Escape rates and physical measures for the infinite horizon Lorentz gas with holes

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Abstract

We study the statistical properties of the infinite horizon Lorentz gas after the introduction of small holes. Our basic approach is to prove the persistence of a spectral gap for the transfer operator associated with the billiard map in the presence of such holes. The new feature here is the interaction between the holes and the infinite horizon corridors, which causes previous approaches to fail. In order to overcome this difficulty, we redefine the Banach spaces on which we consider the action of the transfer operator. In this modified setting, we recover a complete set of results for the open system: Existence of a unified exponential rate of escape and limiting conditionally invariant measure for a large class of initial distributions, the convergence of the physical conditionally invariant measure to the smooth invariant measure for the billiard as the size of the hole tends to zero, and the characterization of the escape rate via a notion of pressure on the survivor set.

1 Introduction

Dynamical systems with holes are examples of systems whose domains are not invariant under the dynamics. They arise quite naturally in the study of metastability and in extended particle systems in which mass or energy leaks between adjacent cells of a long chain. Central questions involve the rate of escape of mass through the hole and the existence of physically relevant conditionally invariant measures which describe the (normalized) distribution of mass before it escapes (see [DY] for a survey of the topic and detailed references).

Despite the popularity of open systems and the successful study of a wide range of specific systems [PY, C, CMS1, CMS2, CM1, CM2, CMT1, CMT2, LiM, CV, D1, D2, DL, BDM] to name a few, the mathematical analysis of billiards with holes has been relatively recent, [DWY1, D3] (with the exception of the earlier paper [LM] which treated a very special case). The paper [DWY1] considers the billiard map associated with the finite horizon Lorentz gas via Markov extensions constructed after the introduction of the hole. The reference [D3] establishes a spectral gap for the transfer operator for the map with holes and generalizes previous results in two ways: by considering more general holes and by allowing a wider class of dispersing billiards (including those with corner points).

Although [D3] formally includes the infinite horizon Lorentz gas, the abstract assumptions on the holes considered there effectively imply that the holes cannot interact with the infinite horizon corridors in a meaningful way. A single hole placed in an infinite horizon corridor of the billiard

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flow can induce a hole in the phase space of the billiard map with countably many connected components that do not satisfy the assumptions of [D3]. Indeed, the characteristic function of the induced hole has infinite norm with respect to the norms used in [D3] and so a new approach is needed.

The purpose of the present paper is to study the behavior of the infinite horizon Lorentz gas in the presence of holes which may interact with infinite horizon corridors. We describe specific examples of such holes in Section 2.2. The geometry of these holes in the phase space of the billiard map forces us to redefine the required Banach spaces for the infinite horizon billiard so that this type of hole can be viewed as a small perturbation with respect to these modified norms. To do this, we will use stable curves and much narrower stable cones that respect the countably many singularity curves of the map which comprise the set $\mathcal{S}_1$. As a consequence, previous arguments regarding the compactness and embeddings of the Banach spaces used in [DZ1, DZ2, D3] no longer hold and must be reformulated taking into account this additional partitioning of the phase space. We show here that one can balance these competing interests in such a way that permits the analysis of this type of open system.

Once the functional analytic framework has been established in this infinite horizon setting, we recover the full set of results proved in [DWY1, D3] in the finite horizon case: Existence of a unified exponential rate of escape and limiting conditionally invariant measure for a large class of initial distributions; the convergence of the physical conditionally invariant measure to the smooth invariant measure for the billiard as the size of the hole tends to zero; a characterization of the escape rate via a notion of pressure on the survivor set and the construction of an invariant measure which achieves the supremum in the associated variational principle.

The paper is organized as follows. In Section 1.1, we recall the fundamental objects associated with open systems and establish notation we shall use throughout the paper. In Section 2, we formulate precisely our setting of an infinite horizon Lorentz gas and our formal assumptions on the types of holes we admit, and state our main results. Section 3 defines the functional analytic framework in which we study the transfer operator associated with the billiard map; in particular we define the modified Banach spaces and establish the key properties which allow us to prove a spectral gap for the map before the introduction of the hole. In Section 4, we show that this spectral gap persists for the map with a hole and prove our main theorems regarding the open system.

1.1 Basic Definitions Regarding Open Systems

We begin by defining the essential concepts that appear throughout this paper. Given a self-map T of a metric space M and an open set $H \subset M$, which we shall call the hole, we define $\mathring{M} = M \setminus H$ and $\mathring{M}^n = \cap_{i=0}^n T^{-i}(M \setminus H)$ to be the set of points in M that have not escaped by time n, $n \geq 0$. Note that $\mathring{M}^0 = \mathring{M}$.

We let $\mathring{T}^n = T^n|\mathring{M}^n$, for $n \geq 1$, denote the map with holes. We will refer to $\mathring{T}^n$ as the iterates of $\mathring{T}$ despite the fact that the domain of $\mathring{T}$ is not invariant: Once a point enters H, it is not allowed to return.

Rates of escape. Let μ be a Borel probability measure on M (not necessarily invariant with respect to T). We define the *exponential rate of escape* with respect to μ to be $-\rho(\mu)$ where

$$\rho(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \mu(\mathring{M}^n), \tag{1.1}$$

when the limit exists.

Conditionally invariant measures. Define $\mathring{T}_*\mu(A) = \mu(T^{-1}A \cap \mathring{M}^1)$ for any Borel $A \subset M$. We say μ is *conditionally invariant* with respect to $\mathring{T}$ if

$$\frac{\mathring{T}_*\mu(A)}{\mathring{T}_*\mu(M)} = \mu(A) \quad \text{for all Borel } A \subseteq M.$$

We will refer to the normalizing constant $\lambda = \mu(\mathring{M}^1)$ as the eigenvalue of μ since iterating the above equation yields $\mathring{T}^n_*\mu(A) = \lambda^n\mu(A)$ for $n \in \mathbb{N}$. In particular, $\mu(\mathring{M}^n) = \mathring{T}^n_*(M) = \lambda^n$ so that $-\log\lambda$ is the escape rate with respect to μ according to (1.1).

Pressure on the survivor set. The survivor set $\mathring{M}^\infty := \cap_{i=-\infty}^{\infty} T^i(M \setminus H)$ is a $\mathring{T}$-invariant (and also T-invariant) set which necessarily supports all the invariant measures that persist after the introduction of the hole.[1] We define the *pressure* on $\mathring{M}^\infty$ with respect to a class of invariant measures $\mathcal{C}$ to be

$$\mathcal{P}_\mathcal{C} = \sup_{\nu\in\mathcal{C}} P_\nu \quad \text{where} \quad P_\nu = h_\nu(T) - \int \chi^+(T)\, d\nu.$$

Here $h_\nu(T)$ denotes the Kolmogorov-Sinai entropy of T with respect to ν and $\chi^+(T)$ represents the sum of positive Lyapunov exponents, counted with multiplicity.

We say the open system satisfies a *variational principal* if $\rho(\mu) = \mathcal{P}_\mathcal{C}$ for some physically relevant reference measure μ and a class of invariant measures $\mathcal{C}$. If there is an invariant measure $\nu \in \mathcal{C}$ such that $\rho(\mu) = P_\nu$, we say that ν satisfies an *escape rate formula*.

2 Setting and Results

In this section, we define our setting of a periodic Lorentz gas, introduce our class of admissible holes and state our main results.

2.1 Specific Setting: Periodic Lorentz Gas

Let Γ_i, $i = 1, \dots d$, denote open convex regions in the 2-torus $\mathbb{T}^2$ such that $\partial\Gamma_i$ is $\mathcal{C}^3$ with strictly positive curvature. The Γ_i are assumed to be pairwise disjoint and the set $Q = \mathbb{T}^2 \setminus \cup_i\Gamma_i$ is referred to as the billiard table. The billiard flow is defined by a point particle traveling at unit speed in Q and reflecting elastically at the boundary. The phase space for the billiard flow is $Q \times \mathbb{S}^1/\sim$ with conventional identifications at the boundaries.

Define $M = \cup_{i=1}^d(\partial\Gamma_i \times [-\pi/2, \pi/2])$. The collision-to-collision map $T : M \to M$ is the billiard map associated with the flow. We will use the canonical coordinates (r, φ) in M, where r is the arclength parameter in $\partial\Gamma_i$ (oriented clockwise) and φ is the angle that the post-collision velocity vector at position r makes with the normal to $\partial\Gamma_i$ pointing into Q. It is well known that T preserves the smooth invariant measure $d\mu_{\text{SRB}} = c\cos\varphi dr d\varphi$, where $c = 1/2|\partial Q|$ is the normalizing constant and $|\partial Q|$ denotes the arclength of ∂Q (see for example, [CM, Section 2.12]).

For $x = (r, \varphi) \in M$, define $\tau(x)$ to be the time of the first collision of the trajectory starting at x under the billiard flow. The billiard table is said to have finite horizon if τ has a uniform upper bound on M. Otherwise, the table is said to have infinite horizon. In this paper, we will be concerned exclusively with tables having infinite horizon. In addition, we assume that the scatterers Γ_i are a positive distance apart. Thus there is some $\tau_{\min} > 0$ such that $\tau(x) \geq \tau_{\min}$ for all $x \in M$.

Define $\mathcal{S}_0 = \{\varphi = \pm\pi/2\}$. Then for $n \geq 1$, $\mathcal{S}_{\pm n} = \cup_{i=0}^n T^{\mp i}\mathcal{S}_0$ are the singularity sets for $T^{\pm n}$ and $T : M \setminus \mathcal{S}_1 \to M \setminus \mathcal{S}_{-1}$ is a piecewise $\mathcal{C}^2$ diffeomorphism. In the infinite horizon case, $\mathcal{S}_{\pm 1}$

[1] In the non-invertible case, we define $\mathring{M}^\infty := \cap_{i=0}^\infty T^{-i}(M \setminus H)$.

consist of countably many curves which accumulate on a finite number of infinite horizon points. See [CM, Section 4.10] for a detailed description of the singularity sets.

The assumption of strictly positive curvature and the existence of $\tau_{\min} > 0$ guarantee the uniform hyperbolicity of T: There are families of stable and unstable cones (defined in Section 3.1) which are strictly invariant under DT^{-1} and DT respectively.

Near $\mathcal{S}_1$, the norm of DT blows up. In order to control distortion, we introduce homogeneity strips, following [BSC1]. Fix $k_0 > 0$ to be determined later and for $k \geq k_0$ define

$$\mathbb{H}_k = \{(r, \varphi) \in M : (k+1)^{-2} \leq \pi/2 - \varphi \leq k^{-2}\}$$

$\mathbb{H}_{-k}$ is defined similarly near $\varphi = -\pi/2$.

We call a curve $W \subset M$ *stable* if the tangent vector to W lies in the stable cone at x for each $x \in W$. Moreover, a stable curve is called *homogeneous* if it lies in a single homogeneity strip. In Section 3.1 we will define a set of homogeneous stable curves $\mathcal{W}^s$ which are invariant under T^{-1} and a set of homogeneous unstable curves $\mathcal{W}^u$ which are invariant under T.

In fact, we shall need finer control of stable curves than the set $\mathcal{W}^s$ and so also in Section 3.1, we define a set $\mathcal{W}^s_1 \subset \mathcal{W}^s$ of homogeneous stable curves each contained in one component of $M \setminus \mathcal{S}_1$. This is the set of curves on which we shall build our function spaces and it effectively repartitions M into a countable number of components, in addition to those already created by the homogeneity strips, $\mathbb{H}_k$.

2.2 Introduction of Holes

We introduce abstract assumptions on the type of holes allowed in the phase space M and then give some examples of holes with physical motivation. Let $N_\varepsilon(A)$ denote the ε-neighborhood of a set A in M.

A hole $H \subset M$ is an open set with countably many connected components. Each component of H has a boundary comprised of finitely many compact smooth curves. In addition, we require that H satisfy the following three conditions.

(H1) (Complexity bound) There exists $B_0 > 0$ such that any stable curve $W \in \mathcal{W}^s_1$ can be cut into at most B_0 connected components by ∂H.

(H2) (Weak transversality) There exist $B_1, t_0 > 0$ such that for any $W \in \mathcal{W}^s_1$, $m_W(N_\varepsilon(\partial H) \cap W) \leq B_1 \varepsilon^{t_0}$ for all $\varepsilon > 0$ sufficiently small, where m_W denotes unnormalized arclength measure on W.

(H3) (Finite partitioning of $\mathbb{H}_k$) For each $k \in \mathbb{N}$, there exists N_k such that $H \cap \mathbb{H}_k$ consists of at most N_k connected components.

We define $\mathcal{H}(B_0, B_1)$ to be the set of all holes satisfying (H1)–(H3) with uniform constants B_0 and B_1. Note that we do not need uniformity in the sequence N_k. The constants B_0 and B_1 will ensure uniform control of our norms along a sequence of holes in $\mathcal{H}(B_0, B_1)$, which guarantees the persistence of a spectral gap for sufficiently small holes.

Although we prove our theorems for holes in M that satisfy these abstract assumptions, we are especially interested in holes ω which are first made in the billiard table Q for the flow and in turn induce holes H_ω in M. Such holes have greater interest since they have a physical interpretation on the level of the billiard flow. Examples of such holes presented in [D3] include: (1) holes consisting of arcs in ∂Q; (2) holes which are open, convex, connected sets in the interior of Q; (3) generalized holes which depend on both the angle and position of a particle to allow escape. The first two types listed above were introduced in [DWY1].

Below, we give two examples of holes in Q which induce holes in M satisfying (H1)–(H3). For each example, we verify that H_ω satisfies (H1)–(H3) and then explain why H_ω does not fit into the framework used in [D3].

Example 1: An open hole in an infinite horizon corridor.

Let ω be an open, convex connected set in Q disjoint from the scatterers. Since ω is disjoint from M, we have a choice as to how to define the induced hole H_ω. We define $H_\omega \subset M$ to be the set of points $(r, \varphi) \in M$ whose trajectories under the flow will enter $\omega \times \mathbb{S}^1$ before making their next collision. This is equivalent to the set of first collisions of $\omega \times \mathbb{S}^1$ under the reverse billiard flow (running the flow backwards). Such holes are called holes of Type II in [DWY1], although for technical reasons, our H_ω here is in fact $T^{-1}H_\omega$ in [DWY1]. Essentially, we define H_ω to be the 'backward shadow' of the hole ω under the flow, while [DWY1] considers it to be the 'forward shadow.'

