_i^n)}|J_{W_j^k}T^k|_{\mathcal{C}^0(W_j^k)}.$$

Thus using Lemma 3.1 from time k to time n,

$$\begin{aligned}\sum_{k=1}^n\sum_{j\in L_k}\sum_{i\in\mathcal{I}_n(W_j^k)}|W|^{-\alpha}|J_{W_i^n}T^n|_{\mathcal{C}^0(W_i^n)} &\le e^{C_d}\sum_{k=1}^n\sum_{j\in L_k}|J_{W_j^k}T^k|_{\mathcal{C}^0(W_j^k)}|W|^{-\alpha}\sum_{i\in\mathcal{I}_n(W_j^k)}\frac{|T^{n-k}W_i^n|}{|W_i^n|} \\ &\le C/\delta_1^\alpha\sum_{k=1}^n\sum_{j\in L_k}\frac{|T^kW_j^k|}{|W_j^k|}\frac{|W_j^k|^\alpha}{|W|^\alpha}\theta_1^{n-k},\end{aligned}$$

since $|W_j^k| \geq \delta_1$. The last two sums are bounded independently of n and W by Lemma 3.3 with $\varsigma = \alpha$.

Finally, for the sum corresponding to $k = 0$, we write

$$\sum_{i \in \mathcal{I}_n(W)} \frac{|W_i^n|^\alpha}{|W|^\alpha} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \leq C \left(\sum_{i \in \mathcal{I}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|} \right)^{1-\alpha} \leq C\theta_1^{n(1-\alpha)},$$

using Lemma 3.1 and Jensen's inequality as in the proof of Lemma 3.3.

Gathering these estimates together, we have

$$\sum_i \frac{1}{|W|^\alpha \cos W} \left| \int_{W_i^n} h |DT^n|^{-1} J_{W_i^n} T^n \, dm_W \right| \;\leq\; C\delta_1^{-\alpha} |h|_w + C\|h\|_s \theta_1^{n(1-\alpha)}. \tag{4.9}$$

Putting together (4.8) and (4.9) proves (2.5),

$$\|\mathcal{L}^n h\|_s \leq C \left(\Lambda^{-qn} + \theta_1^{n(1-\alpha)} \right) \|h\|_s + C\delta_1^{-\alpha} |h|_w.$$

4.3 Estimating the Strong Unstable Norm

Fix $\varepsilon \leq \varepsilon_0$ and consider two curves $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \leq \varepsilon$. For $n \geq 1$, we describe how to partition $T^{-n} W^\ell$ into "matched" pieces U_j^ℓ and "unmatched" pieces V_i^ℓ, $\ell = 1, 2$.

Let ω be a connected component of $W^1 \setminus \mathcal{S}_{-n}^{\mathbb{H}}$. To each point $x \in T^{-n}\omega$, we associate a vertical line segment γ_x of length at most $C\Lambda^{-n}\varepsilon$ such that its image $T^n \gamma_x$, if not cut by a singularity or the boundary of a homogeneity strip, will have length $C\varepsilon$. By [CM, §4.4], all the tangent vectors to $T^i \gamma_x$ lie in the unstable cone $C^u(T^i x)$ for each $i \geq 1$ so that they remain uniformly transverse to the stable cone and enjoy the minimum expansion given by (2.8).

Doing this for each connected component of $W^1 \setminus \mathcal{S}_{-n}^{\mathbb{H}}$, we subdivide $W^1 \setminus \mathcal{S}_{-n}^{\mathbb{H}}$ into a countable collection of subintervals of points for which $T^n \gamma_x$ intersects $W^2 \setminus \mathcal{S}_{-n}^{\mathbb{H}}$ and subintervals for which this is not the case. This in turn induces a corresponding partition on $W^2 \setminus \mathcal{S}_{-n}^{\mathbb{H}}$.

We denote by V_i^ℓ the pieces in $T^{-n} W^\ell$ which are not matched up by this process and note that the images $T^n V_i^\ell$ occur either at the endpoints of W^ℓ or because the vertical segment γ_x has been cut by a singularity. In both cases, the length of the curves $T^n V_i^\ell$ can be at most $C\varepsilon$ due to the uniform transversality of $\mathcal{S}_{-n}^{\mathbb{H}}$ with the stable cone and of $C^s(x)$ with $C^u(x)$.

In the remaining pieces the foliation $\{T^n \gamma_x\}_{x \in T^{-n} W^1}$ provides a one to one correspondence between points in W^1 and W^2. We further subdivide these pieces in such a way that the lengths of their images under T^{-i} are less than δ_0 for each $0 \leq i \leq n$ and the pieces are pairwise matched by the foliation $\{\gamma_x\}$. We call these matched pieces U_j^ℓ. Possibly changing the constant $\delta_0/2$ to δ_0/C for some uniform constant $C > 1$ (depending only on the distortion constant and the angle between stable and unstable cones) in the definition of $\mathcal{G}_n(W^\ell)$, we may arrange it so that $U_j^\ell \subset W_i^{\ell,n}$ for some $W_i^{\ell,n} \in \mathcal{G}_n(W^\ell)$ and $V_k^\ell \subset W_{i'}^{\ell,n}$ for some $W_{i'}^{\ell,n} \in \mathcal{G}_n(W^\ell)$ for all $j, k \geq 1$ and $\ell = 1, 2$. There is at most one U_j^ℓ and two V_j^ℓ per $W_i^{\ell,n} \in \mathcal{G}_n(W^\ell)$.

In this way we write $W^\ell = (\cup_j T^n U_j^\ell) \cup (\cup_i T^n V_i^\ell)$. Note that the images $T^n V_i^\ell$ of the unmatched pieces must be short while the images of the matched pieces U_j^ℓ may be long or short.

Recalling the notation of Section 3.1, we have arranged a pairing of the pieces U_j^ℓ with the following property:

$$\begin{aligned} &\text{If } \; U_j^1 = G_{U_j^1}(I_j) = \{(r, \varphi_{U_j^1}(r)) : r \in I_j\}, \\ &\text{then } \; U_j^2 = G_{U_j^2}(I_j) = \{(r, \varphi_{U_j^2}(r)) : r \in I_j\}, \end{aligned} \tag{4.10}$$

so that the point $x = (r, \varphi_{U_j^1}(r))$ is associated with the point $\bar{x} = (r, \varphi_{U_j^2}(r))$ by the vertical segment $\gamma_x \subset \{(r,s)\}_{s\in[-\pi/2,\pi/2]}$, for each $r \in I_j$.

Remark 4.1. *The fact that we have matched stable curves using vertical line segments is not essential to our argument: we could have matched them using any smooth foliation of curves in the unstable cones. However, a remarkable feature of the present approach is that we do not match stable curves along real unstable manifolds, as is commonly done in coupling arguments, and thus we avoid the technical difficulties associated with the corresponding holonomy map.*

Given ψ_ℓ on W^ℓ with $|\psi_\ell|_{W^\ell,0,p} \le 1$ and $d_q(\psi_1,\psi_2) \le \varepsilon$, with the above construction we must estimate

$$\begin{aligned}\left|\int_{W^1} \mathcal{L}^n h\, \psi_1\, dm_W - \int_{W^2} \mathcal{L}^n h\, \psi_2\, dm_W\right| \;&\le\; \sum_{\ell,i} \left|\int_{V_i^\ell} h|DT^n|^{-1} J_{V_i^\ell}T^n \psi_\ell \circ T^n\, dm_W\right| \\ + \sum_j \left|\int_{U_j^1} h|DT^n|^{-1} J_{U_j^1}T^n \psi_1 \circ T^n\, dm_W - \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2}T^n \psi_2 \circ T^n\, dm_W\right|&.\end{aligned} \tag{4.11}$$

