A FUNCTIONAL ANALYTIC APPROACH TO PERTURBATIONS OF THE LORENTZ GAS

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Abstract. We present a functional analytic framework based on the spectrum of the transfer operator to study billiard maps associated with perturbations of the periodic Lorentz gas. We show that recently constructed Banach spaces for the billiard map of the classical Lorentz gas are flexible enough to admit a wide variety of perturbations, including: movements and deformations of scatterers; billiards subject to external forces; nonelastic reflections with kicks and slips at the boundaries of the scatterers; and random perturbations comprised of these and possibly other classes of maps. The spectra and spectral projections of the transfer operators are shown to vary continuously with such perturbations so that the spectral gap enjoyed by the classical billiard persists and important limit theorems follow.

1. Introduction

The Lorentz gas is known to enjoy strong ergodic properties: both the continuous time dynamics and the billiard maps are completely hyperbolic, ergodic, K-mixing and Bernoulli (see [S, GO, SC, CH] and the references therein). Young [Y] proved exponential decay of correlations for billiard maps corresponding to the finite horizon periodic Lorentz gas using Markov extensions; this technique was subsequently extended to other dispersing billiards [Ch1] and used to obtain important limit theorems such as local large deviation estimates and almost-sure invariance principles [MN1, MN2, RY].

In this setting, it is natural to ask how the statistical properties of dispersing billiard maps vary with the shape and position of the scatterers. Alternatively, one may change the billiard dynamics by introducing an external force between collisions or by considering nonelastic reflections at the boundaries. Such perturbed dynamics lead to nonequilibrium billiards whose invariant measures are singular with respect to Lebesgue measure.

One of the first nonequilibrium physical models that was studied rigorously is the periodic Lorentz gas with a small constant electrical field [CELS1, CELS2] and the well-known Ohm's law was proved for that case. More general external forces were handled in [Ch2, Ch4, CD2] and billiards with kicks at reflections have been studied in [MPS, Z]. Recently, Chernov and Dolgopyat [CD1] used coupling methods to study the motion of a point particle colliding with a moving scatterer. Locally perturbed periodic rearrangements of scatterers have also been the subject of recent studies [DSV]. Despite such successes, the study of perturbations of billiards has thus far been handled on a case by case basis, with methods adapted and developed for each specific type of perturbation considered.

In this paper, we propose a unified framework in which to study a large class of perturbations of dispersing billiards. This framework is based on the spectral analysis of the transfer operator associated with the billiard map and uses the recent work [DZ] which successfully constructed Banach spaces on which the transfer operator for the classical periodic Lorentz gas has a spectral gap.

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We first present abstract conditions under which we have uniform control of spectral data for a given class of perturbed maps. We then prove that four broad classes of perturbations of billiards fit within this framework, namely:

(i) Tables with shifted, rotated or deformed scatterers;
(ii) Billiards under small external forces which bend trajectories during flight;
(iii) Billiards with kicks or twists at reflections, including slips along the disk;
(iv) Random perturbations comprised of maps with uniform properties (including any of the above classes, or a combination of them).

In particular, the results on random perturbations are a version of time-dependent billiards, in which scatterers are allowed to change positions between collisions. The fact that our main theorems, 2.2 and 2.3, are proved in an abstract setting will facilitate the application of this framework to other classes of perturbations as they arise in future works.

The present functional analytic approach uses the Banach spaces constructed in [DZ] as well as the perturbative framework of Keller and Liverani [KL] to prove that the spectral data and spectral projectors, including invariant measures, rates of decay of correlations, variance in the central limit theorem, etc, vary Hölder continuously for the classes of perturbations mentioned above (see [B, L] for expositions of this approach). In addition, this approach yields new results for the perturbed billiard maps in terms of local limit theorems, in particular giving new information about the evolution of noninvariant measures in the context of these limit theorems. For example, applying Corollary 2.4 to billiards under external forces and kicks, we obtain a local large deviation estimate with a rate function that is the same for all probability measures in our Banach space. This implies in particular that Lebesgue measure and the singular SRB measure for the perturbed billiard have the same large deviation rate function.

The paper is organized as follows. In Section 2, we describe our abstract framework, state precisely the applications which serve as our model perturbations and formulate our main results. In Section 3, we lay out our common approach under the general conditions **(H1)-(H5)** which guarantee the required uniform Lasota-Yorke inequalities for Theorem 2.2, proved in Section 4; we also formulate conditions **(C1)-(C4)** to verify that a perturbation is small in the sense of our Banach spaces for Theorem 2.3, proved in Section 5. The investigations of the concrete models are provided in Sections 6 and 7.