The geometry of ∂H for holes of this type can be understood by considering ω as a small scatterer in Q. Then ∂H comprises two types of curves: the backward images of $\partial\omega \times \{\pm\pi/2\}$ under the flow and curves in $\mathcal{S}_1$. The first type are curves in M which have the same properties as curves in $\mathcal{S}_1 \setminus \mathcal{S}_0$, i.e. they are decreasing curves which terminate on other curves in $\mathcal{S}_1$. Thus ∂H comprises curves with negative slopes and parts of $\mathcal{S}_0$, which are horizontal lines. Although ∂H is not uniformly transverse to $C^s(x)$, it does satisfy (H2). We postpone the proof of this fact until Section 4.5.

Now suppose ω lies in an infinite horizon corridor as shown in Figure 1, top. Then the induced hole H_ω intersects a sequence of components of $M \setminus \mathcal{S}_1$ which accumulate on one of the finitely many infinite horizon points. Note that for singularity curves of high index, there can be at most two connected components of H_ω per connected component of $M \setminus \mathcal{S}_1$. Moreover, there is an index N_ω past which H_ω will contain all components of $M \setminus \mathcal{S}_1$ accumulating on the infinite horizon point. See Figure 1, bottom left.

Figure 1, bottom right, shows the image TH_ω near an infinite horizon point. Since TH_ω is comprised of increasing curves and is uniformly transverse to the stable cones, by the injectivity of T, we have (H1) satisfied with $B_0 = 3$.

In Section 4.5, we also prove that this type of hole satisfies (H3) with $N_k = \mathcal{O}(k^4)$. This is because in a homogeneity strip $\mathbb{H}_k$ in a neighborhood of one of the infinite horizon points, there are $\mathcal{O}(k^4)$ components of $M \setminus \mathcal{S}_1$ that intersect $\mathbb{H}_k$.

We remark that if we formulate (H1) for curves in $\mathcal{W}^s$ rather than $\mathcal{W}^s_1$, a single hole of this type will satisfy (H1) with $B_0 = B_0(N_\omega)$. However, we cannot choose holes of this type sufficiently small and belonging to $\mathcal{H}(B_0, B_1)$ for fixed B_0 because as we shrink ω, N_ω and therefore $B_0(N_\omega)$ increases without bound. Indeed, B_0 increases like k^4, where k is the least index of homogeneity strips intersected by those components of $M \setminus \mathcal{S}_1$ with index greater than N_ω. Thus this type of hole cannot be analyzed in the framework of [D3].

Example 2: A slit in an infinite horizon corridor.

A more extreme example is a hole in the form of a slit in the billiard table. If the slit is oriented parallel to the infinite horizon corridor (ω would be a horizontal line segment in Figure 1, top), the induced hole H_ω will have countably many components accumulating on one of the infinite horizon points. In this case, there is no index N_ω as in Example 1; rather the structure of components of H_ω in each component of $M \setminus \mathcal{S}_1$ is the same as that of Example 1 and simply continues without end as it accumulates on the infinite horizon point.

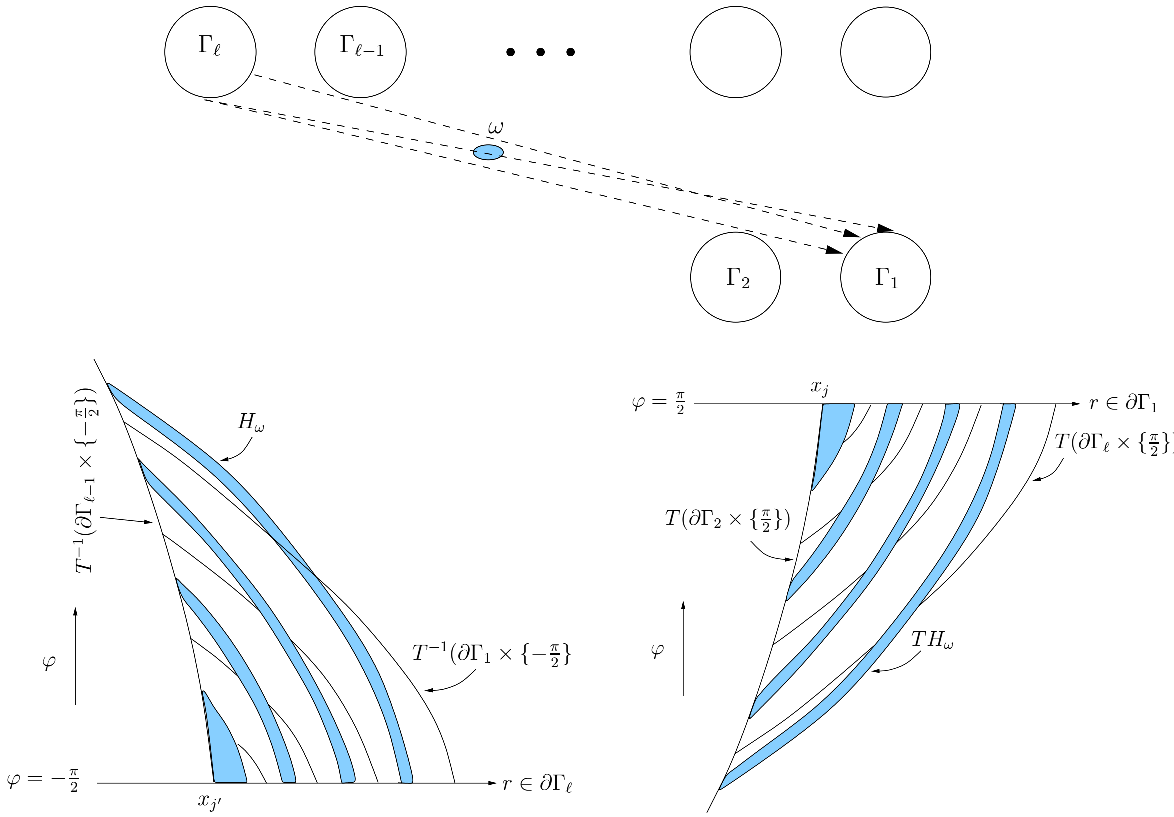

Figure 1: *Top:* A hole ω of Type II in an infinite horizon corridor. Unfolding the billiard table, we number the scatterers Γ_i, $i \geq 1$. Sample trajectories are shown which induce the hole H_ω in $\partial\Gamma_\ell$ and its forward shadow TH_ω in $\partial\Gamma_1$. *Bottom left:* The induced hole H_ω in the component of the phase space M corresponding to Γ_ℓ in a neighborhood of the infinite horizon point $x_{j'}$. Singularity curves belonging to $\mathcal{S}_1$ are also shown; these curves typically pass through H_ω until the index is high enough that H_ω contains all curves in $\mathcal{S}_1$ accumulating on $x_{j'}$. *Bottom right:* The forward shadow TH_ω in the component of the phase space M corresponding to Γ_1 in a neighborhood of the infinite horizon point x_j. Singularity curves in $\mathcal{S}_{-1}$ are also shown.

As before, (H1) is satisfied with $B_0 = 3$, the proof of (H2) is postponed until Section 4.5 and (H3) is satisfied with $N_k = \mathcal{O}(k^4)$.

This hole would fail assumption (H1) if (H1) were formulated for curves in $\mathcal{W}^s$ rather than $\mathcal{W}^s_1$. As explained above, in a neighborhood of an infinite horizon point, a homogeneity strip $\mathbb{H}_k$ intersects $\mathcal{O}(k^4)$ curves in S_1. Thus a homogeneous stable curve could be cut into $\mathcal{O}(k^4)$ pieces in $\mathbb{H}_k$ by ∂H_ω, and taking the supremum over k, we would violate (H1). Thus this type of hole also does not satisfy the assumptions used in [D3].

Holes of Type II which lie entirely outside of infinite horizon corridors also satisfy (H1)–(H3), but the results of [D3] apply to these holes so we do not elaborate on them here. If we take ω to be an arc in ∂Q (called a hole of Type I in [DWY1]), this induces a hole H_ω which is a single vertical rectangle in M. All holes of Type I satisfy (H1)–(H3) above and also fit the framework of [D3]. We leave the interested reader to generate more examples of holes ω in Q which satisfy (H1)–(H3) (see [DWY1, D3] for some ideas).

2.3 Transfer operator

Since the Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$ that we shall work with are spaces of distributions, we begin by defining the *transfer operator* $\mathcal{L}$ associated with T in a general setting. For a smooth test function ψ, $\psi \circ T$ is no longer smooth; in order to bypass this problem, we first define $\mathcal{L}$ acting on scales of spaces defined using the set of stable curves $\mathcal{W}^s_1$, defined in Section 3.1. We will then embed $\mathcal{B}$ and $\mathcal{B}_w$ into these spaces.

Define $T^{-n}\mathcal{W}^s_1$ to be the set of homogeneous stable curves W such that T^n is smooth on W and $T^iW \in \mathcal{W}^s_1$ for $0 \le i \le n$. It follows from the invariance of $\mathcal{W}^s_1$ that $T^{-n}\mathcal{W}^s_1 \subset \mathcal{W}^s_1$. We define $T^{-n}\mathcal{W}^s$ similarly.

Let $\mathcal{F}_b$ denote the set of bounded, measurable complex-valued functions on M. For $W \in T^{-n}\mathcal{W}^s_1$, $\psi \in \mathcal{F}_b$ and $0 < p \le 1$ define $H^p_W(\psi)$ to be the Hölder constant with exponent p of ψ on W measured according to arclength. Define $H^p_n(\psi) = \sup_{W \in T^{-n}\mathcal{W}^s_1} H^p_W(\psi)$ and let $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s) = \{\psi \in \mathcal{F}_b : H^p_n(\psi) < \infty\}$, denote the set of complex-valued functions which are Hölder continuous on elements of $T^{-n}\mathcal{W}^s_1$. The set $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s_1)$ equipped with the norm $|\psi|_{\mathcal{C}^p(T^{-n}\mathcal{W}^s_1)} = |\psi|_\infty + H^p_n(\psi)$ is a Banach space. Recalling that $\mathcal{S}_n$ denotes the singularity set for T^n, we define $\mathcal{C}^p(T^{-n}\mathcal{W}^s_1)$ to be the closure of $\mathcal{C}^1(M \setminus \mathcal{S}_n)$ in $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s_1, \mathbb{C})$.[2]

It follows from uniform hyperbolicity (3.1) that if $\psi \in \tilde{\mathcal{C}}^p(T^{-(n-1)}\mathcal{W}^s_1)$, then $\psi \circ T \in \tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s_1)$. Similarly, if $\xi \in \mathcal{C}^1(M \setminus \mathcal{S}_{n-1})$, then $\xi \circ T \in \mathcal{C}^1(T^{-n}\mathcal{W}^s_1)$. These two facts together imply that for $p < 1$, if $\psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s_1)$, then $\psi \circ T \in \mathcal{C}^p(T^{-n}\mathcal{W}^s_1)$.

If $f \in (\mathcal{C}^p(T^{-n}\mathcal{W}^s_1))'$, is an element of the dual of $\mathcal{C}^p(T^{-n}\mathcal{W}^s_1)$, then $\mathcal{L} : (\mathcal{C}^p(T^{-n}\mathcal{W}^s_1))' \to (\mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s_1))'$ acts on f by

$$\mathcal{L}f(\psi) = f(\psi \circ T) \quad \forall \psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s_1).$$

We denote (normalized) Lebesgue measure on M by m. If $f \in L^1(M, m)$, then f is canonically identified with a signed measure absolutely continuous with respect to Lebesgue, which we shall also call f, i.e.,

$$f(\psi) = \int_M \psi f \, dm.$$

With the above identification, we write $L^1(M, m) \subset (\mathcal{C}^p(T^{-n}\mathcal{W}^s_1))'$ for each $n \in \mathbb{N}$. Then restricted to $L^1(M, m)$, $\mathcal{L}$ acts according to the familiar expression

$$\mathcal{L}^n f = f \circ T^{-n} \, |DT^n(T^{-n})|^{-1}$$

for any $n \ge 0$ and any $f \in L^1(M, m)$, where $|DT^n|$ denotes $|\det DT^n|$ to simplify notation. When we wish to be explicit about the dependence of $\mathcal{L}$ on a map T, we will use the notation $\mathcal{L}_T$.

After the introduction of a hole $H \subset M$, we define the transfer operator $\mathring{\mathcal{L}}$ corresponding to $\mathring{T} = T|_{\mathring{M}^1}$ by

$$\mathring{\mathcal{L}} = \mathbf{1}_{\mathring{M}} \mathcal{L} \mathbf{1}_{\mathring{M}} = \mathcal{L} \mathbf{1}_{\mathring{M}^1}, \tag{2.1}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. For any test function $\psi \in \mathcal{C}^p(\mathcal{W}^s_1)$ and $f \in (\mathcal{C}^p(T^{-1}\mathcal{W}^s_1)'$, we have

$$\mathring{\mathcal{L}} f(\psi) = \mathbf{1}_{\mathring{M}} \mathcal{L}(\mathbf{1}_{\mathring{M}} f)(\psi) = \mathcal{L}(\mathbf{1}_{\mathring{M}} f)(\psi \mathbf{1}_{\mathring{M}}) = f(\psi \circ T \cdot \mathbf{1}_{\mathring{M}^1}),$$

using the fact that $1_{\mathring{M}} \mathbf{1}_{\mathring{M}} \circ T = \mathbf{1}_{\mathring{M}^1}$. Iterating this expression, we have for each $n \in \mathbb{N}$,

$$\mathring{\mathcal{L}}^n f(\psi) = f(\psi \circ T^n \cdot \mathbf{1}_{\mathring{M}^n}).$$

When we wish to be explicit about the dependence of $\mathring{\mathcal{L}}$ on H, we use the notation $\mathring{\mathcal{L}}_H$.

[2]Here by $\tilde{\mathcal{C}}^1$, we mean $\tilde{\mathcal{C}}^p$ with $p = 1$.

2.4 Statement of Results

In Section 3.4 we shall define Banach spaces $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}_w, |\cdot|_w)$ similar to those used in [DZ1, DZ2], but using the smaller set of stable curves $\mathcal{W}^s_1$. We will prove that $\mathcal{L}_T$ has a spectral gap on $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ using these norms and use this to prove a spectral gap for $\mathring{\mathcal{L}}_H$. We postpone the definition of these norms and first state our main results.