We do the estimate over the unmatched pieces V_i^ℓ first using the strong stable norm. Note that by (4.3), $|\psi_\ell \circ T^n|_{\mathcal{C}^q(T^{-n}V_i^\ell)} \le C|\psi_\ell|_{\mathcal{C}^p(W^\ell)} \le C(\cos W^\ell)^{-1}$. We estimate as in Section 4.2, using the fact that $|T^n V_i^\ell| \le C\varepsilon$,

$$\begin{aligned}\sum_{\ell,i} \left|\int_{V_i^\ell} h|DT^n|^{-1} J_{V_i^\ell}T^n \psi_\ell \circ T^n\, dm_W\right| &\le C \sum_{\ell,i} \|h\|_s |V_i^\ell|^\alpha \big||DT^n|^{-1} J_{V_i^\ell}T^n\big|_{\mathcal{C}^q} \frac{\cos V_i^\ell}{\cos W^\ell} \\ \le C\|h\|_s \sum_{\ell,i} |V_i^\ell|^\alpha \frac{|T^n V_i^\ell|}{|V_i^\ell|} &\le C\varepsilon^\alpha \|h\|_s \sum_{\ell,i} \frac{|T^n V_i^\ell|^{1-\alpha}}{|V_i^\ell|^{1-\alpha}} \le C\varepsilon^\alpha \|h\|_s C_1^n,\end{aligned} \tag{4.12}$$

where we have applied Lemma 3.4 with $\varsigma = 1-\alpha > 5/6$ since there are at most two V_i^ℓ corresponding to each element $W_i^{\ell,n} \in \mathcal{G}_n^\ell(W)$ as defined in Section 3.2 and by bounded distortion, $\frac{|T^n V_i^\ell|}{|V_i^\ell|} \le e^{C_d} \frac{|T^n W_i^{\ell,n}|}{|W_i^{\ell,n}|}$.

Next, we must estimate

$$\sum_j \left|\int_{U_j^1} h|DT^n|^{-1} J_{U_j^1}T^n\, \psi_1 \circ T^n\, dm_W - \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2}T^n\, \psi_2 \circ T^n\, dm_W\right|.$$

Recalling the notation defined by (4.10), we fix j and estimate the difference. Define

$$\phi_j = (|DT^n|^{-1} J_{U_j^1}T^n\, \psi_1 \circ T^n) \circ G_{U_j^1} \circ G_{U_j^2}^{-1}.$$

The function ϕ_j is well-defined on U_j^2 and we can estimate,

$$\begin{aligned}&\left|\int_{U_j^1} h|DT^n|^{-1} J_{U_j^1}T^n\, \psi_1 \circ T^n - \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2}T^n\, \psi_2 \circ T^n\right| \\ &\le \left|\int_{U_j^1} h|DT^n|^{-1} J_{U_j^1}T^n\, \psi_1 \circ T^n - \int_{U_j^2} h\, \phi_j\right| + \left|\int_{U_j^2} h(\phi_j - |DT^n|^{-1} J_{U_j^2}T^n\, \psi_2 \circ T^n)\right|.\end{aligned} \tag{4.13}$$

We estimate the first term in equation (4.13) using the strong unstable norm. The distortion bounds given by (A.1) and the estimates of (4.3) and (4.4) imply that

$$\begin{aligned} | \, |DT^n|^{-1} J_{U_j^1} T^n \cdot \psi_1 \circ T^n|_{U_j^1, 0, p} &\le \cos(U_j^1) | \, |DT^n|^{-1} J_{U_j^1} T^n \cdot \psi_1 \circ T^n|_{\mathcal{C}^p(U_j^1)} \\ &\le C \frac{\cos(U_j^1)}{\cos W^1} | \, |DT^n|^{-1} J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \le C |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)}. \end{aligned} \tag{4.14}$$

Similarly, since $|G_{U_j^1} \circ G_{U_j^2}^{-1}|_{\mathcal{C}^1} \le C_g$, where C_g is from (3.18),

$$\cos(U_j^2) |\phi_j|_{\mathcal{C}^p(U_j^2)} \le C \frac{\cos(U_j^2)}{\cos W^2} | \, |DT^n|^{-1} J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \le C |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)},$$

where $\frac{\cos(U_j^2)}{\cos W^2} \le C_0^2 \frac{\cos(U_j^1)}{\cos W^1}$ by Lemma 3.6 since the corresponding curves lie in the same homogeneity strips. By the definition of ϕ_j and $d_q(\cdot, \cdot)$,

$$d_q(|DT^n|^{-1} J_{U_j^1} T^n \psi_1 \circ T^n, \phi_j) = \left| \left[|DT^n|^{-1} J_{U_j^1} T^n \psi_1 \circ T^n \right] \circ G_{U_j^1} - \phi_j \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} = 0.$$

To complete the estimate on the first term of (4.13), we need the following lemma.

Lemma 4.2. *There exists $C > 0$, independent of W_1 and W_2, such that for each j,*

$$d_{\mathcal{W}^s}(U_j^1, U_j^2) \le C \Lambda^{-n} n \varepsilon =: \varepsilon_1.$$

We postpone the proof of the lemma until Section 4.3.1 and use it to complete the estimate of the first term of (4.13).

In view of (4.14), we renormalize the test functions by $R_j = C |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)}$. Then we apply the definition of the strong unstable norm with ε_1 in place of ε. Thus,

$$\begin{aligned} &\sum_j \left| \int_{U_j^1} h |DT^n|^{-1} J_{U_j^1} T^n \, \psi_1 \circ T^n - \int_{U_j^2} h \, \phi_j \right| \\ &\le C \varepsilon_1^\beta \|h\|_u \sum_j |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \le C \|h\|_u \Lambda^{-n\beta} n^\beta \varepsilon^\beta \sum_j \frac{|T^n U_j^1|}{|U_j^1|}, \end{aligned} \tag{4.15}$$

where the sum is $\le C_s$ by Lemma 3.2 since there is at most one matched piece U_j^1 corresponding to each component of $T^{-n} W^1$, $W_i^{1,n} \in \mathcal{G}_n(W^1)$.

It remains to estimate the second term in (4.13) using the strong stable norm. We need the following lemma.

Lemma 4.3. *There exists $C > 0$ such that for each $j \ge 1$,*

$$|(|DT^n|^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} - (|DT^n|^{-1} J_{U_j^2} T^n) \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le C | |DT^n|^{-1} J_{U_j^2} T^n|_{C^0(U_j^2)} \varepsilon^{1/3 - q}.$$

Proof. Throughout the proof, for ease of notation we write J_ℓ^n for $|DT^n|^{-1} J_{U_j^\ell} T^n$.

For any $r \in I_j$, $x = G_{U_j^1}(r)$ and $\bar{x} = G_{U_j^2}(r)$ lie on a common vertical segment γ_x. Thus $T^n(x)$ and $T^n(\bar{x})$ also lie on the element $T^n \gamma_x \in \mathcal{W}^u$ which intersects W^1 and W^2 and has length at most $C\varepsilon$. By (A.3) and (A.4),

$$|J_1^n(x) - J_2^n(\bar{x})| \le C |J_2^n|_{\mathcal{C}^0(U_j^2)} (d(T^n x, T^n \bar{x})^{1/3} + \theta(T^n x, T^n \bar{x})),$$

where $\theta(T^n x, T^n \bar{x})$ is the angle between the tangent line to W^1 at $T^n x$ and the tangent line to W^2 at $T^n \bar{x}$. Let $y \in W^2$ be the unique point in W^2 which lies on the same vertical segment as $T^n x$. Since by assumption $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$, we have $\theta(T^n x, y) \le \varepsilon$. Due to the uniform transversality of curves in $\mathcal{W}^u$ and $\mathcal{W}^s$ and the fact that W^1 and W^2 are graphs of C^2 functions with uniformly bounded C^2 norms, we have $\theta(y, T^n \bar{x}) \le C\varepsilon$ and so $\theta(T^n x, T^n \bar{x}) \le C\varepsilon$. Similarly, $d_W(T^n x, T^n \bar{x}) \le C\varepsilon$ so that

$$|J_1^n(x) - J_2^n(\bar{x})| \le C\varepsilon^{1/3} |J_2^n|_{\mathcal{C}^0(U_j^2)}. \tag{4.16}$$

Using this estimate and the fact that $|G_{U_j^\ell}|_{\mathcal{C}^1(I_j)} \le C_g$, we write for $r, s \in I_j$,

$$\frac{|(J_1^n \circ G_{U_j^1}(r) - J_2^n \circ G_{U_j^2}(r)) - (J_1^n \circ G_{U_j^1}(s) - J_2^n \circ G_{U_j^2}(s))|}{|r-s|^q} \le \frac{2C\varepsilon^{1/3}|J_2^n|_{\mathcal{C}^0(U_j^2)}}{|r-s|^q}. \tag{4.17}$$

Also, using (A.1) since $G_{U_j^\ell}(r)$ and $G_{U_j^\ell}(s)$ lie on the same stable curve,

$$\frac{|(J_1^n \circ G_{U_j^1}(r) - J_1^n \circ G_{U_j^1}(s)) - (J_2^n \circ G_{U_j^2}(r) - J_2^n \circ G_{U_j^2}(s))|}{|r-s|^q} \le 2C|J_2^n|_{\mathcal{C}^0(U_j^2)} |r-s|^{1/3-q}. \tag{4.18}$$

Putting (4.17) and (4.18) together implies that the Hölder constant of $J_1^n \circ G_{U_j^1} - J_2^n \circ G_{U_j^2}$ is bounded by

$$H^q(J_1^n \circ G_{U_j^1} - J_2^n \circ G_{U_j^2}) \le C|J_2^n|_{\mathcal{C}^0(U_j^2)} \sup_{r,s \in I_j} \min\{\varepsilon^{1/3}|r-s|^{-q}, |r-s|^{1/3-q}\}.$$

This expression is maximized when $\varepsilon^{1/3}|r-s|^{-q} = |r-s|^{1/3-q}$, i.e., when $\varepsilon = |r-s|$. Thus the Hölder constant satisfies, $H^q(J_1^n \circ G_{U_j^1} - J_2^n \circ G_{U_j^2}) \le C|J_2^n|_{\mathcal{C}^0(U_j^2)} \varepsilon^{1/3-q}$, which, together with (4.16), concludes the proof of the lemma. $\square$

Using the strong stable norm, we estimate the second term in (4.13) by

$$\left| \int_{U_j^2} h(\phi_j - |DT^n|^{-1} J_{U_j^2} T^n \psi_2 \circ T^n) \right| \le \|h\|_s |U_j^2|^\alpha \cos(U_j^2) \left| \phi_j - |DT^n|^{-1} J_{U_j^2} T^n \psi_2 \circ T^n \right|_{\mathcal{C}^q(U_j^2)}. \tag{4.19}$$

In order to estimate the $\mathcal{C}^q$-norm of the function in (4.19), we split it up into two differences. Since $|G_{U_j^2}|_{\mathcal{C}^1}, |G_{U_j^2}^{-1}|_{\mathcal{C}^1} \le C_g$, we obtain

$$\begin{aligned}
&|\phi_j - (|DT^n|^{-1} J_{U_j^2} T^n) \cdot \psi_2 \circ T^n|_{\mathcal{C}^q(U_j^2)} \\
&\le C \left| \left[(|DT^n|^{-1} J_{U_j^1} T^n) \cdot \psi_1 \circ T^n \right] \circ G_{U_j^1} - \left[(|DT^n|^{-1} J_{U_j^2} T^n) \cdot \psi_2 \circ T^n \right] \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} \\
&\le C \left| (|DT^n|^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} \left[\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2} \right] \right|_{\mathcal{C}^q(I_j)} \\
&\quad + C \left| \left[(|DT^n|^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} - (|DT^n|^{-1} J_{U_j^2} T^n) \circ G_{U_j^2} \right] \psi_2 \circ T^n \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} \\
&\le C| \, |DT^n|^{-1} J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \left| \psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} \\
&\quad + C(\cos W^2)^{-1} \left| (|DT^n|^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} - (|DT^n|^{-1} J_{U_j^2} T^n) \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)}.
\end{aligned} \tag{4.20}$$

Note that the second term can be bounded using Lemma 4.3. To bound the first term, we prove the following lemma.

Lemma 4.4. $|\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le C(\cos W^2)^{-1}\, \varepsilon^{p-q}$.

We postpone the proof of the lemma to Section 4.3.1 and show how this completes the estimate on the strong unstable norm. Notice that $||DT^n|^{-1} J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \le C||DT^n|^{-1} J_{U_j^2} T^n|_{\mathcal{C}^0(U_j^2)}$ by the distortion bounds (A.3) and (A.4). Then using Lemmas 4.3 and 4.4 together with (4.20) yields by (4.19)

$$\begin{aligned}
&\sum_j \Big| \int_{U_j^2} h(\phi_j - |DT^n|^{-1} J_{U_j^2} T^n \psi_2 \circ T^n)\, dm_W \Big| \\
&\le C\|h\|_s \sum_j |U_j^2|^\alpha \frac{\cos U_j^2}{\cos W^2} ||DT^n|^{-1} J_{U_j^2} T^n|_{\mathcal{C}^0(U_j^2)}\, \varepsilon^{p-q} \le C\|h\|_s \varepsilon^{p-q} \sum_j \frac{|T^n U_j^2|}{|U_j^2|},
\end{aligned} \tag{4.21}$$

where again the sum is finite as in (4.15). This completes the estimate on the second term in (4.13). Now we use this bound, together with (4.12) and (4.15) to estimate (4.11)

$$\left| \int_{W^1} \mathcal{L}^n h\, \psi_1\, dm_W - \int_{W^2} \mathcal{L}^n h\, \psi_2\, dm_W \right| \le CC_1^n \|h\|_s \varepsilon^\alpha + C\|h\|_u \Lambda^{-n\beta} n^\beta \varepsilon^\beta + C\|h\|_s \varepsilon^{p-q}.$$

Since $p - q \ge \beta$ and $\alpha \ge \beta$, we divide through by ε^β and take the appropriate suprema to complete the proof of (2.6).

4.3.1 Proof of Lemmas 4.2 and 4.4

Proof of Lemma 4.2. Note that by construction U_j^1 and U_j^2 lie in the same homogeneity strip. Also, they are both defined on the same interval I_j so the length of the symmetric difference of their r-intervals is 0. Recalling the definition of $d_{\mathcal{W}^s}(U_j^1, U_j^2)$, we see that it remains only to estimate $|\varphi_{U_j^1} - \varphi_{U_j^2}|_{\mathcal{C}^1(I_j)}$ for their defining functions $\varphi_{U_j^\ell}$.

The fact that $|\varphi_{U_j^1} - \varphi_{U_j^2}|_{\mathcal{C}^0(I_j)} \le C\Lambda^{-n}\varepsilon$ follows from the fact that U_j^1 and U_j^2 are connected by a foliation of vertical segments $\{\gamma_x\}$ and $T^i\gamma_x$ lies in the enlarged unstable cone $\hat{C}^u(x) = \{(dr, d\varphi) \in \mathcal{T}_x M : \mathcal{K}_{\min} \le \frac{d\varphi}{dr} \le \infty\}$, for $0 \le i \le n$. Since any vector in $\hat{C}^u(x)$ undergoes the uniform expansion[3] given by (2.8) under iteration by T (see [CM, §4.4]) and $|T^n\gamma_x| \le C\varepsilon$ by assumption on W^1 and W^2, we have $|\gamma_x| \le C\Lambda^{-n}\varepsilon$.