For all our results, we assume our map T is the billiard map of an infinite horizon Lorentz gas with the properties described in Section 2.1. Our first result establishes that the usual characterization of our Banach spaces still holds despite the countable partition of the phase space that we have adopted.

Theorem 2.1. *Let $\beta, p > 0$ be from the definition of the norms, Section 3.4. For $s > \beta/(1-\beta)$ and each $n \geq 0$, there is a sequence of continuous, injective embeddings, $\mathcal{C}^s(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^p(T^{-n}\mathcal{W}^s)'$. Moreover, the unit ball of $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is compactly embedded in $(\mathcal{B}_w, |\cdot|_w)$.*

$\mathcal{L}$ is well-defined as a continuous operator on both $\mathcal{B}$ and $\mathcal{B}_w$. In addition, $\mathcal{L}$ has a spectral gap on $\mathcal{B}$, i.e. 1 is a simple eigenvalue (whose normalized eigenvector is μ_{SRB}) and all other eigenvalues have modulus bounded by σ_0 for some $\sigma_0 < 1$.

In order to obtain information about the spectrum of $\mathring{\mathcal{L}}$ from the spectrum of $\mathcal{L}$, we will use the perturbative framework of Keller and Liverani [KL]. This framework requires two ingredients: (1) uniform Lasota-Yorke inequalities along a sequence of holes; (2) smallness of the perturbation in the following norm:

$$|||\mathcal{L}||| := \{|\mathcal{L}f|_w : \|f\|_{\mathcal{B}} \leq 1\}. \tag{2.2}$$

The following two propositions establish these ingredients.

Proposition 2.2. *Fix $B_0, B_1 > 0$ and let $\mathcal{H}(B_0, B_1)$ denote the corresponding family of holes satisfying (H1)–(H3). Then there exist constants $C > 0$, $\sigma < 1$ depending only on T, B_0 and B_1 such that for all $H \in \mathcal{H}(B_0, B_1)$ and $n \in \mathbb{N}$,*

$$\|\mathring{\mathcal{L}}^n_H f\|_{\mathcal{B}} \leq C\sigma^n \|f\|_{\mathcal{B}} + C|f|_w \quad \text{for all } f \in \mathcal{B}; \tag{2.3}$$

$$\|\mathcal{L}^n f\|_{\mathcal{B}} \leq C\sigma^n \|f\|_{\mathcal{B}} + C|f|_w \quad \text{for all } f \in \mathcal{B}; \tag{2.4}$$

$$|\mathring{\mathcal{L}}^n_H f|_w \leq C|f|_w \quad \text{for all } f \in \mathcal{B}_w; \tag{2.5}$$

$$|\mathcal{L}^n f|_w \leq C|f|_w \quad \text{for all } f \in \mathcal{B}_w. \tag{2.6}$$

Let $|W|$ denote the arclength of a curve $W \subset M$. For a hole $H \subset M$, we define $\mathrm{diam}_s(H) = \sup_{W \in \mathcal{W}^s_1} |W \cap H|$ and refer to this quantity as the *stable diameter* of H. We define the unstable diameter $\mathrm{diam}_u(H)$ similarly using curves in $\mathcal{W}^u$.

Proposition 2.3. *Suppose H is a hole satisfying (H1)–(H3) and let $h = \mathrm{diam}_s(H)$. Then there exists $C > 0$, depending only on B_0, B_1 and T, such that*

$$|||\mathcal{L} - \mathring{\mathcal{L}}_H||| \leq Ch^{\alpha-\gamma},$$

where $0 < \gamma < \alpha$ are from the norms, Section 3.4.

Theorem 2.4. *Fix $B_0, B_1 > 0$. Then for all $H \in \mathcal{H}(B_0, B_1)$ with $\mathrm{diam}_s(H)$ sufficiently small, $\mathring{\mathcal{L}}_H$ has a spectral gap on $\mathcal{B}$. Its eigenvalue of maximum modulus $\lambda_H < 1$ is real and its associated eigenvector $\mu_H \in \mathcal{B}$ is a conditionally invariant measure for $\mathring{T}$ that is singular with respect to Lebesgue measure.*

Moreover, for any probability measure $\mu \in \mathcal{B}$ such that $\lim_{n\to\infty} \lambda_H^{-n} \mathring{\mathcal{L}}^n_H \mu \neq 0$, we have, *(a)* $\rho(\mu) = \log \lambda_H$*;*

(b) $\left\| \frac{\mathring{\mathcal{L}}_H^n \mu}{|\mathring{\mathcal{L}}_H^n \mu|} - \mu_H \right\|_{\mathcal{B}} \leq C\sigma_1^n$*, for some* $C > 0$*,* $\sigma_1 < 1$*.*

In particular, both Lebesgue measure and the smooth invariant measure μ_{SRB} *for* T *have the same escape rate and converge to* μ_H *under the normalized action of* $\mathring{T}$*.*

We remark that the characterization of μ_H via the limit in (b) above is very important in open systems. Infinitely many conditionally invariant measures have been shown to exist under quite general conditions for any $0 \leq \lambda < 1$ [DY] so that we are not interested in existence results for such measures. Rather, we are interested in conditionally invariant measures with physical properties: Measures that can be realized as the limit of (renormalized) Lebesgue measure or other physically relevant initial distributions (as in (b) above) or that describe the rate of escape with respect to a large class of reference measures (as in (a)).

Theorem 2.5. *Let* H_ε *be a sequence of holes in* $\mathcal{H}(B_0, B_1)$ *such that* $diam_s(H_\varepsilon) \leq \varepsilon$*. Let* μ_ε *denote the conditionally invariant measures corresponding to* λ_ε *from Theorem 2.4. Then*

$$\lim_{\varepsilon \to 0} |\mu_\varepsilon - \mu_{SRB}|_w = 0,$$

and $\lambda_\varepsilon \to 1$ *as* $\varepsilon \to 0$*.*

Note that convergence in the weak norm $|\cdot|_w$ implies the weak convergence of measures.

Next we proceed to study the connection between escape rate and pressure on the survivor set. Let $\mathcal{M}_{\mathring{T}}$ denote the set of ergodic, $\mathring{T}$-invariant probability measures supported on $\mathring{M}^\infty$. Following [DWY2], we define a class of invariant measures by

$$\mathcal{G}_H = \{\nu \in \mathcal{M}_{\mathring{T}} : \exists C, r > 0 \text{ such that } \forall \varepsilon > 0, \nu(N_\varepsilon(\mathcal{S}_0 \cup \partial H)) \leq C\varepsilon^r\}. \tag{2.7}$$

If $H = \emptyset$, the condition on $N_\varepsilon(\mathcal{S}_0)$ is the same as that used in [KS] to ensure the existence of Lyapunov exponents and stable and unstable manifolds for ν-a.e. point. Thus this restriction, or something like it, on the class of invariant measures is necessary for maps with singularities.

Theorem 2.6. *Suppose* $H \in \mathcal{H}(B_0, B_1)$ *satisfies the assumptions of Theorem 2.4. If* $diam_u(H)$ *is sufficiently small, then*

$$\rho(m) = \log \lambda_H = \sup_{\nu \in \mathcal{G}_H} \{h_\nu(T) - \chi_\nu^+(T)\}.$$

Moreover, we may define a measure ν_H *via the limit,*

$$\nu_H(\psi) = \lim_{n \to \infty} \lambda_H^{-n} \mu_H(\psi \cdot \mathbf{1}_{\mathring{M}^n}) \quad \textit{for all } \psi \in \mathcal{C}^0(M),$$

and ν_H *is an invariant probability measure for* $\mathring{T}$ *belonging to* $\mathcal{G}_H$ *that achieves the supremum in the variational principle above, i.e.* $\rho(m) = h_{\nu_H}(T) - \chi_{\nu_H}^+(T)$*.*

Escape rate formulas have been proved in several of the references mentioned in the introduction; for such systems, variational principles are often formulated in terms of the associated symbolic dynamics. Here, we follow the general strategy of [DWY2], which contains variational principles and inequalities for nonuniformly hyperbolic systems without appealing to symbolic dynamics.

3 Functional Analytic Framework

In this section, we define the Banach spaces on which the transfer operators $\mathcal{L}$ and $\mathring{\mathcal{L}}_H$ have a spectral gap. We focus on establishing some fundamental properties of those spaces in order to prove Theorem 2.1. We leave the analysis of the map with holes to Section 4. We begin by recalling some important properties of T and defining the families of stable and unstable curves we shall use.

3.1 Hyperbolicity and Singularities

Let $\mathcal{K}_{\min} > 0$ and $\mathcal{K}_{\max} < \infty$ denote the minimum and maximum curvatures of ∂B_i. Following [CM], we may define global stable and unstable cones in the tangent spaces $\mathcal{T}_x M$ by

$$\tilde{C}^s(x) = \{(dr, d\varphi) \in \mathcal{T}_x M : -\mathcal{K}_{\max} - \tau_{\min}^{-1} \le d\varphi/dr \le -\mathcal{K}_{\min}\},$$
$$C^u(x) = \{(dr, d\varphi) \in \mathcal{T}_x M : \mathcal{K}_{\min} \le d\varphi/dr \le \mathcal{K}_{\max} + \tau_{\min}^{-1}\},$$

so that $DT(C^u(x)) \subset C^u(Tx)$ and $DT^{-1}\tilde{C}^s(x) \subset \tilde{C}^s(T^{-1}x)$ wherever DT and DT^{-1} are defined. Moreover, there exist constants $C_e > 0$, $\Lambda > 1$ such that

$$\|DT^n(x)v\| \ge C_e \Lambda^n \|v\| \;\; \forall v \in C^u(x), \; ; \;\text{ and }\; \|DT^{-n}(x)v\| \ge C_e \Lambda^n \|v\| \;\; \forall v \in \tilde{C}^s(x), \tag{3.1}$$

where $\|\cdot\|$ denotes the Euclidean norm.

In order to better control the set of stable curves for our norms, we will define and use a set of narrower stable cones

$$C^s(x) := DT_{Tx}^{-1}(\tilde{C}^s(Tx)) \subset \tilde{C}^s(x).$$

Note that the family $C^s(x)$ is continuous on each connected component of $M \setminus \mathcal{S}_1$.

We define $\mathcal{W}_1^s$ to be those homogeneous stable curves contained in a single component of $M \setminus \mathcal{S}_1$ whose tangent vectors at each point lie inside the narrower cones $C^s(x)$ and whose curvature is bounded by a uniform constant $B_c > 0$. It is shown in [CM, Section 5.10] that by choosing B_c sufficiently large, we can ensure that the connected components of $T^{-1}W$ are again in $\mathcal{W}_1^s$ for each $W \in \mathcal{W}_1^s$. Finally, we require that curves in $\mathcal{W}_1^s$ have length no greater than δ_0, where δ_0 is chosen in (3.3).

Since we do not need as precise control on unstable curves, we define $\mathcal{W}^u$ to be the set of homogeneous unstable curves whose tangent vectors lie in the global unstable cones $C^u(x)$.

Next we recall the structure of singularity sets of T near infinite horizon points as described in [CM, Section 4.10]. Let x_j denote one of the finitely many infinite horizon points on $\mathcal{S}_0$ in one component M_i of M. There is one curve in $\mathcal{S}_1$, which we denote by $S_{j,0}$, containing x_j as an endpoint and running the full height of M_i to the other boundary in $\mathcal{S}_0$. $S_{j,0}$ corresponds to tangential collisions with a scatterer adjacent to Γ_i. There are countably many curves $S_{j,\ell} \subset \mathcal{S}_1$ which accumulate on x_j. These curves have one endpoint on $S_{j,0}$ and the other on $\mathcal{S}_0$. The distance from $S_{j,\ell}$ to x_j is of order ℓ^{-1} along $\mathcal{S}_0$ and of order $\ell^{-1/2}$ along $S_{j,0}$. The region bounded by $S_{j,0}$, $S_{j,\ell}$, $S_{j,\ell+1}$ and $\mathcal{S}_0$ is called an ℓ-cell and denoted by $D_{j,\ell}^+$. The flight time $\tau(x) \sim \ell$ for $x \in D_{j,\ell}^+$.

Similarly, one can define ℓ-cells $D_{j,\ell}^-$ whose boundaries consist of curves in $\mathcal{S}_{-1}$. It is shown in [CM] that $T^{-1}D_{j,\ell}^- = D_{j',\ell}^+$. If we ignore the hole in Figure 1, then we can see that Figure 1, bottom left, shows singularity curves in $\mathcal{S}_1$ forming the $D_{j',\ell}^+$ cells and Figure 1, bottom right, shows singularity curves in $\mathcal{S}_{-1}$ forming the $D_{j,\ell}^-$ cells.

3.2 Uniform Properties of T

In this section we recall the uniform properties (A1)–(A5) for a hyperbolic map T used in [DZ2] to prove the required Lasota-Yorke inequalities for $\mathcal{L}_T$. The following properties refer to a map without a hole. The importance of the estimates in [DZ2] for us is that the constants appearing in the Lasota-Yorke inequalities depend only on the quantities appearing in (A1)–(A5) and the choices of parameters in the norms. The properties as written in [DZ2] are formulated in an abstract setting. Rather than reformulate this abstract setting, we translate (A1)–(A5) into the concrete setting of the infinite horizon Lorentz gas which we have adopted here. This will make the properties easier to verify and will avoid cumbersome additional notation which serves no purpose. Translated into this setting, the abstract assumptions read as well-known facts about dispersing billiards.

(A1) Jacobian. $|DT(x)| := |\det DT(x)| = \cos\varphi(x)/\cos\varphi(Tx)$ wherever $DT(x)$ exists.

(A2) Hyperbolicity. The set $\mathcal{S}_0$ consists of finitely many curves, although $\mathcal{S}_{\pm n}$, $n \ge 1$, may be finite or countable. There exist families of stable and unstable cones, $\tilde{C}^s(x)$ and $C^u(x)$, continuous on the closure of each component of $M \setminus \mathcal{S}_0$, such that the angle between $\tilde{C}^s(x)$ and $C^u(x)$ is uniformly bounded away from 0 on M. Furthermore, there exist constants $C > 0$, $\Lambda > 1$ such that the following hold.

1. $DT(C^u(x)) \subset C^u(Tx)$ and $DT^{-1}(\tilde{C}^s(x)) \subset \tilde{C}^s(T^{-1}x)$ whenever DT and DT^{-1} exist.
2. $\|DT(x)v\|_* \ge \Lambda\|v\|_*, \forall v \in C^u(x)$ and $\|DT^{-1}(x)v\|_* \ge \Lambda\|v\|_*, \forall v \in \tilde{C}^s(x)$, where $\|\cdot\|_*$ is an adapted norm, uniformly equivalent to the Euclidean norm, $\|\cdot\|$.

(A3) Structure of Singularities.

1. There exists $C_1 > 0$ such that for all $x \in M$,

$$C_1 \frac{\tau(T^{-1}x)}{\cos\varphi(T^{-1}x)} \le \frac{\|DT^{-1}(x)v\|}{\|v\|} \le C_1^{-1} \frac{\tau(T^{-1}x)}{\cos\varphi(T^{-1}x)}, \quad \forall v \in \tilde{C}^s(x).$$

   Let $\exp_x$ denote the exponential map from the tangent space $\mathcal{T}_xM$ to M. Then,

$$\|D^2T^{-1}(x)v\| \le C_1^{-1}\tau^2(T^{-1}x)(\cos\varphi(T^{-1}x))^{-3},$$

   for all $v \in \mathcal{T}_xM$ such that $T^{-1}(\exp_x(v))$ and $T^{-1}x$ lie in the same homogeneity strip.

   Let x_∞ denote one of the finitely many infinite horizon points and let $\{S_\ell\} \subset \mathcal{S}_{-1}$ denote the sequence of curves in $\mathcal{S}_{-1}$ converging to x_∞. Let $M_{\ell,k}$ denote the set of points between S_ℓ and $S_{\ell+1}$ and whose image under T^{-1} lies in $\mathbb{H}_k$. Then there exists a constant $c_s > 0$ such that $k \ge c_s \ell^{1/4}$.

2. There exists $C_2, t_0 > 0$ such that for any stable curve $W \in \mathcal{W}^s_1$ and any smooth curve $S \subset \mathcal{S}_{-n}$, we have $m_W(N_\varepsilon(S) \cap W) \le C_2 \varepsilon^{t_0}$ for all $\varepsilon > 0$ sufficiently small.
3. $\partial\mathbb{H}_k$ are uniformly transverse to the stable cones.
4. There exists $C_3 > 0$ such that for all $W \in \mathcal{W}^s_1$, $W \subset \mathbb{H}_k$, we have $|W| \le C_3 k^{-3}$.
5. For $k \ge k_0$, choose $W_k \subset \mathbb{H}_k$, $W_k \in \mathcal{W}^s_1$. Define $\cos W_k$ to be the average value of $\cos\varphi$ on W_k integrated according to arclength. Then for any such choice of W_k, $\sum_{k \ge k_0} \cos W_k < \infty$.

(A4) Invariant families of stable and unstable curves. There are invariant families of curves $\mathcal{W}^s_1$ and $\mathcal{W}^u$ with the properties described in Sect. 3.1. Moreover, we require the following distortion bounds along stable curves.

There exists $C_d > 0$ such that for any $W \in \mathcal{W}^s_1$ with $T^iW \in \mathcal{W}^s$ for $i = 0, 1, \dots, n$, and any $x, y \in W$,

$$\left| \frac{J_W T^n(x)}{J_W T^n(y)} - 1 \right| \le C_d d_W(x,y)^{1/3} \quad \text{and} \quad \left| \frac{|DT^n(x)|}{|DT^n(y)|} - 1 \right| \le C_d d_W(x,y)^{1/3},$$

where $J_W T(x)$ denotes the Jacobian of T along W.

We also require an analogous distortion bound along unstable curves. If $T^iW \in \mathcal{W}^u$ for $0 \le i \le n$, then for any $x, y \in W$,

$$\left| \frac{|DT^n(x)|}{|DT^n(y)|} - 1 \right| \le C_d d_W(T^n x, T^n y)^{1/3}.$$

(A5) One-step expansion. Let $W \in \mathcal{W}^s_1$ and partition the connected components of $T^{-1}W$ into maximal pieces V_i such that each V_i is a homogeneous stable curve. Let $|J_{V_i}T|_*$ denote the minimum contraction on V_i under T in the metric induced by the adapted norm from (A2)(2). There exists a choice of k_0 for the homogeneity strips such that

$$\lim_{\delta \to 0} \sup_{\substack{W \in \mathcal{W}^s_1 \\ |W| < \delta}} \sum_i |J_{V_i} T|_* < 1. \tag{3.2}$$

We remark that in the context of the infinite horizon Lorentz gas, (A3)(3) and (A3)(4) follow immediately from the definition of $\tilde{C}^s(x)$ and $\mathbb{H}_k$. Also, $\cos W_k \sim k^{-2}$ so that (A3)(5) holds easily as well.

3.3 Representation of Admissible Stable Curves

In light of (A5), we may fix $\delta_0 > 0$ and $\theta_* < 1$ such that

$$\sup_{\substack{W \in \mathcal{W}^s_1 \\ |W| < \delta_0}} \sum_i |J_{V_i} T|_* = \theta_*. \tag{3.3}$$

Note that this also fixes the choice of k_0 for the remainder of the paper.

Since the stable cone $C^s(x)$ is bounded away from the vertical direction, any curve $W \in \mathcal{W}^s_1$ can be viewed as the graph of a function $\varphi_W(r)$ of the arclength coordinate r with derivative uniformly bounded above. For each homogeneous stable curve W, let I_W denote the r-interval on which φ_W is defined and define $G_W(r) = (r, \varphi_W(r))$ so that $W = \{G_W(r) : r \in I_W\}$.[3]

With this view of stable curves, we may redefine $\mathcal{W}^s_1$ to be the set of homogeneous stable curves satisfying $|W| \le \delta_0$ and $|\frac{d^2\varphi_W}{dr^2}| \le B_c$ for some $B_c > 0$. From this point forward, we fix a choice of $B_c > 0$ such that $\mathcal{W}^s_1$ is invariant under T^{-1} in the sense described in Section 3.1.

[3]Our treatment of stable curves here differs from that in [DZ2]. In that abstract setting, stable curves are defined via graphs in charts of the given manifold. In the present more concrete setting, we dispense with charts and use the global (r, φ) coordinates.

We define a distance in $\mathcal{W}_1^s$ as follows. Let $W_i = G_{W_i}(I_i)$, $i = 1,2$ be two curves in $\mathcal{W}_1^s$ with defining functions φ_{W_i}. We denote by $\ell(I_1 \triangle I_2)$ the length of the symmetric difference of the r-intervals on which they are defined. Then the distance between W_1 and W_2 is defined as

$$d_{\mathcal{W}_1^s}(W_1, W_2) = \eta(W_1, W_2) + \ell(I_1 \triangle I_2) + |\varphi_{W_1} - \varphi_{W_2}|_{\mathcal{C}^1(I_1 \cap I_2)},$$

where $\eta(W_1, W_2) = 0$ if W_1 and W_2 lie in the same homogeneity strip in the same component of $M \setminus \mathcal{S}_1$ and $\eta = \infty$ otherwise.

For two functions $\psi_i \in \mathcal{C}^p(W_i)$, we define the distance between them to be,

$$d_q(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathcal{C}^q(I_1 \cap I_2)},$$

where $q < 1$ is from the definition of the strong stable norm in Section 3.4.

3.4 Definition of the Norms

Given a curve $W \in \mathcal{W}_1^s$ and $0 \le p \le 1$, we define $\tilde{\mathcal{C}}^p(W)$ to be the set of complex valued Hölder continuous functions on W with exponent p, with distance measured in the Euclidean metric along W. We set $\mathcal{C}^p(W)$ to be the closure of $\tilde{\mathcal{C}}^1(W)$ in the $\tilde{\mathcal{C}}^p$-norm: $|\psi|_{\mathcal{C}^p(W)} = |\psi|_{\mathcal{C}^0(W)} + H_W^p(\psi)$, where $H_W^p(\psi)$ denotes the Hölder constant of ψ on W as in Sect. 2.3. $\tilde{\mathcal{C}}^p(M)$ and $\mathcal{C}^p(M)$ are defined similarly.

For $\alpha, p \ge 0$, define the following norms for test functions,

$$|\psi|_{W,\alpha,p} := |W|^\alpha \cdot \cos W \cdot |\psi|_{\mathcal{C}^p(W)},$$

where $\cos W$ denotes the average value of $\cos\varphi$ along W integrated with respect to arclength.

We choose constants to define our norms as follows. Choose $\alpha, \gamma > 0$ such that $\gamma < \alpha < \frac{1}{3}$. Next choose $p, q > 0$ such that $q < p < \gamma$ and note that $p < \frac{1}{3}$ necessarily by the restriction on γ. Finally, choose $\beta > 0$ such that $\beta < \min\left\{\frac{t_0(\alpha-\gamma)}{3}, p - q, \frac{1}{3} - \alpha\right\}$.

Given a function $f \in \mathcal{C}^1(M)$, define the *weak norm* of f by

$$|f|_w := \sup_{W \in \mathcal{W}_1^s} \sup_{\substack{\psi \in \mathcal{C}^p(W) \\ |\psi|_{W,\gamma,p} \le 1}} \int_W f\psi \, dm_W.$$

We define the *strong stable norm* of f by

$$\|f\|_s := \sup_{W \in \mathcal{W}_1^s} \sup_{\substack{\psi \in \mathcal{C}^q(W) \\ |\psi|_{W,\alpha,q} \le 1}} \int_W f\psi \, dm_W,$$

and the *strong unstable norm* of f by

$$\|h\|_u := \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}_1^s \\ d_{\mathcal{W}_1^s}(W_1, W_2) < \varepsilon}} \sup_{\substack{\psi_i \in \mathcal{C}^p(W_i) \\ |\psi_i|_{W_i,\gamma,p} \le 1 \\ d_q(\psi_1, \psi_2) < \varepsilon}} \varepsilon^{-\beta} \left| \int_{W_1} f\psi_1 \, dm_W - \int_{W_2} f\psi_2 \, dm_W \right|,$$

where $\varepsilon_0 > 0$ is chosen less than δ_0, the maximum length of $W \in \mathcal{W}_1^s$. We then define the *strong norm* of f by

$$\|f\|_{\mathcal{B}} = \|f\|_s + z\|f\|_u,$$

where z is a small constant chosen in (3.13).

We define $\mathcal{B}$ to be the completion of $\mathcal{C}^1(M)$ in the strong norm and $\mathcal{B}_w$ to be the completion of $\mathcal{C}^1(M)$ in the weak norm.

3.5 Properties of the Banach spaces

We review the essential properties of the Banach spaces proved in [DZ1, DZ2]. Most we will not have to reprove, but several will require separate proofs since our Banach spaces are defined using the smaller set of curves $\mathcal{W}_1^s$ so that we have partitioned the space into a countable number of components according to $M \setminus \mathcal{S}_1$.

We first recall two lemmas from [DZ2] which we shall not have to reprove, but whose explicit statements we shall need to invoke later.

Lemma 3.1. *([DZ2, Lemma 3.5]) Let $\mathcal{P}$ be a (mod 0) countable partition of M into open, simply connected sets such that (1) for each $k \in \mathbb{N}$, there is an $N_k < \infty$ such that at most N_k elements $P \in \mathcal{P}$ intersect $\mathbb{H}_k$; (2) there are constants $K, C_3, t_0 > 0$ such that for each $P \in \mathcal{P}$ and $W \in \mathcal{W}_1^s$, $P \cap W$ comprises at most K connected components and for any $\varepsilon > 0$, $m_W(N_\varepsilon(\partial P) \cap W) \leq C_3 \varepsilon^{t_0}$.*

Let $s > \beta/(1-\beta)$. If $f \in \mathcal{C}^s(P)$ for each $P \in \mathcal{P}$ and $\sup_{P \in \mathcal{P}} |f|_{\mathcal{C}^s(P)} < \infty$, then $f \in \mathcal{B}$. In particular, $\mathcal{C}^s(M) \subset \mathcal{B}$ and both Lebesgue measure and the smooth SRB measure for T are in $\mathcal{B}$.

The lemma above is proved in [DZ2] with $\mathcal{W}^s$ in place of $\mathcal{W}_1^s$, but this change is not essential to the approximation argument given there so we do not repeat its proof.

Lemma 3.2. *([DZ2, Lemma 5.3]) If $f \in \mathcal{B}$ and $\psi \in \mathcal{C}^s(M)$, $s > \max\{\beta/(1-\beta), p\}$, then $\psi f \in \mathcal{B}$. Moreover, $\|\psi f\|_{\mathcal{B}} \leq C \|f\|_{\mathcal{B}} |\psi|_{\mathcal{C}^p(M)}$ for some $C > 0$ independent of ψ and f.*

Our next lemma, which is key to establishing the spectral decomposition of $\mathcal{L}$ on $\mathcal{B}$, goes through with only minor modifications.

Lemma 3.3. *The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.*

Proof. Although our set of curves $\mathcal{W}_1^s$ is much more refined than the set $\mathcal{W}^s$ used in [DZ2], the compactness argument remains essentially unchanged due to the fact that although $M \setminus \mathcal{S}_1$ has countably many components, for any $\varepsilon > 0$, only finitely many of these components have stable curves with length greater than ε.

First notice that on a fixed $W \in \mathcal{W}_1^s$, $|\cdot|_{W,\gamma,p}$ is equivalent to $|\cdot|_{\mathcal{C}^p(W)}$ and $|\cdot|_{W,\alpha,q}$ is equivalent to $|\cdot|_{\mathcal{C}^q(W)}$ so that $p > q$ implies that the unit ball of $|\cdot|_{W,\gamma,p}$ is compactly embedded in $|\cdot|_{W,\alpha,q}$. Since $\|\cdot\|_s$ is the dual of $|\cdot|_{W,\alpha,q}$ and $|\cdot|_w$ is the dual of $|\cdot|_{W,\gamma,p}$ on each curve $W \in \mathcal{W}_1^s$, the unit ball of $\|\cdot\|_s$ is compactly embedded in $|\cdot|_w$ on W. It remains to compare the weak norm on different curves.

Let $0 < \varepsilon \leq \varepsilon_0$ be fixed. Letting $M_1(x)$ denote the homogeneous component of $M \setminus \mathcal{S}_1$ containing x, we split M into two parts:

$$A = \{x \in M : M_1(x) \text{ has at least one stable curve } W \text{ with } |W| \geq \varepsilon\}$$

and $B = A^c$. Note that A contains only finitely many homogeneous components of $M \setminus \mathcal{S}_1$ due to the structure of the singularity sets described in Section 3.1. In particular, there exists $k_\varepsilon \in \mathbb{N}$ such that $\cup_{k > k_\varepsilon} \mathbb{H}_k \subset B$, due to (A3)(4).