Finally, we must estimate $|\varphi'_{U_j^1} - \varphi'_{U_j^2}|$, where $\varphi'_{U_j^\ell}$ denotes the derivative of $\varphi_{U_j^\ell}$ with respect to r. For $x \in U_j^\ell$, let $\phi(x)$ denote the angle that $G_{U_j^\ell}$ makes with the positive r-axis at x. For $x \in U_j^1$ and $\bar{x} = \gamma_x \cap U_j^2$, let $\theta(x, \bar{x})$ denote the angle between the tangent vectors to U_j^1 and U_j^2 at the points x and $\bar{x}$, respectively. We have

$$|\varphi'_{U_j^1}(x) - \varphi'_{U_j^2}(\bar{x})| = |\tan\phi(x) - \tan\phi(\bar{x})| \le \big[\sup_{z \in U_j^\ell} \sec^2\phi(z)\big] |\phi(x) - \phi(\bar{x})| = \big[\sup_{z \in U_j^\ell} \sec^2\phi(z)\big] \theta(x, \bar{x}).$$

Since the slopes of vectors in $C^s(x)$ are uniformly bounded away from 0 and $-\infty$, we have $\sec^2\phi(z)$ uniformly bounded above for any $z \in U_j^\ell$. It follows from (A.5) that

$$\theta(x, \bar{x}) \le C\Lambda^{-n}(n\, d(T^n x, T^n \bar{x}) + \theta(T^n x, T^n \bar{x})).$$

Since $T^n x \in W^1$, $T^n \bar{x} \in W^2$, it follows from the assumption $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$ that $d(T^n x, T^n \bar{x}) + \theta(T^n x, T^n \bar{x}) \le C\varepsilon$, which proves the lemma. □

[3]Indeed, all the uniformly hyperbolic properties of Section 2.5.1 hold in the larger cone $\hat{C}^u(x)$. The reason we define the narrower cones $C^s(x)$ and $C^u(x)$ is to maintain uniform transversality of curves in $\mathcal{W}^s$ and $\mathcal{W}^u$.

Proof of Lemma 4.4. Let φ_{W^ℓ} be the function whose graph is W^ℓ, defined for $r \in I_{W^\ell}$, and set $f_j^\ell := G_{W^\ell}^{-1} \circ T^n \circ G_{U_j^\ell}$, $\ell = 1, 2$. Notice that since $|G_{W^\ell}^{-1}|_{\mathcal{C}^1}, |G_{U_j^\ell}|_{\mathcal{C}^1} \le C_g$, and due to the uniform contraction along stable curves, we have $|f_j^\ell|_{\mathcal{C}^1(I_j)} \le C$, where C is independent of W^ℓ and j. We may assume that $f_j^\ell(I_j) \subset I_{W^1} \cap I_{W^2}$ since if not, by the transversality of $C^u(x)$ and $C^s(x)$, we must be in a neighborhood of one of the endpoints of W^ℓ of length at most $C\varepsilon$; such short pieces may be estimated as in (4.12) using the strong stable norm. Thus

$$\begin{aligned} |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le\; & |\psi_1 \circ G_{W^1} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^1|_{\mathcal{C}^q(I_j)} \\ & + |\psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2|_{\mathcal{C}^q(I_j)}. \end{aligned} \tag{4.22}$$

Using the above observation about f_j^1, we estimate the first term of (4.22) by

$$|\psi_1 \circ G_{W^1} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^1|_{\mathcal{C}^q(I_j)} \le C|\psi_1 \circ G_{W^1} - \psi_2 \circ G_{W^2}|_{\mathcal{C}^q(f_j^1(I_j))} \le C\varepsilon. \tag{4.23}$$

To estimate the second term of (4.22), notice that since $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$, we have $|f_j^1 - f_j^2|_{\mathcal{C}^0(I_j)} \le C\varepsilon$. Thus for $r \in I_j$,

$$|\psi_2 \circ G_{W^2} \circ f_j^1(r) - \psi_2 \circ G_{W^2} \circ f_j^2(r)| \le C|\psi_2|_{\mathcal{C}^p} |f_j^1(r) - f_j^2(r)|^p \le C|\psi_2|_{\mathcal{C}^p} \varepsilon^p. \tag{4.24}$$

Using (4.24), we write for $r, s \in I_j$,

$$|(\psi_2 \circ G_{W^2} \circ f_j^1(r) - \psi_2 \circ G_{W^2} \circ f_j^2(r)) - (\psi_2 \circ G_{W^2} \circ f_j^1(s) - \psi_2 \circ G_{W^2} \circ f_j^2(s))| \le 2C|\psi_2|_{\mathcal{C}^p} \varepsilon^p.$$

On the other hand, notice that for $k = 1, 2$,

$$|\psi_2 \circ G_{W^2} \circ f_j^k(r) - \psi_2 \circ G_{W^2} \circ f_j^k(s)| \le C|\psi_2|_{\mathcal{C}^p} |f_j^k(r) - f_j^k(s)|^p \le C|\psi_2|_{\mathcal{C}^p} |r - s|^p,$$

using the fact that $|f_j^k|_{\mathcal{C}^1} \le C$. These estimates together imply that the Hölder constant of $\psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2$ is bounded by $C|\psi_1|_{\mathcal{C}^p} \sup_{r,s \in I_j} \min\{\varepsilon^p |r - s|^{-q}, |r - s|^{p-q}\}$. The minimum is attained when the two bounds are equal, i.e., when $\varepsilon = |r - s|$. This, together with (4.24), implies

$$|\psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2|_{\mathcal{C}^q(I_j)} \le C|\psi_2|_{\mathcal{C}^p} \varepsilon^{p-q}.$$

This estimate combined with (4.23) proves the lemma since $|\psi_2|_{\mathcal{C}^p(W^2)} \le (\cos W^2)^{-1}$. □

5 Proof of Theorem 2.5

The Lasota-Yorke estimate (2.7) and the compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ imply via the standard Hennion argument that the spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is bounded by 1 and the essential spectral radius is bounded by $\sigma < 1$ (see for example [B1]). Indeed, the spectral radius is 1, since if it were smaller than 1, by Lemma 3.9, we would obtain the following contradiction,

$$1 = m(1) = \lim_{n \to \infty} |\mathcal{L}^n m(1)| \le C \lim_{n \to \infty} \|\mathcal{L}^n m\|_{\mathcal{B}} = 0. \tag{5.1}$$

Our proof of Theorem 2.5 follows very closely that in [DL, Section 5]. Although our proofs in Sections 3 and 4 were different from those in [DL] due to the countable number of singularities and the additional cutting to maintain bounded distortion, the norms are in fact very similar (excluding the additional weights of $\cos W$) so that once the spectral gap is proved, the subsequent characterization of the peripheral spectrum of $\mathcal{L}$ follows from the same rather general arguments.[4] We include some of the arguments here for completeness and to point out which properties follow from our functional analytic setup and which follow from previously known properties of billiards.

[4] See also [BG1, Appendix B] for a general strategy to prove the characterization of the peripheral spectrum once the Lasota-Yorke inequalities and several of the lemmas of Section 3.3 have been established.

5.1 Peripheral Spectrum

Let $\mathbb{V}_\theta$ be the eigenspace of $\mathcal{L}$ associated with the eigenvalue $e^{2\pi i\theta}$ and let Π_θ be the eigenprojector onto $\mathbb{V}_\theta$. We begin by proving the following characterization of the peripheral spectrum of $\mathcal{L}$.

Lemma 5.1. *Let $\mathbb{V} = \oplus_\theta \mathbb{V}_\theta$. Then,*

(i) $\mathcal{L}$ restricted to $\mathbb{V}$ has semi-simple spectrum (no Jordan blocks);

(ii) $\mathbb{V}$ consists of signed measures;

(iii) all measures in $\mathbb{V}$ are absolutely continuous with respect to $\overline{\mu} := \Pi_0 m$. Moreover, 1 is in the spectrum of $\mathcal{L}$.