Let $h \in \mathcal{C}^1(M)$ with $\|h\|_{\mathcal{B}} \leq 1$. First we estimate the weak norm of h on curves W in B. If $W \subset B$, and $|\psi|_{W,\gamma,p} \leq 1$, then

$$\int_W h\psi \, dm_W \leq \|h\|_s |\psi|_{W,\alpha,q} \leq \|h\|_s |W|^\alpha f(W) |\psi|_{\mathcal{C}^q(W)} \leq C \|h\|_s \varepsilon^{\alpha - \gamma}. \tag{3.4}$$

Now on A, we can use the same approximation argument used in [DZ2, Lemma 3.9] since there are only finitely many homogeneous connected components of $M \setminus \mathcal{S}_1$ in A.

Thus we can find finitely many curves $W_i \in \mathcal{W}^s_1$, $i = 1, \dots N_\varepsilon$, and finitely many functions $\psi_{i,j} \in \mathcal{C}^p(W_i)$, $j = 1, \dots L_\varepsilon$, such that for any $W \in \mathcal{W}^s_1$, $W \subset A$, and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,\gamma,p} \le 1$, we have

$$\left| \int_W h\psi \, dm_W - \int_{W_i} h\psi_{i,j} \, dm_W \right| \le C\varepsilon^\beta \|h\|_u,$$

for some uniform constant $C > 0$ and some choice of i, j. Putting this together with (3.4), we conclude that for each $0 < \varepsilon \le \varepsilon_0$, there exist finitely many bounded linear functionals $\ell_{i,j}$, $\ell_{i,j}(h) = \int_{W_i} h\psi_{i,j} dm_W$, such that

$$|h|_w \le \max_{i \le N_\varepsilon, j \le L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C\|h\|_u + \varepsilon^{\alpha-\gamma} C\|h\|_s \le \max_{i \le N_\varepsilon, j \le L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C z^{-1} \|h\|_{\mathcal{B}},$$

which implies the required compactness. □

3.6 Dynamical Estimates and Proof of Theorem 2.1

In order to prove Theorem 2.1, we first need to describe the action of T on stable curves and recall some growth lemmas proved in [DZ2].

Let $W \in \mathcal{W}^s$ and let V_i denote the maximal connected components of $T^{-1}W$ after cutting due to singularities and the boundaries of the homogeneity strips. To ensure that each component of $T^{-1}W$ is in $\mathcal{W}^s_1$, we subdivide any of the long pieces V_i whose length is $> \delta_0$, where δ_0 was chosen in (3.3). This process is then iterated so that given $W \in \mathcal{W}^s$, we construct the components of $T^{-n}W$, which we call the n^{th} generation $\mathcal{G}_n(W)$, inductively as follows. Let $\mathcal{G}_0(W) = \{W\}$ and suppose we have defined $\mathcal{G}_{n-1}(W) \subset \mathcal{W}^s$. First, for any $W' \in \mathcal{G}_{n-1}(W)$, we partition $T^{-1}W'$ into at most countably many pieces W'_i so that T is smooth on each W'_i and each W'_i is a homogeneous stable curve. If any W'_i have length greater than δ_0, we subdivide those pieces into pieces of length between $\delta_0/2$ and δ_0. We define $\mathcal{G}_n(W)$ to be the collection of all pieces $W^n_i \subset T^{-n}W$ obtained in this way. Note that even if $W \in \mathcal{W}^s \setminus \mathcal{W}^s_1$, it is still the case that each W^n_i is in $\mathcal{W}^s_1$ for $n \ge 1$.

We recall the results of [DZ1, Section 3.2] in the following growth lemma, which is proved by iterating the one-step expansion (A5) using mainly combinatorial arguments. It does not need to be reproved since the sets $\mathcal{G}_n(W)$ are the same as those appearing in [DZ2].

Lemma 3.4. *([DZ1]) Let $W \in \mathcal{W}^s$ and for $n \ge 0$, let $\mathcal{G}_n(W)$ be defined as above. There exists $C_4 > 0$, independent of W and depending only on the constants appearing in (A1)–(A5), such that for any $n \ge 0$ and any $0 \le \varsigma \le 1$,*

$$\sum_{W^n_i \in \mathcal{G}_n(W)} \frac{|W^n_i|^\varsigma}{|W|^\varsigma} \cdot |J_{W^n_i} T^n|_{\mathcal{C}^0(W^n_i)} \le C_4.$$

The final lemma of this section is needed to prove the continuity of the embedding $\mathcal{B}_w \to (\mathcal{C}^p(T^{-n}\mathcal{W}^s_1))'$ and is also essential to showing that the peripheral spectrum of $\mathcal{L}$ on $\mathcal{B}$ is comprised of measures. It does need to be reproved since the argument is heavily affected by the additional cutting we have introduced according to $\mathcal{S}_1$.

Lemma 3.5. *There exists $C > 0$ such that for each $f \in \mathcal{C}^1(M)$, $n \ge 0$, and $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s_1)$, we have*

$$\left| \int_M f\psi \, dm \right| \le C|f|_w(|\psi|_\infty + H^p_n(\psi)).$$

Proof. Define $\mathbb{H}_0 = M \setminus (\cup_{k \ge k_0} \mathbb{H}_k)$. We partition each component of $\mathbb{H}_0 \cap (M \setminus \mathcal{S}_1)$ into finitely many boxes B_j whose boundary curves are elements of $\mathcal{W}^s_1$ and $\mathcal{W}^u$ as well as the boundary of $\mathbb{H}_0 \cap (M \setminus \mathcal{S}_1)$. We construct the boxes so that each B_j has diameter $\le \delta_0$ and is foliated by curves $W \in \mathcal{W}^s_1$. On each B_j, we choose a smooth foliation $\{W_\xi\}_{\xi \in E_j} \subset \mathcal{W}^s_1$, each of whose elements completely crosses B_j in the approximate stable direction. This is possible since the stable cones $C^s(x)$ are continuous up to the closure of each component of $M \setminus \mathcal{S}_1$.

We decompose Lebesgue measure on B_j into $dm = \hat{m}(d\xi) dm_\xi$, where m_ξ is the conditional measure of m on W_ξ and $\hat{m}$ is the transverse measure on E_j. We normalize the measures so that $m_\xi(W_\xi) = |W_\xi|$. Since the foliation is smooth, $dm_\xi = \rho_\xi dm_W$ where $C^{-1} \le |\rho_\xi|_{\mathcal{C}^1(W_\xi)} \le C$ for some constant C independent of ξ. Note that $\hat{m}(E_j) \le C\delta_0$ due to the transversality of curves in $\mathcal{W}^s$ and $\mathcal{W}^u$.

Next in each homogeneity strip $\mathbb{H}_k$, $k \ge k_0$, we define $D^k_{j,\ell} := D^+_{j,\ell} \cap \mathbb{H}_k$, where $D^+_{j,\ell}$ is defined as in Section 3.1. On each $D^k_{j,\ell}$, we choose a smooth foliation $\{W_\xi\}_{\xi \in E^k_{j,\ell}} \subset \mathcal{W}^s_1$ whose elements fully cross $D^k_{j,\ell}$. We again decompose m on each $D^k_{j,\ell}$ into $dm = \hat{m}(d\xi) dm_\xi$, $\xi \in E^k_{j,\ell}$, and $dm_\xi = \rho_\xi dm_W$ is normalized as above.

Recall that the boundary curves of $D^+_{j,\ell}$ belong to $\mathcal{S}_1$ and so are uniformly transverse to the horizontal boundaries of the homogeneity strips. Each $D^+_{j,\ell}$ is subdivided into countably many components $D^k_{j,\ell}$ for $k \ge c_s \ell^{1/4}$ for some uniform constant $c_s > 0$ by (A3)(1). Thus since typical curves $\{W_\xi\}_{\xi \in E^k_{j,\ell}}$ have length $\mathcal{O}(k^{-3})$, we must have $\hat{m}(E^k_{j,\ell})$ at most $C\ell^{-1}$ due to the spacing between boundary curves of $D^+_{j,\ell}$ described in Section 3.1.

There will also be large parts of $\mathbb{H}_k$ that do not belong to any $D^+_{j,\ell}$. These we also foliate with families of curves in $\mathcal{W}^s_1$ which completely cross $\mathbb{H}_k$ completely in the stable direction. We label these index sets E_k, note that there are only finitely many of them for each $\mathbb{H}_k$ and that the transverse measure $\hat{m}(E_k)$ is of order 1.

Now let $h \in \mathcal{C}^1(M)$ and $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s_1)$. Notice that since $M = T^{-n}M$ (mod 0), we have $\int_M h\psi \, dm = \int_M \mathcal{L}^n h \, \psi \circ T^{-n} \, dm$. We estimate the second integral on each connected component M_s of M, $s = 1, \dots d$, where d is the number of scatterers.

$$
\begin{aligned}
\int_{M_s} \mathcal{L}^n h \, \psi \circ T^{-n} \, dm &= \sum_j \int_{B_j} \mathcal{L}^n h \, \psi \circ T^{-n} \, dm + \sum_{k \ge k_0} \int_{\mathbb{H}_k} \mathcal{L}^n h \, \psi \circ T^{-n} \, dm \\
&= \sum_j \int_{E_j} \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W d\hat{m}(\xi) + \sum_{k \ge k_0} \int_{E_k} \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W d\hat{m}(\xi) \\
&\quad + \sum_{j,\ell} \sum_{k > c_s \ell^{1/4}} \int_{E^k_{j,\ell}} \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W d\hat{m}(\xi)
\end{aligned}
\tag{3.5}
$$

We change variables and estimate the integrals on one W_ξ at a time. Letting $W^n_{\xi,i}$ denote the components of $\mathcal{G}_n(W_\xi)$ defined at the beginning of this section, we define $J_{W^n_{\xi,i}} T^n$ to be the stable Jacobian of T^n along the curve $W^n_{\xi,i}$, and write

$$
\begin{aligned}
\int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, dm_W &= \sum_{i \in \mathcal{G}_n(W_\xi)} \int_{W^n_{\xi,i}} h\psi |DT^n|^{-1} J_{W^n_{\xi,i}} T^n \, \rho_\xi \circ T^n \, dm_W \\
&\le \sum_{i \in \mathcal{G}_n(W_\xi)} |h|_w \cos(W^n_{\xi,i}) |W^n_{\xi,i}|^\gamma |\psi|_{\mathcal{C}^p(W^n_{\xi,i})} |\rho_\xi \circ T^n|_{\mathcal{C}^p(W^n_{\xi,i})} ||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^p(W^n_{\xi,i})}.
\end{aligned}
$$

Since $p \le 1/3$, the distortion bounds given by (A4) imply that

$$||DT^n|^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^p(W^n_{\xi,i})} \le (1+2C_d)||DT^n|^{-1} J_{W^n_i} T^n|_{\mathcal{C}^0(W^n_{\xi,i})}. \tag{3.6}$$

Since $\cos\varphi(x)$ has bounded distortion on each $\mathbb{H}_k$, by (A1) there exists a uniform constant $C > 0$ such that

$$\cos(W^n_{\xi,i})\, ||DT^n|^{-1}|_{C^0(W^n_{\xi,i})} \le C\cos(W_\xi). \tag{3.7}$$

Moreover, for $x, y \in W^n_{\xi,i}$, it follows from (3.1) (or equivalently (A2)) that

$$\frac{|\rho_\xi(T^n x) - \rho_\xi(T^n y)|}{d_W(T^n x, T^n y)^p} \cdot \frac{d_W(T^n x, T^n y)^p}{d_W(x,y)^p} \le |\rho_\xi|_{\mathcal{C}^p(W_\xi)} C_e \Lambda^{-pn} \tag{3.8}$$

and so $|\rho_\xi \circ T^n|_{\mathcal{C}^p(W^n_{\xi,i})} \le C|\rho_\xi|_{\mathcal{C}^p(W_\xi)} \le C$ for some uniform constant C. Putting these estimates together yields,

$$\begin{aligned}\int_{W_\xi} \mathcal{L}^n h\, \psi\circ T^{-n}\, \rho_\xi\, dm_W &\le C|h|_w(|\psi|_\infty + H^p_n(\psi))\cos(W_\xi)|W_\xi|^\gamma \sum_i \frac{|W^n_{\xi,i}|^\gamma}{|W_\xi|^\gamma} |J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})} \\ &\le C|h|_w(|\psi|_\infty + H^p_n(\psi))\cos(W_\xi)|W_\xi|^\gamma,\end{aligned} \tag{3.9}$$

where in the last line we have used Lemma 3.4 with $\varsigma = \gamma$ for the sum .

Now we use (3.9) to estimate each of the three sums appearing in (3.5). For the first sum in (3.5), we combine (3.9) with the fact that there are only finitely many index sets E_j to write

$$\begin{aligned}\sum_j \int_{E_j}\int_{W_\xi} \mathcal{L}^n h\, \psi\circ T^{-n}\, \rho_\xi\, dm_W d\hat{m}(\xi) &\le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \sum_j \int_{E_j} \cos(W_\xi)|W_\xi|^\gamma d\hat{m}(\xi) \\ &\le C|h|_w(|\psi|_\infty + H^p_n(\psi)),\end{aligned} \tag{3.10}$$

since $\hat{m}(E_j)$ is of order 1 for each j.

For the second sum in (3.5), we use the fact that $\cos(W_\xi) \le Ck^{-2}$ and $|W_\xi| \le Ck^{-3}$ for $W_\xi \subset \mathbb{H}_k$ together with (3.9),

$$\begin{aligned}\sum_{k\ge k_0} \int_{E_k}\int_{W_\xi} \mathcal{L}^n h\, \psi\circ T^{-n}\, \rho_\xi\, dm_W d\hat{m}(\xi) &\le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \sum_{k\ge k_0} \int_{E_k} \cos(W_\xi)|W_\xi|^\gamma d\hat{m}(\xi) \\ &\le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \sum_{k\ge k_0} k^{-2-3\gamma}\hat{m}(E_k),\end{aligned} \tag{3.11}$$

and the sum is finite since $\hat{m}(E_k)$ is of order 1 for each k.

Finally we estimate the third sum in (3.5), always using (3.9) and the previously recalled fact that $\hat{m}(E^k_{j,\ell}) \le C\ell^{-1}$.

$$\begin{aligned}\sum_{j,\ell}\sum_{k>c_s\ell^{1/4}} \int_{E^k_{j,\ell}}\int_{W_\xi} \mathcal{L}^n h\, \psi\circ T^{-n}\, \rho_\xi\, dm_W d\hat{m}(\xi) &\le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \sum_{j,\ell}\sum_{k>c_s\ell^{1/4}} k^{-2-3\gamma}\hat{m}(E^k_{j,\ell}) \\ &\le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \sum_{j,\ell} \ell^{(-1-3\gamma)/4}\ell^{-1},\end{aligned} \tag{3.12}$$

and the last sum is finite since there are only finitely many infinite horizon points x_j.

Putting together (3.10), (3.11) and (3.12) with (3.5) yields

$$\Big|\int_{M_s} \mathcal{L}^n h\, \psi \circ T^{-n}\, dm\Big| \le C|h|_w(|\psi|_\infty + H_n^p(\psi))$$

from which the lemma follows since there are only finitely many components M_s. □

We are finally ready to give the proof of Theorem 2.1, which follows from the facts we have established and several others proved in [DZ2].

Proof of Theorem 2.1. First we show that there is a sequence of continuous and injective embeddings $\mathcal{C}^s(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^p(M))'$ for all $s > \beta/(1-\beta)$. The continuity of the first embedding follows from Lemma 3.1 and that of the second follows from the definition of the norms since $|\cdot|_w \le \|\cdot\|_{\mathcal{B}}$. The continuity of the last embedding follows from Lemma 3.5. The injectivity of the first embedding is obvious while the injectivity of the second relies on the fact that we use test functions in $\mathcal{C}^q(W)$ in the strong stable norm rather than $\tilde{\mathcal{C}}^q(W)$. Finally, the injectivity of the third embedding follows from [DZ2, Lemma 3.8].

Now we turn to the action of $\mathcal{L}$ on $\mathcal{B}$. Lemma 3.1 implies that $\mathcal{L}f \in \mathcal{B}$ whenever $f \in \mathcal{C}^1(M)$ and the proof can be taken word for word from [DZ2, Lemma 3.6]. Moreover, under assumptions (A1)–(A5), it is proved in [DZ2] that

Lemma 3.6. *[DZ2, Prop. 2.2] There exists $C > 0$ such that for all $f \in \mathcal{B}$ and $n \ge 0$,*

$$\begin{aligned}
&|\mathcal{L}^n f|_w \le C|f|_w\,,\\
&\|\mathcal{L}^n f\|_s \le C(\theta_*^{n(1-\alpha)} + \Lambda^{-qn})\|f\|_s + C\delta_0^{\gamma-\alpha}|f|_w\,,\\
&\|\mathcal{L}^n f\|_u \le C\Lambda^{-\beta n}\|f\|_u + Cn\|f\|_s\,,
\end{aligned}$$

where C depends only on the constants appearing in (A1)–(A5) and the choices of constants in the norms.

Although these inequalities are proved for norms defined on the larger set of curves $\mathcal{W}^s$ in [DZ2], they do not have to be reproved here. This is because $\mathcal{W}^s_1 \subset \mathcal{W}^s$ and both sets are closed under the action of T^{-1}. Thus the same inequalities hold when we take the supremum over curves in $\mathcal{W}^s_1$.

The above lemma implies (2.6) immediately and the traditional Lasota-Yorke inequality (2.4) also holds using the following standard argument. Choose $1 > \sigma > \max\{\Lambda^{-\beta}, \Lambda^{-q}, \theta_*^{n(1-\alpha)}\}$. Then there exists $N > 0$ large enough such that

$$\begin{aligned}
\|\mathcal{L}^N f\|_{\mathcal{B}} = \|\mathcal{L}^N f\|_s + z\|\mathcal{L}^N f\|_u &\le \frac{\sigma^N}{2}\|f\|_s + C\delta_0^{\gamma-\alpha}|f|_w + z\sigma^N\|f\|_u + zCN\|f\|_s\\
&\le \sigma^N\|f\|_{\mathcal{B}} + C\delta_0^{\gamma-\alpha}|f|_w,
\end{aligned} \tag{3.13}$$

provided z is chosen small enough with respect to N. This, together with the compactness lemma, Lemma 3.3, yields the quasi-compactness of $\mathcal{L}$ via the standard Hennion argument [HH].

Once quasi-compactness has been established, the peripheral spectrum of $\mathcal{L}$ is proved in [DZ2, Section 5] to consist of finitely many cyclic groups whose eigenspaces consist of measures. Moreover, all physical measures belong to $\mathcal{B}$ and they form a basis for the eigenspace corresponding to 1 in $\mathcal{B}$.[4] These arguments are general and do not have to be repeated here.

[4]Recall that a physical measure for T is an ergodic, invariant probability measure μ for which there exists a positive Lebesgue measure set B_μ, with $\mu(B_\mu) = 1$, such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \psi(T^i x) = \mu(\psi)$ for all $x \in B_\mu$ and all continuous functions ψ.

Finally, since μ_{SRB} is known to be a mixing invariant measure for T and μ_{SRB} has full support in M, it follows that $\mathcal{L}$ has a spectral gap on $\mathcal{B}$, i.e. 1 is a simple eigenvalue and all other eigenvalues have modulus strictly less than 1. □

4 Extension to Open Systems

In this section we will prove Propositions 2.2 and 2.3 and Theorems 2.4 and 2.5. As already explained in the proof of Theorem 2.1, Lemma 3.6 yields the required inequalities for Proposition 2.2 for $\mathcal{L}$. We begin by explaining how to extend those inequalities to $\mathring{\mathcal{L}}_H$.

In this section, we *assume* that we have a map T (without holes) satisfying (A1)–(A5) and then show how to choose a higher iterate of T once we introduce the additional cuts made by ∂H so that (A1)–(A5) are still satisfied, although some of the constants will have changed.

We fix $B_0, B_1 > 0$ and the set of holes $\mathcal{H}(B_0, B_1)$ satisfying (H1)–(H3). We choose $H \in \mathcal{H}(B_0, B_1)$ and define $\mathring{T}$ as in Section 1.1.

We want to think of ∂H as an extended singularity set for T. To this end, we define a map $\hat{T}$ which is equal to T everywhere, except $\hat{T}$ has the expanded singularity set induced by $\mathcal{S}_0 \cup \partial H$. Thus when iterating $\hat{T}$, we introduce artificial cuts according to ∂H. When we want to consider the map with a hole, we simply drop the pieces that would have entered H.

Note that by (H2) and (A3)(2), ∂H has the same properties as $\mathcal{S}_{-1}$. Since $\hat{T}$ and T are the same map everywhere, properties (A1)–(A4) hold for $\hat{T}$ with essentially the same constants as for T (we may have to replace C_2 by B_1 in (A3)(2), but taking C_2' to be the larger of these two numbers, we note that both maps satisfy (A3)(2) with respect to C_2').

Thus the only thing which we need to address is (A5) and in particular (3.2) which may fail for $\hat{T}$ due to the additional cuts. Note that since ∂H increases the sums in (A5) by at most a factor of B_0, both sums are still finite. This is sufficient to ruin contraction in the Lasota-Yorke inequalities (by making $\theta_* > 1$), but still yields a finite bound on $\|\mathcal{L}_{\hat{T}}\|_{\mathcal{B}}$ via Lemma 3.6 even with $\theta_* > 1$, where $\mathcal{L}_{\hat{T}}$ denotes the transfer operator corresponding to $\hat{T}$. Thus if we can establish a spectral gap for an iterate $\mathcal{L}_{\hat{T}}^n$, it will follow that $\mathcal{L}_{\hat{T}}$ has a spectral gap as well.

4.1 Complexity Bound and Proof of Proposition 2.2

Before proving Proposition 2.2, we prove the following lemmas, which will allow us to regain (A5) for a higher iterate of $\hat{T}$. The proofs of these lemmas are similar to those in [D3], but we include them here to keep the exposition self-contained.

Lemma 4.1. *There exists a sequence $\delta_n \downarrow 0$ such that*

$$\sup_{\substack{W \in \mathcal{W}_1^s \\ |W| \le \delta_n}} \sum_i |J_{V_i^n} T^n|_* \le \theta_*^n, \tag{4.1}$$

where V_i^n denote the maximal homogeneous stable curves in $T^{-n}W$ on which T^n is smooth.

Proof. We prove the lemma by induction on n. The case $n = 1$ follows from (A5) and (3.3) by taking $\delta_1 = \delta_0$.

Now assume (4.1) holds for all $0 \le k \le n$. In order to extend this inequality to $n+1$, we claim that $\delta_{n+1} \le \delta_n$ can be chosen so small that $|V_i^n| \le \delta_0$ whenever $|W| \le \delta_{n+1}$. In this way, V_i^n will belong to $\mathcal{W}_1^s$ and we may apply (A5) to each such curve without additional artificial subdivisions.

Let $A(V_i^n)$ comprise those indices j such that $TV_j^{n+1} \subset V_i^n$. Then grouping V_j^{n+1} according to the sets $A(V_i^n)$, we have

$$\sum_j |J_{V_j^{n+1}} T^{n+1}|_* \le \sum_i \sum_{j \in A(V_i^n)} |J_{V_j^{n+1}} T|_* |J_{V_i^n} T^n|_* \le \sum_i |J_{V_i^n} T^n|_* \theta_* \le \theta_*^{n+1},$$

as required. It remains to prove the claim.

The claim follows from the fact that if T^{-1} is smooth on a stable curve W, then there exists a uniform constant C, depending only on T, such that $|T^{-1}W| \le C|W|^{1/3}$ [CM, Sect. 4.9].[5] Thus the lemma follows if we inductively choose $\delta_{n+1} = \delta_n^3$. □

For $W \in \mathcal{W}_1^s$, Let $\hat{V}_i^n$ denote the maximal homogeneous stable curves in $\hat{T}^{-n}W$ on which $\hat{T}^n$ is smooth.

Lemma 4.2. *For $n \in \mathbb{N}$, let δ_n be from Lemma 4.1. Then*

$$\sup_{\substack{W \in \mathcal{W}_1^s \\ |W| \le \delta_n}} \sum_i |J_{\hat{V}_i^n} \hat{T}^n|_* \le (1 + n(B_0 - 1))\theta_*^n.$$

Proof. Fix $W \in \mathcal{W}_1^s$ with $|W| \le \delta_n$. Each V_i^n comprises one or more $\hat{V}_j^n$ due to the expanded singularity set for $\hat{T}$. For a fixed V_i^n, we must estimate the cardinality of the curves $\hat{V}_j^n \subset V_i^n$.

Let $U_i^n = T^n V_i^n$ and $\hat{U}_j^n = T^n \hat{V}_j^n$ for each i and j. Note that if $\hat{V}_{j_1}^n$ and $\hat{V}_{j_2}^n$ belong to the same curve V_i^n, then in fact $T^{-k}\hat{U}_{j_1}^n$ and $T^{-k}\hat{U}_{j_2}^n$ belong to the same smooth curve $T^{-k}U_i^n$ for each $0 \le k \le n$ since the additional cuts due to $\hat{T}$ are artificial and do not change the orbits of points. Also, $|T^{-k}U_i^n| \le \delta_0$ for each $k \le n$ by choice of δ_n from the proof of Lemma 4.1.

Applying (H2) to $T^{-k+1}U_i^n$, the total number of new cuts in $T^{-k}U_i^n$ compared to $T^{-k+1}U_i^n$ can be no more than $B_0 - 1$. Inductively, the total number of cuts introduced into V_i^n by time n can be no more than $n(B_0 - 1)$. Thus the cardinality of the set of j such that $\hat{V}_j^n \subset V_i^n$ is at most $1 + n(B_0 - 1)$. This, plus the fact that $|J_{\hat{V}_j^n} \hat{T}^n|_* \le |J_{V_i^n} T^n|_*$ whenever $\hat{V}_j^n \subset V_i^n$ completes the proof of the lemma. □

Proof of Proposition 2.2. Now we choose n_0 such that $(1 + n_0(B_0 - 1))\theta_*^{n_0} = \theta_0 < 1$. Then setting $\hat{T}_0 = \hat{T}^{n_0}$, and choosing δ_{n_0} from Lemma 4.1 to be the maximum length scale of curves in $\mathcal{W}_1^s$, we have (A1)-(A5) satisfied for $\hat{T}_0$. Thus the results of [DZ2] imply the uniform Lasota-Yorke inequalities for $\mathcal{L}_{\hat{T}_0}$ given by Lemma 3.6 with δ_{n_0} in place of δ_0 and θ_0 in place of θ_* with the same choices of constants in the norms.

Notice that we do not need to change the definition of the Banach spaces $\mathcal{B}$ and $\mathcal{B}_w$. This is because once the uniform Lasota-Yorke inequalities hold for $|W| \le \delta_{n_0}$, we can extend them to $|W| \le \delta_0$ by subdividing such curves into at most $[\delta_0/\delta_{n_0}] + 1$ pieces of length at most δ_{n_0} and then applying the estimates on the shorter pieces. This has the effect of multiplying all the inequalities in Lemma 3.6 by the factor $[\delta_0/\delta_{n_0}] + 1$ which affects neither the essential spectral radius nor the spectral radius of $\mathcal{L}_{\hat{T}_0}$.

Since $\mathcal{L}_{\hat{T}}$ is bounded as an operator on $\mathcal{B}$ as mentioned previously, this implies that $\mathcal{L}_{\hat{T}}$ also satisfies a uniform set of Lasota-Yorke inequalities with the essential spectral radius increased by the exponent $1/n_0$.

[5]Indeed, [CM] shows only the bound $|W|^{1/2}$ in the finite horizon case, but a quick calculation shows that an exponent of $1/3$ is in fact needed in the infinite horizon case.

Now the transfer operator $\mathring{\mathcal{L}}_H$ corresponding to the map with a hole satisfies the same Lasota-Yorke inequalities as $\mathcal{L}_{\hat{T}}$ with the same constants since the pieces we must sum over are fewer (we drop the pieces that pass through H), but the estimates on each surviving piece remain the same (the maps T, $\mathring{T}$ and $\hat{T}$ are all the same on such pieces). The equalities of Proposition 2.2 now follow from Lemma 3.6 with constants depending only on (A1)–(A5), B_0 and B_1, as required. □

4.2 Proof of Proposition 2.3

First we prove two preliminary lemmas. We will use repeatedly that there exists a constant $C_c > 0$ such that

$$C_c^{-1} \le \frac{\cos \varphi(x)}{\cos \varphi(y)} \le C_c \tag{4.2}$$

whenever x and y lie in the same homogeneity strip. Our first lemma shows that the indicator functions $\mathbf{1}_H$ and $\mathbf{1}_{\mathring{M}}$ are bounded multipliers in both spaces $\mathcal{B}$ and $\mathcal{B}_w$.