(iv) Let $\mathcal{S}^{\mathbb{H}}_{\pm n,\varepsilon}$ denote the ε-neighborhood of $\mathcal{S}^{\mathbb{H}}_{\pm n}$. Then for each $\nu \in \mathbb{V}$, $n \in \mathbb{N}$, we have $\nu(\mathcal{S}^{\mathbb{H}}_{\pm n,\varepsilon}) \le C_n \varepsilon^\alpha$, for some constants $C_n > 0$.

Proof. (i) Suppose there exists $z \in \mathbb{C}$, $|z| = 1$, and $h_1, h_2 \in \mathcal{B}$, $h_1 \neq 0$, such that $\mathcal{L}h_1 = zh_1$ and $\mathcal{L}h_2 = zh_2 + h_1$. Then $\mathcal{L}^n h_2 = z^n h_2 + nz^{n-1}h_1$ so that

$$\|\mathcal{L}^n h_2\|_{\mathcal{B}} \ge n\|h_1\|_B - \|h_2\|_{\mathcal{B}}, \quad \text{for each } n \ge 0,$$

which contradicts the fact that $\|\mathcal{L}^n\|_{\mathcal{B}}$ remains bounded for all n due to (2.7).

(ii) Recall that for $\psi \in \mathcal{C}^p(\mathcal{W}^s)$, we have $\psi \circ T^n \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$. Thus by Lemma 3.9, for $h \in \mathcal{B}$,

$$|\mathcal{L}^n h(\psi)| = |h(\psi \circ T^n)| \le C\|h\|_{\mathcal{B}}(|\psi|_\infty + H^p_n(\psi \circ T^n)) \le C\|h\|_{\mathcal{B}}(|\psi|_\infty + \Lambda^{-pn} H^p_0(\psi)), \tag{5.2}$$

where as usual, $H^p_n(\cdot)$ is the Hölder constant with exponent p measured along curves in $T^{-n}\mathcal{W}^s \subseteq \mathcal{W}^s$ and we have used (4.3).

Suppose $\nu \in \mathbb{V}$ with $\mathcal{L}\nu = z\nu$, for some $z \in \mathbb{C}$, $|z| = 1$. Then by (5.2), for each $n \ge 0$,

$$|\nu(\psi)| = |z|^{-n}|\mathcal{L}^n\nu(\psi)| \le C\|\nu\|_{\mathcal{B}}(|\psi|_\infty + \Lambda^{-pn} H^p_0(\psi)).$$

Taking the limit as $n \to \infty$ yields $|\nu(\psi)| \le C\|\nu\|_{\mathcal{B}}|\psi|_\infty$ for all $\psi \in \mathcal{C}^p(\mathcal{W}^s)$, so that ν is a measure.

(iii) By density, $\mathbb{V}_\theta = \Pi_\theta \mathcal{C}^1(M)$. So for each $\nu \in \mathbb{V}_\theta$, there exists $h \in \mathcal{C}^1(M)$ such that $\Pi_\theta h = \nu$. Now for each $\psi \in \mathcal{C}^p(M)$,

$$|\nu(\psi)| = |\Pi_\theta h(\psi)| \le |h|_\infty \Pi_0 1(|\psi|) = |h|_\infty \overline{\mu}(|\psi|).$$

Thus ν is absolutely continuous with respect to $\overline{\mu}$. Moreover, letting $h_\nu = \frac{d\nu}{d\overline{\mu}}$, we have $h_\nu \in L^\infty(M, \overline{\mu})$. This implies that $\overline{\mu} \neq 0$ since then the spectral radius of $\mathcal{L}$ would be strictly less than 1, leading to the contradiction given by (5.1). Since $\mathcal{L}\overline{\mu} = \overline{\mu}$, $\overline{\mu} \neq 0$, then 1 belongs to the spectrum of $\mathcal{L}$.

(iv) We give a different proof here from that in [DL] due to the fact that our singularity set is countable rather than finite.

Let $\nu \in \mathbb{V}$ and fix $n \ge 0$. Let $\mathcal{S}^{\mathbb{H}}_{-n,\varepsilon}$ denote the ε-neighborhood of $\mathcal{S}^{\mathbb{H}}_{-n}$ and let h_k be a sequence of $\mathcal{C}^1$ functions converging to ν in $\mathcal{B}$; then since $\mathcal{L}$ is bounded, $\mathcal{L}^n h_k$ converges to $\mathcal{L}^n \nu$ in $\mathcal{B}$. It is straightforward to check (applying Lemma 3.7) that $(\mathcal{L}^n h_k)_\varepsilon(\psi) := \mathcal{L}^n h_k(1_{\mathcal{S}^{\mathbb{H}}_{-n,\varepsilon}}\psi)$ belongs to $\mathcal{B}_w$ due to the uniform transversality of curves in $\mathcal{S}^{\mathbb{H}}_{-n}$ to the stable cone. Then, for $\psi \in \mathcal{C}^p(M)$ and $W \in \mathcal{W}^s$,

$$\int_W (\mathcal{L}^n h_k)_\varepsilon \psi \, dm_W = \int_W \mathcal{L}^n h_k \, 1_{\mathcal{S}^{\mathbb{H}}_{-n,\varepsilon}} \psi \, dm_W = \sum_i \int_{W^n_i \cap T^{-n}\mathcal{S}^{\mathbb{H}}_{-n,\varepsilon}} h_k |DT^n|^{-1} J_{W^n_i} T^n \psi \circ T^n dm_W.$$

Notice that since W_i^n are created by intersections of W with $\mathcal{S}_{-n}^{\mathbb{H}}$, it follows that there are at most two connected components in each $W_i^n \cap T^{-n}\mathcal{S}_{-n,\varepsilon}^{\mathbb{H}}$ and $|T^n W_i^n \cap \mathcal{S}_{-n,\varepsilon}^{\mathbb{H}}| \le C\varepsilon$. Consequently, we estimate the above expression following (4.12),

$$\begin{aligned}\left|\int_W (\mathcal{L}^n h_k)_\varepsilon \psi \, dm_W\right| &\le C\|h_k\|_s \sum_i |W_i^n \cap T^{-n}\mathcal{S}_{-n,\varepsilon}^{\mathbb{H}}|^\alpha \frac{|T^n W_i^n|}{|W_i^n|} \\ &\le C\varepsilon^\alpha \|h_k\|_s \sum_i \frac{|T^n W_i^n|^{1-\alpha}}{|W_i^n|^{1-\alpha}} \le C\varepsilon^\alpha \|h_k\|_s C_1^n,\end{aligned}$$

by Lemma 3.4 with $\varsigma = 1-\alpha$. Similarly, $(\mathcal{L}^n h_k)_\varepsilon$ is a Cauchy sequence in $\mathcal{B}_w$ and so must converge to $(\mathcal{L}^n \nu)_\varepsilon(\psi) := \mathcal{L}^n \nu(1_{\mathcal{S}_{-n,\varepsilon}^{\mathbb{H}}} \psi)$. Then by Lemma 3.9, we have $|\mathcal{L}^n \nu(\mathcal{S}_{-n,\varepsilon}^{\mathbb{H}})| \le C_n \|\nu\|_s \varepsilon^\alpha$. But since $\mathcal{L}^n \nu = z^n \nu$ for some $z \in \mathbb{C}$, $|z| = 1$, we have the same bound for $\nu(\mathcal{S}_{-n,\varepsilon}^{\mathbb{H}})$. The bound for $\nu(\mathcal{S}_{n,\varepsilon}^{\mathbb{H}})$ follows since $T^{-n}\mathcal{S}_{-n}^{\mathbb{H}} = \mathcal{S}_n^{\mathbb{H}}$ for each $n \ge 0$. □

Since the spectrum outside the circle of radius $\sigma < 1$ consists of only finitely many eigenvalues of finite multiplicity and there are no Jordan blocks, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-2\pi i\theta k} \mathcal{L}^k = \Pi_\theta \tag{5.3}$$

is well-defined in the uniform topology of $L(\mathcal{B}, \mathcal{B})$.