Lemma 4.3. *Suppose $f \in \mathcal{B}$ and $H \in \mathcal{H}(B_0, B_1)$. There exists $C > 0$, depending only on B_0, B_1 and (A1)–(A5), such that $\|\mathbf{1}_H f\|_{\mathcal{B}} \le C\|f\|_{\mathcal{B}}$ and $|\mathbf{1}_H f|_w \le C|f|_w$. Similar bounds hold for $\mathbf{1}_{\mathring{M}} f$.*

Proof. We begin by checking that the partition $\mathcal{P}$ formed by the open, connected components of H and $M \setminus H$ satisfies assumptions (1) and (2) of Lemma 3.1. Assumption (1) holds by (H3). Also, assumption (2) of Lemma 3.1 is satisfied due to (H2) and (A3)(2) with $C_3 = \max\{C_0, C_2\}$ and $K = B_0$.

By density of $C^1(M)$ in $\mathcal{B}$, it suffices to prove the lemma for $f \in \mathcal{C}^1(M)$. We fix $f \in \mathcal{C}^1(M)$ and note that $\mathbf{1}_H f$ and $\mathbf{1}_{\mathring{M}} f$ have the type of singularity admitted in Lemma 3.1 so that $\mathbf{1}_H f, \mathbf{1}_{\mathring{M}} f \in \mathcal{B}$.

We will estimate $\|\mathbf{1}_H f\|_{\mathcal{B}} \le C\|f\|_{\mathcal{B}}$. The estimate for the weak norm is similar to that for the strong stable norm and is omitted. From these, the estimates for $\mathbf{1}_{\mathring{M}} f$ follow by linearity since $\mathbf{1}_{\mathring{M}} = \mathbf{1}_M - \mathbf{1}_H$.

To estimate $\|\mathbf{1}_H f\|_s$, let $W \in \mathcal{W}_1^s$ and $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \le 1$. Note that $|\psi|_{\mathcal{C}^q(W)} \le (\cos W)^{-1}|W|^{-\alpha}$. Then since $W \cap H$ comprises at most B_0 curves $W_i \in \mathcal{W}_1^s$ according to (H1), we have

$$\begin{aligned}\int_W \mathbf{1}_H f \, \psi \, dm_W &= \sum_i \int_{W_i} f \, \psi \, dm_W \le \|f\|_s \sum_i |W_i|^\alpha \cos W_i \, |\psi|_{\mathcal{C}^q(W)} \\ &\le \|f\|_s \sum_i \frac{|W_i|^\alpha}{|W|^\alpha} \frac{\cos W_i}{\cos W} \le \|f\|_s C_c B_0,\end{aligned}$$

since $\cos W_i / \cos W$ is uniformly bounded by (4.2) and $|W_i| \le |W|$. Taking the supremum over $W \in \mathcal{W}_1^s$ and $\psi \in \mathcal{C}^q(W)$ yields $\|\mathbf{1}_H f\|_s \le C_c B_0 \|f\|_s$.

Next we estimate $\|1_H f\|_u$. Let $\varepsilon \le \varepsilon_0$ and choose $W^1, W^2 \in \mathcal{W}_1^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$. For $\ell = 1, 2$, let $\psi_\ell \in W^\ell$ with $|\psi_\ell|_{W^\ell,\gamma,p} \le 1$ and $d_q(\psi_1, \psi_2) \le \varepsilon$. We must estimate

$$\int_{W^1} \mathbf{1}_H f \, \psi_1 \, dm_W - \int_{W^2} \mathbf{1}_H f \, \psi_2 \, dm_W.$$

Recalling the notation of Section 3.3, we consider W^ℓ as graphs of functions of their position coordinates, $\varphi_{W^\ell}(r)$, and write $W^\ell = G_{W^\ell}(I_{W^\ell})$, $\ell = 1, 2$. We subdivide $W^1 \cap H$ and $W^2 \cap H$ into matched pieces U_j^1 and U_j^2 and unmatched pieces V_k^1 and V_k^2 respectively using a foliation of vertical line segments in M. Thus U_j^1 and U_j^2 are matched if both are defined over the same r-interval I_j.

Due to (H1), there are at most B_0 matched pieces U_j^ℓ and $2B_0+2$ unmatched pieces V_k^ℓ created by ∂H and near the endpoints of W^ℓ. Note that due to (H2) and (A3)(2) we have $|V_k^\ell| \le C_2' \varepsilon^{t_0}$, where $C_2' = \max\{C_0, C_2\}$, since $d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon$. We split the estimate into matched and unmatched pieces,

$$\begin{aligned} \int_{W^1} \mathbf{1}_H f\, \psi_1\, dm_W - \int_{W^2} \mathbf{1}_H f\, \psi_2\, dm_W \;\; &= \;\; \sum_j \int_{U_j^1} f\, \psi_1\, dm_W - \int_{U_j^2} f\, \psi_2\, dm_W \\ &\quad + \sum_{\ell,k} \int_{V_k^\ell} f\, \psi_\ell\, dm_W. \end{aligned}$$

We estimate the integrals on unmatched pieces first,

$$\begin{aligned} \int_{V_k^\ell} f\, \psi_\ell\, dm_W &\le \|f\|_s |V_k^\ell|^\alpha \cos V_k^\ell |\psi_\ell|_{\mathcal{C}^q(W^\ell)} \le \|f\|_s \frac{|V_k^\ell|^\alpha \cos V_k^\ell}{|W^\ell|^\gamma \cos W^\ell} \\ &\le C_c C_2' \|f\|_s \varepsilon^{(\alpha-\gamma)t_0}, \end{aligned} \tag{4.3}$$

where we have used (4.2) and $|V_k^\ell| \le |W^\ell|$ in the last step.

To estimate the integrals on matched pieces, note that

$$d_{\mathcal{W}^s}(U_j^1, U_j^2) \le d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon.$$

Also,

$$|\psi_1 \circ G_{U_j^1} - \psi_2 \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le |\psi_1 \circ G_{W^1} - \psi_2 \circ G_{W^2}|_{\mathcal{C}^q(I_{W^1} \cap I_{W^2})} \le \varepsilon,$$

since $\varphi_{U_j^1}$ and $\varphi_{U_j^2}$ are simply the restrictions of φ_{W^1} and φ_{W^2} to I_j respectively. Thus,

$$\left| \int_{U_j^1} f\, \psi_1\, dm_W - \int_{U_j^2} f\, \psi_2\, dm_W \right| \le \|h\|_u \varepsilon^\beta.$$

Putting this estimate together with (4.3) and using the fact that the number of matched and unmatched pieces are finite as mentioned earlier, we obtain,

$$\left| \int_{W^1} \mathbf{1}_H f\, \psi_1\, dm_W - \int_{W^2} \mathbf{1}_H f\, \psi_2\, dm_W \right| \le C_c C_2' (2B_0+2) \|h\|_s \varepsilon^{(\alpha-\gamma)t_0} + B_0 \|h\|_u \varepsilon^\beta,$$

which, since $\beta \le (\alpha-\gamma)t_0$, means we may divide through by ε^β to complete the estimate on the strong unstable norm and the proof of the lemma. □

Our second lemma shows that the hole is a small perturbation in the sense of the norm $|||\cdot|||$ defined by (2.2).

Lemma 4.4. *If $f \in \mathcal{B}$ and $H \in \mathcal{H}(B_0, B_1)$, then $|\mathbf{1}_H f|_w \le C h^{\alpha-\gamma} \|f\|_s$, where $h = diam_s(H)$ and $C > 0$ depends only on B_0.*

Proof. As with Lemma 4.3, by density it suffices to prove this estimate for $f \in \mathcal{C}^1(M)$.

Let $f \in \mathcal{C}^1(M)$ and $W \in \mathcal{W}_1^s$. Take $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,\gamma,p} \le 1$. Let W_i denote the at most B_0 connected components of $W \cap H$. Then each W_i belongs to $\mathcal{W}_1^s$ and $|W_i| \le h$ by definition of the stable diameter. We thus estimate,

$$\begin{aligned} \int_W \mathbf{1}_H f\, \psi\, dm_W &= \sum_i \int_{W_i} f \psi\, dm_W \le \sum_i \|f\|_s |W_i|^\alpha \cos W_i |\psi|_{\mathcal{C}^q(W_i^1)} \\ &\le \sum_i \|f\|_s h^{\alpha-\gamma} \frac{|W_i|^\gamma \cos W_i}{|W|^\gamma \cos W} \le B_0 C_c \|f\|_s h^{\alpha-\gamma} \end{aligned}$$

where we have used (4.2) in the last line.

Taking the supremum over $W \in \mathcal{W}_1^s$ and $\psi \in \mathcal{C}^p(W)$, the lemma is proved. $\square$

Now fix $H \in \mathcal{H}(B_0, B_1)$ with $\mathrm{diam}_s(H) \le h$. We must estimate $|||\mathcal{L} - \mathring{\mathcal{L}}|||$. We do this estimate for T directly rather than some power of T that we worked with in Section 4.1. This is because we do not need contraction for the present estimate, but only use the smallness of the hole. For our purposes then, T satisfies (A1)–(A5) with the sum in (A5) finite, but not contracting.

We choose $f \in \mathcal{B}$ and recalling (2.1), we estimate,

$$\begin{aligned} |\mathcal{L}f - \mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_{\mathring{M}}f)|_w &\le |\mathcal{L}f - \mathbf{1}_{\mathring{M}}\mathcal{L}f|_w + |\mathbf{1}_{\mathring{M}}\mathcal{L}f - \mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_{\mathring{M}}f)|_w \\ &\le |\mathbf{1}_H\mathcal{L}f|_w + |\mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_H f)|_w, \end{aligned} \tag{4.4}$$

by linearity since $H = M \setminus \mathring{M}$.

Let $C_w = \sup\{|\mathcal{L}f|_w : f \in \mathcal{B}_w, |f|_w \le 1\}$ and $C_B = \sup\{\|\mathcal{L}f\|_{\mathcal{B}} : f \in \mathcal{B}, \|f\|_{\mathcal{B}} \le 1\}$ denote the norm of $\mathcal{L}$ in the spaces $\mathcal{B}_w$ and $\mathcal{B}$ respectively. Then using Lemmas 4.3 and 4.4 together with (4.4), we have,

$$\begin{aligned} |\mathcal{L}f - \mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_{\mathring{M}}f)|_w &\le |\mathbf{1}_H\mathcal{L}f|_w + |\mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_H f)|_w \\ &\le Ch^{\alpha-\gamma}\|\mathcal{L}f\|_{\mathcal{B}} + CC_w|\mathbf{1}_H f|_w \\ &\le CC_B\|f\|_{\mathcal{B}}h^{\alpha-\gamma} + CC_w h^{\alpha-\gamma}\|f\|_{\mathcal{B}}. \end{aligned}$$

Now taking the supremum over $f \in \mathcal{B}$, $\|f\|_{\mathcal{B}} \le 1$, completes the proof of Proposition 2.3.

4.3 Proof of Theorems 2.4 and 2.5

Proof of Theorem 2.4. Using Propositions 2.2 and 2.3, we now apply the perturbative framework of Keller and Liverani [KL]. Fix $B_0, B_1 > 0$ and consider the family of holes $\mathcal{H}(B_0, B_1)$. Proposition 2.2 guarantees uniform Lasota-Yorke inequalities for $\mathring{\mathcal{L}}_H$ for all $H \in \mathcal{H}(B_0, B_1)$. Then for $H \in \mathcal{H}(B_0, B_1)$ with $\mathrm{diam}_s(H)$ sufficiently small, Proposition 2.3 and [KL, Corollary 1] imply that the spectra outside the disk of radius $\sigma < 1$ and the corresponding spectral projectors of $\mathring{\mathcal{L}}_H$ move Hölder continuously for $H \in \mathcal{H}(B_0, B_1)$. Thus for $\mathrm{diam}_s(H)$ sufficiently small, $\mathring{\mathcal{L}}_H$ has a spectral gap. We prove the remainder of the theorem assuming that $\mathring{\mathcal{L}}_H$ has a spectral gap in this context.

Since $\mathring{\mathcal{L}}_H$ is real, its eigenvalue of maximum modulus, λ_H, must persist in being real and positive for small holes. To see that its corresponding eigenvector $\mu_H \in \mathcal{B}$ is a measure, note that the spectral decomposition of $\mathring{\mathcal{L}}_H$ implies that for each $f \in \mathcal{B}$, there exists a constant c_f such that

$$\lim_{n\to\infty} \lambda_H^{-n}\mathring{\mathcal{L}}_H^n f(\psi) = c_f\mu_H(\psi), \quad \forall \psi \in \mathcal{C}^p(M). \tag{4.5}$$

The limit above defines the spectral projector Π_{λ_H} onto the eigenspace corresponding to λ_H for $\mathring{\mathcal{L}}_H$. Letting Π_1 denote the eigenprojector onto the eigenspace corresponding to eigenvalue 1 for $\mathcal{L}$, we know that these projectors vary Hölder continuously in the $|||\cdot|||$-norm from (2.2) according to [KL, Corollary 2]. Recall that μ_{SRB} denotes the smooth invariant measure for T before the introduction of the hole (see Sect. 2.1). Then since before the introduction of the hole, $\Pi_1 m(1) = \mu_{\mathrm{SRB}}(1) = 1$, where m denotes Lebesgue measure, we must have that $\Pi_{\lambda_H} m(1) = c_m\mu_H(1) > 0$ holds for sufficiently small holes. Indeed, the positivity of $\mathring{\mathcal{L}}$ requires both $c_m > 0$ and $\mu_H(1) > 0$.

Now (4.5) with 1 (the density of m) in place of f implies

$$|\mu_H(\psi)| = c_m^{-1}\lim_{n\to\infty}|\lambda_H^{-n}\mathring{\mathcal{L}}_H^n 1(\psi)| \le c_m^{-1}|\psi|_\infty \lim_{n\to\infty}\lambda_H^{-n}\mathring{\mathcal{L}}_H^n 1(1) = |\psi|_\infty\mu_H(1),$$

which implies that μ_H is a measure. Since $\mu_H(1) > 0$ by the positivity of $\mathring{\mathcal{L}}_H$ we may normalize μ_H to be a probability measure, $\mu_H(1) = 1$. It is now clear that μ_H is a conditionally invariant measure for $\mathring{T}$.