Further information about the measures corresponding to the peripheral spectrum of $\mathcal{L}$ can be proved using similar techniques as in Lemma 5.1: In other words, they are proved using properties of the Banach spaces we have defined without relying on specific properties of the billiard map. We summarize these results in our next lemma, which we state without proof since the proof can be found in [DL, Lemmas 5.5 and 5.7].

Recall that an ergodic invariant probability measure ν is called a *physical measure* if there exists a positive Lebesgue measure invariant set B_ν, with $\nu(B_\nu) = 1$, such that, for each continuous function f,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \nu(f) \quad \forall x \in B_\nu.$$

Lemma 5.2. *(i) There exist a finite number of $q_i \in \mathbb{N}$ such that the spectrum of $\mathcal{L}$ on the unit circle is $\cup_k \{e^{2\pi i \frac{p}{q_k}} : 0 \le p < q_k,\, p \in \mathbb{N}\}$. In addition, the set of ergodic probability measures absolutely continuous with respect to $\overline{\mu}$ form a basis of $\mathbb{V}_0$.*

(ii) T admits only finitely many physical probability measures and they belong to $\mathbb{V}_0$.

(iii) The ergodic decomposition with respect to Lebesgue and with respect to $\overline{\mu}$ coincide. In addition, the ergodic decomposition with respect to Lebesgue corresponds to the supports of the physical measures.

The only properties of T that are used in the proof of the preceding lemma in [DL] are the invertibility of T and the items in Lemma 5.1.

At this point it is useful to invoke some well-known facts about the Lorentz gas that simplify the spectral picture greatly. Recall that T has a smooth invariant measure $d\mu = \rho \, dm$ where $\rho = c\cos\varphi$ and c is a normalizing constant. Since $\rho \in \mathcal{C}^1(M)$, we have $\mu \in \mathcal{B}$. So by Lemma 5.1(iii), μ is absolutely continuous with respect to $\overline{\mu}$ and since the support of μ is all of M, it must be that $\mu = \overline{\mu}$.

Now the ergodicity and mixing properties of T imply that the peripheral spectrum of $\mathcal{L}$ consists of just the simple eigenvalue at 1 with μ as its unique normalized eigenvector. Thus the spectral projectors Π_θ are all zero except for Π_0 which can be recharacterized by $\Pi_0 h = \lim_{n\to\infty} \mathcal{L}^n h$. It thus follows that any probability measure $\nu \in \mathcal{B}$ satisfies $\Pi_0 \nu = \mu$ and this convergence occurs at an exponential rate given by the spectral radius of $\mathcal{L} - \Pi_0$ on $\mathcal{B}$. This proves item (1) of Theorem 2.5.

5.2 Statistical Properties

We prove items (2) and (3) of Theorem 2.5. Given $\phi \in \mathcal{C}^\gamma(M)$, $\gamma > 2\beta$ and $\psi \in \mathcal{C}^p(\mathcal{W}^s)$, we define the correlation functions by

$$C_{\phi,\psi}(n) := \mu(\phi\,\psi \circ T^n) - \mu(\phi)\mu(\psi).$$

Define $\mu_\phi = \phi\mu$. Since $\phi \cos\varphi \in \mathcal{C}^\gamma(M)$, by Lemma 3.7 we have $\mu_\phi \in \mathcal{B}$. Thus by Theorem 2.5(1), $\Pi_0 \mu_\phi = \mu(\phi)\mu$ and so

$$|\mu(\phi\,\psi \circ T^n) - \mu(\phi)\mu(\psi)| = |(\mathcal{L}^n \mu_\phi - \mu(\phi)\mu)(\psi)| \le C\|\mathcal{L}^n \mu_\phi - \mu(\phi)\mu\|_{\mathcal{B}}(|\psi|_\infty + H_0^p(\psi))$$

and again the exponential rate of convergence is given by the spectral radius of $\mathcal{L} - \Pi_0$ on $\mathcal{B}$ as in item (1). Item (2) of Theorem 2.5 follows by noting that $\|\mu_\phi\|_{\mathcal{B}} \le C|\phi|_{\mathcal{C}^\gamma(M)}$ by (3.24) in the proof of Lemma 3.7.

If we assume $\phi, \psi \in \mathcal{C}^{p'}(M)$, where $p' > \max\{p, 2\beta\}$, we can define the Fourier transform of the correlation function,[5]

$$\hat{C}_{\phi,\psi}(z) := \sum_{n\in\mathbb{Z}} z^n C_{\phi,\psi}(n).$$

The importance of this function stems from the connection between its poles and the Ruelle resonances, which are in principal measurable in physical systems, [Ru1, Ru2, PP1, PP2, L2].

Given the spectral picture we have established, it follows by standard arguments (see for example [DL, Section 5.3]) that the function is convergent in a neighborhood of $|z| = 1$ and admits a meromorphic extension in the annulus $\{z \in \mathbb{C} : \sigma < |z| < \sigma^{-1}\}$ where σ is from (2.7). It follows that the poles of the correlation function are in a one-to-one correspondence (including multiplicity) with the spectrum of $\mathcal{L}$ outside the disk of radius σ. This is item (3) of Theorem 2.5.

6 Proofs of Limit Theorems

In this section, we show how Theorem 2.6 follows from the established spectral picture. Choose $\gamma = \max\{p, 2\beta + \varepsilon\}$ for some $\varepsilon > 0$. Let $g \in \mathcal{C}^\gamma(M)$ and define $S_n g = \sum_{j=0}^{n-1} g \circ T^j$. We define the generalized transfer operator $\mathcal{L}_g$ on $\mathcal{B}$ by, $\mathcal{L}_g h(\psi) = h(e^g \psi \circ T)$ for all $h \in \mathcal{B}$. It is then immediate that

$$\mathcal{L}_g^n h(\psi) = h(e^{S_n g} \psi \circ T^n), \quad \text{for all } n \ge 0.$$

The main element in the proofs of the limit theorems is that $\mathcal{L}_{zg}$, $z \in \mathbb{C}$, is an analytic perturbation of $\mathcal{L} = \mathcal{L}_0$ for small $|z|$.

Lemma 6.1. *For $g \in \mathcal{C}^p(M)$, the map $z \mapsto \mathcal{L}_{zg}$ is analytic for all $z \in \mathbb{C}$.*

[5] Here we need that $\mu_\phi := \phi\mu \in \mathcal{B}$ and $\mu_\psi := \psi\mu$ belongs to the corresponding space of distributions for T^{-1}, which, given the invertibility of T, is simply $\mathcal{B}$ with the roles of $\mathcal{W}^s$ and $\mathcal{W}^u$ reversed.

Proof. The lemma will follow once we show that our strong norm $\|\cdot\|_{\mathcal{B}}$ is continuous with respect to multiplication by e^{zg}. We will prove that for $h \in \mathcal{B}$ and $f \in \mathcal{C}^\gamma(M)$, $\|hf\|_{\mathcal{B}} \le C|f|_{\mathcal{C}^p(M)}\|h\|_{\mathcal{B}}$ for some uniform constant C. Then defining the operator $\mathcal{P}_n h = \mathcal{L}(g^n h)$, $h \in \mathcal{B}$, the claim implies that

$$\|\mathcal{P}_n(h)\|_{\mathcal{B}} = \|\mathcal{L}(g^n h)\|_{\mathcal{B}} \le C\|\mathcal{L}\| \, \|h\|_{\mathcal{B}} \, |g^n|_{\mathcal{C}^p(M)},$$

which allows us to conclude that the operator $\sum_{n=0}^\infty \frac{z^n}{n!}\mathcal{P}_n$ is well-defined on $\mathcal{B}$ and equals $\mathcal{L}_{zg}$ since

$$\sum_{n=0}^\infty \frac{z^n}{n!}\mathcal{P}_n h(\psi) = h\left(\sum_{n=0}^\infty \frac{z^n}{n!} g^n \cdot \psi \circ T\right) = h(e^{zg}\psi) = \mathcal{L}_{zg}h(\psi), \quad \text{for } \psi \in \mathcal{C}^p(\mathcal{W}^s),$$

once we know the sum converges. We proceed to prove our claim.

By density, it suffices to prove the claim for $h \in \mathcal{C}^1(M)$ and $f \in \mathcal{C}^\gamma(M)$. By Lemma 3.7, $hf \in \mathcal{B}$. To estimate the strong stable norm of hf, let $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \le 1$. Then

$$\int_W hf\psi \, dm_W \le \|h\|_s |W|^\alpha \cos W |f|_{\mathcal{C}^q(W)} |\psi|_{\mathcal{C}^q(W)} \le \|h\|_s |f|_{\mathcal{C}^q(W)}.$$

Next we estimate the strong unstable norm of hf. Let $\varepsilon \le \varepsilon_0$ and choose $W_1, W_2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1, W_2) < \varepsilon$. For $\ell = 1, 2$, let $\psi_\ell \in W_\ell$ with $|\psi_\ell|_{\mathcal{C}^p(W_\ell)} \le 1$ and $d_q(\psi_1, \psi_2) \le \varepsilon$. We must estimate

$$\int_{W_1} hf\,\psi_1 \, dm_W - \int_{W_2} hf\,\psi_2 \, dm_W.$$

Recalling the notation of Section 3.1, we write $W_\ell = G_{W_\ell}(I_{W_\ell}) = \{(r, \varphi_{W_\ell}(r)) : r \in I_{W_\ell}\}$ and let $I = I_{W_1} \cap I_{W_2}$. Then,

$$\begin{aligned} d_q(f\psi_1, f\psi_2) &:= |(f\psi_1) \circ G_{W_1} - (f\psi_2) \circ G_{W_2}|_{\mathcal{C}^q(I)} \\ &\le |f \circ G_{W_1}|_{\mathcal{C}^q(I)} d_q(\psi_1, \psi_2) + |\psi_2 \circ G_{W_2}|_{\mathcal{C}^q(I)} |f \circ G_{W_1} - f \circ G_{W_2}|_{\mathcal{C}^q(I)}. \end{aligned}$$

The first term above is bounded by $C|f|_{\mathcal{C}^q(W_1)}\varepsilon$ by assumption on ψ_1 and ψ_2, where C depends only on the maximum slope in $C^s(x)$. Similarly, $|\psi_2 \circ G_{W_2}|_{\mathcal{C}^q(I)} \le C|\psi_2|_{\mathcal{C}^q(W_2)}$.

For $r \in I$, we have $d(G_{W_1}(r), G_{W_2}(r)) \le \varepsilon$ by definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$. Thus $|f \circ G_{W_1}(r) - f \circ G_{W_2}(r)| \le |f|_{\mathcal{C}^p(M)}\varepsilon^p$, and so by the proof of Lemma 4.3, we have $|f \circ G_{W_1} - f \circ G_{W_2}|_{\mathcal{C}^q(I)} \le |f|_{\mathcal{C}^p(M)}\varepsilon^{p-q}$. Putting these estimates together yields,

$$d_q(f\psi_1, f\psi_2) \le C|f|_{\mathcal{C}^p(M)}\varepsilon^{p-q}.$$

Since $p - q \ge \beta$ and $p > q$, we may thus estimate,

$$\varepsilon^{-\beta}\left|\int_{W_1} hf\,\psi_1 \, dm_W - \int_{W_2} hf\,\psi_2 \, dm_W\right| \le C\|h\|_u |f|_{\mathcal{C}^p(M)},$$

which completes the estimate on the strong unstable norm and the proof of the lemma. □

With the analyticity of $z \mapsto \mathcal{L}_{zg}$ established, it follows from analytic perturbation theory [Ka] that both the discrete spectrum and the corresponding spectral projectors of $\mathcal{L}_{zg}$ vary smoothly with z. Thus, since $\mathcal{L}_0$ has a spectral gap, then so does $\mathcal{L}_{zg}$ for $z \in \mathbb{C}$ sufficiently close to 0.

Proof of Theorem 2.6(a). We follow [RY], making necessary modifications to generalize to non-invariant measures $\nu \in \mathcal{B}$. (See also [D].)

Let $\nu \in \mathcal{B}$ be a probability measure. Assume $|z|$ is sufficiently small so that $\mathcal{L}_{zg}$ has a spectral gap. Let λ_z be the eigenvalue of maximum modulus and denote by Π_{λ_z} the associated eigenprojector. Since $\Pi_{\lambda_0}\nu(1) = 1$ and the spectral projectors vary continuously, we have $\Pi_{\lambda_z}\nu(1) \neq 0$ for z sufficiently small, say $|z| < \gamma$ for some $\gamma > 0$. We define the moment generating function $q(z)$ by

$$q(z) := \lim_{n\to\infty} \frac{1}{n} \log \nu(e^{zS_n g}) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}^n_{zg}\nu(1) = \log \lambda_z,$$

where the second and third equalities follows from the spectral gap of $\mathcal{L}_{zg}$ and the fact that $\Pi_{\lambda_z}\nu(1) \neq 0$. Notice that $q(z)$ is independent of ν and is analytic in $|z| < \gamma$.

Let ς^2 denote the limit of the variance of $n^{-1/2}S_n g$ as $n \to \infty$ where $\{g \circ T^k\}_{j\in\mathbb{N}}$ is distributed according to the invariant measure μ. (Such a ς exists and is finite whenever the auto-correlations $C_{g,g}(k)$ are summable). One can show as in [RY, Theorem 4.3] that in fact $q'(0) = \mu(g)$, $q''(0) = \varsigma^2$ and q is strictly convex for real z whenever $\varsigma^2 > 0$.

Now let $I(u)$ be the Legendre transform of $q(z)$. Then it follows from the Gartner-Ellis theorem [DZ] that for any interval $[a, b] \subset [q'(-\gamma), q'(\gamma)]$ and for any probability measure $\nu \in \mathcal{B}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) \in [a, b] \right) = - \inf_{u\in[a,b]} I(u),$$

which is the desired large deviation estimate with uniform rate function I. □

Proof of Theorem 2.6(b). We assume $\mu(g) = 0$ and distribute $(g \circ T^j)_{j\in\mathbb{N}}$ according to μ. As above, let ς^2 denote the variance of $n^{-1/2}S_n g$ as $n \to \infty$. We consider purely imaginary $z = it$ with $|t| < \gamma$. Since $\mathcal{L}_{zg}$ depends analytically on z, it follows from standard perturbation theory that the leading eigenvalue of $\mathcal{L}_{itg}$ is given by $1 - \frac{\varsigma^2 t^2}{2} + \mathcal{O}(t^3)$ for $|t| < \gamma$. It then follows using the weak dependence of $g \circ T^j$ that,

$$\lim_{n\to\infty} \mu(e^{-i\frac{t}{\sqrt{n}}S_n g}) = \lim_{n\to\infty} \left(1 - \frac{\varsigma^2 t^2}{2n}\right)^n = e^{-\varsigma^2 t^2/2},$$

which is the Central Limit Theorem.

The extension to non-invariant probability measures follows easily as well. Let $\nu \in \mathcal{B}$ be a probability measure. We still require $\mu(g) = 0$, but now distribute $(g \circ T)_{j\in\mathbb{N}}$ according to ν. As j goes to ∞, the asymptotic mean is $\nu(g \circ T^j) = \mathcal{L}^j\nu(g) \to \mu(g) = 0$ and the asymptotic variance is still ς^2 as above. At this point, there are a variety of references at our disposal, but we choose to cite [G] as a recent work since it proves both the Central Limit Theorem and the almost-sure invariance principle using spectral methods.