To see that μ_H is singular with respect to Lebesgue, note that since T is injective, it follows from the definition of conditional invariance (see Section 1.1) that μ_H cannot be supported on any of the forward images of the hole, $\cup_{i\geq 0} T^i(H)$. Since this set has full Lebesgue measure, μ_H must be singular.

Now suppose that $\mu \in \mathcal{B}$ is a probability measure such that $c_\mu > 0$. Then by (4.5),

$$c_\mu \mu_H(1) = \lim_{n\to\infty} \lambda_H^{-n} \mathring{\mathcal{L}}_H^n \mu(1) = \lim_{n\to\infty} \lambda_H^{-n} \mu(\mathring{M}^n),$$

so that the escape rates with respect to μ_H and μ are equal, i.e., $\rho(\mu) = -\log \lambda_H$. Moreover,

$$\frac{\mathring{\mathcal{L}}_H^n \mu}{|\mathring{\mathcal{L}}_H^n \mu|} = \frac{\mathring{\mathcal{L}}_H^n \mu}{\lambda_H^n} \frac{\lambda_H^n}{\mathring{\mathcal{L}}_H^n \mu(1)} = c_\mu \mu_H \cdot \frac{1}{c_\mu \mu_H(1)} = \mu_H,$$

and the convergence is at an exponential rate in $\mathcal{B}$ due to the spectral decomposition of $\mathring{\mathcal{L}}_H$.

We complete the proof by remarking that $c_m, c_{\mu_{\mathrm{SRB}}} > 0$ by continuity of the spectral projectors so that Lebesgue and the smooth SRB measure for T are both included in this class of measures in $\mathcal{B}$. $\square$

Proof of Theorem 2.5. With Propositions 2.2 and 2.3 established, the convergence of μ_H to μ_{SRB} and λ_H to 1 follows immediately from the continuity of the spectral projectors corresponding to $\mathring{\mathcal{L}}_H$ given by [KL, Corollary 2] as long as we take a sequence of holes in $\mathcal{H}(B_0, B_1)$ with B_0 and B_1 fixed. $\square$

4.4 Proof of Theorem 2.6

It follows from the general variational inequalities of [DWY2] that for holes satisfying (H1)–(H3) and maps satisfying (A1)–(A5), we have

$$\rho(m) \geq \sup_{\nu \in \mathcal{G}_H} \{h_\nu(T) - \chi_\nu^+(T)\},$$

where m is Lebesgue measure and $\mathcal{G}_H$ is defined by (2.7) (see [DWY2, Theorem C]). (Indeed, the assumptions on both H and T for this inequality are much weaker than those we are using here.) In order to complete the proof of Theorem 2.6, we construct a measure $\nu_H \in \mathcal{G}_H$ which satisfies $\rho(m) = P_{\nu_H}$. The following construction of ν_H is by now standard (see e.g. [CMS1]).

We assume that H satisfies the assumptions of Theorem 2.4 so that there is a well defined measure μ_H with eigenvalue λ_H according to that theorem. Letting $s > \max\{\beta/(1-\beta), p\}$, we define a linear functional on $\mathcal{C}^s(M)$ by

$$\mathcal{F}(\psi) = \lim_{n\to\infty} \lambda_H^{-n} \mu_H(\mathbf{1}_{\mathring{M}^n} \psi), \qquad \forall \psi \in \mathcal{C}^s(M).$$

Note that $\psi\mu_H \in \mathcal{B}$ by Lemma 3.2. Since $\mu_H(\mathbf{1}_{\mathring{M}^n}\psi) = \mathring{\mathcal{L}}_H^n(\psi\mu_H)(1)$, it follows from (4.5) that $\mathcal{F}$ is well-defined. Indeed, $\mathcal{F}(\psi) = c_{\psi\mu_H}$ in the notation of (4.5).

Since $|\mathcal{F}(\psi)| \leq |\psi|_\infty$, $\mathcal{F}$ extends to a bounded linear functional on $\mathcal{C}^0(M)$ and so by the Riesz representation theorem, there is a Borel measure ν_H such that $\nu_H = \mathcal{F}$ as a functional on $\mathcal{C}^0(M)$. Since $\mathcal{F} \geq 0$ and $\mathcal{F}(1) = 1$, ν_H is a probability measure. It follows from the limit definition of $\mathcal{F}$ that ν_H is necessarily supported on the survivor set $\mathring{M}^\infty$.

To see that ν_H is invariant, note that $\mathbf{1}_{\mathring{M}^{n-1}} \circ \mathring{T} = \mathbf{1}_{\mathring{M}^n}$. Then for any $\psi \in \mathcal{C}^0(M)$,

$$\begin{aligned}\nu_H(\psi \circ \mathring{T}) &= \lim_{n\to\infty} \lambda_H^{-n} \mu_H(\mathbf{1}_{\mathring{M}^n} \psi \circ \mathring{T}) = \lim_{n\to\infty} \lambda_H^{-n} \mathring{\mathcal{L}} \mu_H(\mathbf{1}_{\mathring{M}^{n-1}} \psi) \\ &= \lim_{n\to\infty} \lambda_H^{-(n-1)} \mu_H(\mathbf{1}_{\mathring{M}^{n-1}} \psi) = \nu_H(\psi).\end{aligned}$$

Thus ν_H is invariant with respect to $\mathring{T}$, and also with respect to T since $\mathring{T} = T$ on $\mathring{M}^\infty$.

The proof that ν_H constructed in this way satisfies $\rho(m) = P_{\nu_H}$ is contained in [D3, Section 5]. The proof has two components: (1) based on the work of Chernov [Ch], a map satisfying (A1)–(A5) admits a Young tower (a type of Markov extension) in which the hole lifts to a countable union of partition elements; (2) for an open system in which such a tower has been constructed and for holes with sufficiently small unstable diameter $\mathrm{diam}_u(H)$, ν_H defined as above satisfies the required variational principle [DWY2, Theorem D]. Using these two elements, the proof given in [D3] holds in the present setting and will not be repeated here.

4.5 Holes in infinite horizon corridors satisfy conditions (H1)–(H3)

In this section, we complete the proof that the holes described in Examples 1 and 2 of Section 2.2 satisfy conditions (H1)–(H3). We have already noted that they satisfy (H1) since each $D^+_{j,\ell}$ cell contains at most two connected components of H_ω.

To see that H_ω satisfies (H3), we recall the structure of $\mathcal{S}_1$ near an infinite horizon point as described in Section 3.1. We need to calculate the number of $D^+_{j',\ell}$ cells that can intersect a single homogeneity strip $\mathbb{H}_k$ in a neighborhood of the infinite horizon point $x_{j'}$.

First, we estimate how many curves $S_{j',\ell}$ intersect the curve $S_{j',0}$ in a homogeneity strip $\mathbb{H}_k$ (using the notation of Section 3.1). Since the boundary of $\mathbb{H}_k$ is distance k^{-2} from $\mathcal{S}_0$ and the furthest that the boundary of $D^+_{j',\ell}$ can be from $\mathcal{S}_0$ is on the order of $\ell^{-1/2}$ when it intersects $S_{j',0}$ as described in Section 3.1, a simple calculation shows that between k^{-2} and $(k+1)^{-2}$, there are $\mathcal{O}(k^3)$ curves $S_{j',\ell'}$ that intersect $\mathbb{H}_k$.

Next, we estimate the number of curves $S_{j',\ell}$ intersecting $\mathbb{H}_k$ at the other end of $D^+_{j'\ell}$ near $\mathcal{S}_0$. From (A3)(1), it follows that if $\mathbb{H}_k \cap D^+_{j',\ell}$ is nonempty, then $k \geq c_s \ell^{1/4}$. Thus $\mathbb{H}_k$ intersects at most $\mathcal{O}(k^4)$ curves $S_{j',\ell}$.

Since each $D^+_{j,\ell}$ cell contains at most two connected components of H_ω as observed above, it follows by combining the cases above that (H3) holds with $N_k = \mathcal{O}(k^4)$.

For the type of hole in Example 2, this estimate holds for every k. For the type of hole in Example 1, this is only an upper bound: For all ℓ past an index N_ω, H_ω contains $D^+_{j',\ell}$ so that no further partitioning of $\mathbb{H}_k$ occurs.

It remains to prove the transversality condition (H2). We will prove it for the hole of Example 1. The proof for the hole of Example 2 is similar and is omitted.

In order to estimate the angle between a curve $W \in \mathcal{W}^s_1$, $W \subset D^+_{j,\ell}$, and ∂H_ω, we will use two facts. (1) Due to the definition of C^s and $\tilde{C}^s$, we know that TW is a stable curve with respect to the larger cones $\tilde{C}^s$ (not necessarily lying in one homogeneity strip). (2) $\partial(TH_\omega) \setminus \mathcal{S}_0$ is comprised of unstable curves, i.e. curves whose positive slopes are uniformly bounded away from 0 and infinity and which are uniformly transverse to the stable cones C^s and $\tilde{C}^s$ (see Figure 1, bottom right).

Recall that $DT^{-1}x$ is given by [CM, Section 2.11],

$$DT^{-1}(x) = \frac{-1}{\cos\varphi_{-1}} \begin{bmatrix} \tau_{-1}\mathcal{K} + \cos\varphi & \tau_{-1} \\ \tau_{-1}\mathcal{K}(\mathcal{K}_{-1} + \cos\varphi_{-1}) + \mathcal{K}_{-1}\cos\varphi & \tau_{-1}\mathcal{K}_{-1} + \cos\varphi_{-1} \end{bmatrix}$$

where $\tau, \mathcal{K}$ and φ denote the quantities at x while τ_{-1}, $\mathcal{K}_{-1}$ and φ_{-1} denote the quantities at $T^{-1}x$. By (1) above, we can regard a tangent vector to W as the image under DT^{-1} of a vector with negative slope, $\begin{bmatrix} 1 \\ -a \end{bmatrix}$ for some $a > 0$. By (2) we can regard a tangent vector to ∂H_ω as the image of a vector with positive slope $\begin{bmatrix} 1 \\ b \end{bmatrix}$ for some $b > 0$. Taking the image of these two vectors under DT^{-1}, we compute the angle θ between them using the cross product to obtain that

$$\theta \geq C \frac{\cos\varphi \cos\varphi_{-1}}{\tau_{-1}^2},$$

for some uniform constant $C > 0$.

Now suppose $W \subset D^+_{j,\ell} \cap \mathbb{H}_{k_{-1}}$ for some $k_{-1} \geq k_0$. Let $s > 0$ be a constant to be determined below. For $\varepsilon > 0$, at a point $x_{-1} \in W \cap N_\varepsilon(\partial H_\omega)$, we have two cases to consider.

Case 1. $\frac{\cos\varphi_0 \cos\varphi_{-1}}{\tau_{-1}^2} \geq \varepsilon^s$. Here $\varphi_0 = \varphi(T(x_{-1}))$, $\varphi_{-1} = \varphi(x_{-1})$ and $\tau_{-1} = \tau(x_{-1})$. Then the angle between W and ∂H_ω is bounded below by $C\varepsilon^s$. Thus $|W \cap N_\varepsilon(\partial H_\omega)| \leq C\varepsilon^{1-s}$.

Case 2. $\frac{\cos\varphi_0 \cos\varphi_{-1}}{\tau_{-1}^2} < \varepsilon^s$. Since $W \subset D^+_{j,\ell}$, we have $|W| \leq Ck_{-1}^{-3}$ and $\tau_{-1} \approx \ell$, up to a uniform constant, by the definition of $D^+_{j,\ell}$.

Although TW may cross several homogeneity strips, it is still a stable curve in the wider family of cones $\tilde{C}^s(x)$. Thus, letting $(TW)_k := TW \cap \mathbb{H}_k$ denote a homogeneous component of TW, we have

$$|T^{-1}(TW)_k| \leq C|(TW)_k| \frac{\tau_{-1}}{\cos\varphi_{-1}} \tag{4.6}$$

due to uniform expansion in the stable cone given by (A3)(1). Summing over relevant k, we obtain $|W| \leq C|TW| \frac{\tau_{-1}}{\cos\varphi_{-1}}$ for some uniform constant C.

We now consider the point $x_{-1} \in W$ for which $\cos\varphi_0$ is largest. It follows that any smaller $\cos\varphi_0$ also satisfies the assumption of this case since $\cos\varphi_{-1}$ and τ_{-1} are essentially constant on $D^+_{j,\ell} \cap \mathbb{H}_{k_{-1}}$. Thus $|TW| \leq C\cos\varphi_0$. Putting this together with (4.6) and following yields,

$$|W| \leq C\cos\varphi_0 \frac{\tau_{-1}}{\cos\varphi_{-1}}. \tag{4.7}$$

Finally, we recall that $k_{-1} \geq c_s \ell^{1/4}$ by (A3)(1) and so

$$|W|^4 \leq C(k_{-1}^{-3})^4 \leq Cc_s^{-12}\tau_{-1}^{-3}. \tag{4.8}$$

Putting (4.7) together with (4.8), we estimate,

$$\varepsilon^s > \frac{\cos\varphi_0 \cos\varphi_{-1}}{\tau_{-1}^2} = \frac{\cos\varphi_0 \ \tau_{-1}}{\cos\varphi_{-1}} \cdot \cos\varphi_{-1}^2 \cdot \frac{1}{\tau_{-1}^3} \geq C|W||W|^{4/3}|W|^4 \geq C|W|^{19/3}. \tag{4.9}$$

This yields $|W| \leq C\varepsilon^{3s/19}$.

The estimates from Cases 1 and 2 are maximized when the two exponents are equal, i.e. when $s = 19/22$. Thus $|W \cap N_\varepsilon(\partial H_\omega)| \leq C\varepsilon^{3/22}$.

For W outside $D^+_{j,\ell}$, i.e. for W bounded away from the infinite horizon corridors, τ_{-1} is bounded away from 0 and infinity and so the estimate (4.9) still holds, although we may disregard the factor τ_{-1}^{-3}. This yields the better estimate $|W| \leq C\varepsilon^{3s/7}$, which combined with Case 1 yields $s = 7/10$.

Using the worst of these estimates, we have proved that in all cases $|W \cap N_\varepsilon(\partial H_\omega)| \leq C\varepsilon^{t_0}$ for some uniform constant C and $t_0 = 3/22$.

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