Since the transfer operator $\mathcal{L}_{itg}$ codes the characteristic function of the process $(g \circ T^j)$ in the sense of [G, Section 2.1], i.e. $\nu(e^{-itS_n g}) = \mathcal{L}^n_{itg}\nu(1)$, and we have proved that $\mathcal{L}_{itg}$ satisfies the assumptions of strong continuity in [G, Section 2.2], we may apply [G, Theorem 2.1] to conclude that $\frac{1}{\sqrt{n}}S_n g$ converges in distribution to a normal random variable with mean 0 and variance ς^2, as required. □

Proof of Theorem 2.6(c). The almost-sure invariance principle follows from the analyticity of the map $z \to \mathcal{L}_{zg}$ and the resulting persistence of the spectral gap in a neighborhood of the origin, similar to the proofs of the previous two limit theorems. Indeed, the invariance principle holds under much weaker conditions than those present here. As in the proof of (b), noting that we have proved our operators $\mathcal{L}_{itg}$ satisfy the assumptions of strong continuity in [G, Section 2.2], we may apply [G, Theorem 2.1] to conclude the almost-sure invariance principle in the context of the functional analytic framework we have constructed here. □

A Distortion Bounds

The following are distortion bounds used in deriving the Lasota-Yorke estimates which hold for the Lorentz gas with both finite and infinite horizon. There exists a constant $C_d > 0$ with the following properties. Let $W' \in \mathcal{W}^s$ and for any $n \in \mathbb{N}$, let $x, y \in W$ for some connected component $W \subset T^{-n}W'$ such that T^iW is a homogeneous stable curve for each $0 \le i \le n$. Then,

$$\left| \frac{|DT^n(x)|}{|DT^n(y)|} - 1 \right| \le C_d d_W(x,y)^{1/3} \qquad \text{and} \qquad \left| \frac{J_W T^n(x)}{J_W T^n(y)} - 1 \right| \le C_d d_W(x,y)^{1/3}. \tag{A.1}$$

In particular, these bounds imply that $||DT^n|^{-1}|_{C^p(W)} \le C_d ||DT^n|^{-1}|_{\mathcal{C}^0(W)}$ and $|J_W T^n|_{C^p(W)} \le C_d |J_W T^n|_{\mathcal{C}^0(W)}$ for any $0 \le p \le 1/3$.

The second inequality in (A.1) is equivalent to (3.1) and is a standard distortion bound for billiards (see [BSC1, BSC2] or [Ch2] for both the finite and infinite horizon cases). In the proof of the distortion bound, the main idea is to prove that along a stable curve $W \in \mathcal{W}^s$, for any $x, y \in W$

$$\frac{d(x,y)}{\cos\varphi(x)} \le C d(x,y)^{1/3}, \tag{A.2}$$

which follows from the definition of the homogeneity strips $\mathbb{H}_k$ and the uniform transversality of the stable cone to horizontal lines. Indeed the first inequality of (A.1) directly follows from the estimate (A.2). More precisely, for any x, y belonging to a stable curve W,

$$\begin{aligned} \left| \ln \frac{|DT(x)|}{|DT(y)|} \right| &= \left| \ln \frac{\cos\varphi(x)}{\cos\varphi(y)} + \ln \frac{\cos\varphi(Ty)}{\cos\varphi(Tx)} \right| \\ &\le C_1 \frac{d(x,y)}{\cos\varphi(x)} + C_2 \frac{d(Tx,Ty)}{\cos\varphi(Tx)} \le C d(x,y)^{1/3}, \end{aligned}$$

where C_1, C_2, C are positive constants and we used the hyperbolicity (2.8) in the last inequality.

By an entirely analogous argument (due to the time reversibility of the billiard map), if $W \in \mathcal{W}^u$ is an unstable curve such that T^iW is a homogeneous unstable curve for $0 \le i \le n$, then for any $x, y \in W$,

$$\left| \frac{|DT^n(x)|}{|DT^n(y)|} - 1 \right| \le C_d d(T^n x, T^n y)^{1/3}. \tag{A.3}$$

In Section 4.3, we will need to compare the stable Jacobian of T along different stable curves. For this, the following distortion bound is essential. Let $W^1, W^2 \in \mathcal{W}^s$ and suppose there exist $U^\ell \subset T^{-n}W^\ell$, $\ell = 1, 2$, such that T^iU^ℓ is a homogeneous stable curve for $0 \le i \le n$, and U^1 and U^2 can be put into a 1-1 correspondence by a smooth foliation $\{\gamma_x\}_{x \in U^1}$ of curves $\gamma_x \in \mathcal{W}^u$ such that $\{T^n\gamma_x\} \subset \mathcal{W}^u$ creates a 1-1 correspondence between T^nU^1 and T^nU^2. Let $J_{U^\ell}T^n$ denote the stable Jacobian of T^n along the curve U^ℓ. Then for $x \in U^1$, $\bar{x} \in \gamma_x \cap U^2$, we have

$$\left| \frac{J_{U^1}T^n(x)}{J_{U^2}T^n(\bar{x})} - 1 \right| \le C_1 d(T^n x, T^n \bar{x})^{1/3} + C_2 \theta(T^n x, T^n \bar{x}), \tag{A.4}$$

where $\theta(T^n x, T^n \bar{x})$ is the angle formed by the tangent lines of T^nU^1 and T^nU_2 at $T^n x$ and $T^n \bar{x}$, respectively.

This distortion bound is proved as part of [Ch2, Theorem 8.1] (see also [CM, §5.8]). We explain the argument briefly, modifying the notation as necessary since the proof in [Ch2] is written for unstable curves mapped by T while we need these bounds for stable curves mapped backwards by T^{-1}. In addition, the time reversal of the setup in [Ch2] would have x and $\bar{x}$ lying on the same unstable manifold, while in our setting, x and $\bar{x}$ just lie on a common unstable curve. The reason these estimates hold is because we are able to choose our foliation $\{\gamma_x\}$ after fixing n.

We relabel $T^n U^1 = V^1$, $T^n U^2 = V^2$ and $\{\omega_z\}_{z\in V^1} = \{T^n\gamma_x\}_{x\in U^1}$, with the identification $z = T^n x$. For any $z \in V^1$, $\bar{z} \in \omega_z \cap V^2$, and $i = 0, \cdots, n$, we denote $V_i^\ell = T^{-i}V^\ell$, $\ell = 1, 2$ and $z_i = T^{-i}z$, $\bar{z}_i = T^{-i}\bar{z}$. By [Ch2, eq (8.6)], we have

$$\left| \ln \frac{J_{V_i^1} T^{-1}(z_i)}{J_{V_i^2} T^{-1}(\bar{z}_i)} \right| \le C(\rho_{i+1}^{\frac{1}{3}} + \theta_i + \theta_{i+1} + \rho_i), \quad i = 0, 1, \dots, n-1,$$

where C is a uniform constant, $\rho_i = |T^{-i}\omega_z|$ and θ_i is the angle made by the tangent lines of V_i^1 and V_i^2 at z_i, $\bar{z}_i$, respectively.

It is important for the uniformity of this estimate that $\omega_z = T^n(\gamma_{T^{-n}z})$ so that $T^{-i}\omega_z$ remains in $\mathcal{W}^u$ for each $i = 0, 1, \dots, n$. In addition, it is a consequence of [Ch2, eq (8.9)] that

$$\theta_i \le C(\rho_0 i \Lambda^{-i} + \theta_0 \Lambda^{-i}). \tag{A.5}$$

Combining this with the fact that $\rho_i \le C\rho_0\Lambda^{-i}$ due to uniform hyperbolicity, we get

$$|\ln J_{V_i^1} T^{-1}(z_i) - \ln J_{V_i^2} T^{-1}(\bar{z}_i)| \le \text{const.} \left(\rho_0^{\frac{1}{3}} \Lambda^{-i/3} + \rho_0 i \Lambda^{-i} + \theta_0 \Lambda^{-i} \right).$$

Summing over $i = 0, \dots, n-1$, we obtain (A.4) with $x = T^{-n}z, \bar{x} = T^{-n}\bar{z}$.

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