2. Setting and Results

In this section, we describe the abstract framework into which we will place our perturbations and formulate precisely the classes of concrete deterministic perturbations to which our results apply. We also formulate a class of random perturbations with maps drawn from any mixture of the deterministic perturbations described below. We postpone until Section 3 a precise description of the Banach spaces and the formal requirements on the abstract class of maps $\mathcal{F}$.

2.1. **Perturbative framework.** We recall here the perturbative framework of Keller and Liverani [KL]. Suppose there exist two Banach spaces $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}_w, |\cdot|_w)$ with the unit ball of $\mathcal{B}$ compactly embedded in $\mathcal{B}_w$, $|\cdot|_w \le \|\cdot\|_{\mathcal{B}}$, and a family of bounded linear operators $\{\mathcal{L}_\varepsilon\}_{\varepsilon \ge 0}$ defined on both $\mathcal{B}_w$ and $\mathcal{B}$ such that the following holds.[1] There exist constants $C, \eta > 0$ and $\sigma < 1$ such that for all $\varepsilon \ge 0$ and $n \ge 0$,

$$
\begin{aligned}
|\mathcal{L}_\varepsilon^n h|_w &\le C\eta^n |h|_w \qquad \text{for all } h \in \mathcal{B}_w, \\
\|\mathcal{L}_\varepsilon^n h\|_{\mathcal{B}} &\le C\sigma^n \|h\|_{\mathcal{B}} + C\eta^n |h|_w \qquad \text{for all } h \in \mathcal{B}.
\end{aligned}
\tag{2.1}
$$

[1] The results of [KL] hold in a more general setting, but we only state the version we need for our purposes.

If $\sigma < \eta$, the operators $\mathcal{L}_\varepsilon$ are quasi-compact with essential spectral radius bounded by σ and spectral radius at most η (see for example [B]). Suppose further that

$$|||\mathcal{L}_\varepsilon - \mathcal{L}_0||| := \sup\{|\mathcal{L}_\varepsilon h - \mathcal{L}_0 h|_w : h \in \mathcal{B}, \|h\|_{\mathcal{B}} \le 1\} \le \rho(\varepsilon), \tag{2.2}$$

where $\rho(\varepsilon)$ is a non-increasing upper semicontinuous function satisfying $\lim_{\varepsilon\to 0} \rho(\varepsilon) = 0$.

The main result of [KL] is the following. Let $\mathrm{sp}(\mathcal{L}_0)$ denote the spectrum of $\mathcal{L}_0$. For any $\sigma_1 > \sigma$, by quasi-compactness, $\mathrm{sp}(\mathcal{L}_0) \cap \{z \in \mathbb{C} : |z| \ge \sigma_1\}$ consists of finitely many eigenvalues $\varrho_1, \dots, \varrho_k$ of finite multiplicity. Thus there exists $t_* > 0$ and we may choose σ_1 such that $|\varrho_i - \varrho_j| > t_*$ for $i \ne j$ and $\mathrm{dist}(\mathrm{sp}(\mathcal{L}_0), \{|z| = \sigma_1\}) > t_*$. For $t < t_*$ and $\varepsilon \ge 0$, define the spectral projections,

$$\Pi_\varepsilon^{(j)} := \frac{1}{2\pi i} \int_{|z-\varrho_j|=t} (z - \mathcal{L}_\varepsilon)^{-1}\, dz \qquad \text{and}$$

$$\Pi_\varepsilon^{(\sigma_1)} := \frac{1}{2\pi i} \int_{|z|=\sigma_1} (z - \mathcal{L}_\varepsilon)^{-1}\, dz.$$

Theorem 2.1. *([KL]) Assume that (2.1) and (2.2) hold. Then for each $t \le t_*$ and $s < 1 - \frac{\log \sigma_1}{\log \sigma}$, there exist $\varepsilon_1, C > 0$ such that for any $0 \le \varepsilon < \varepsilon_1$, the spectral projections $\Pi_\varepsilon^{(j)}$ and $\Pi_\varepsilon^{(\sigma_1)}$ are well defined and satisfy, for each $j = 1, \dots k$,*

1. $|||\Pi_\varepsilon^{(j)} - \Pi_0^{(j)}||| \le C\rho(\varepsilon)^s$ *and* $|||\Pi_\varepsilon^{(\sigma_1)} - \Pi_0^{(\sigma_1)}||| \le C\rho(\varepsilon)^s$;
2. *rank*$(\Pi_\varepsilon^{(j)}) =$ *rank*$(\Pi_0^{(j)})$;
3. $\|\mathcal{L}_\varepsilon^n \Pi_\varepsilon^{(\sigma_1)}\|_{\mathcal{B}} \le C\sigma_1^n$, *for all* $n \ge 0$.

We say an operator $\mathcal{L}$ has a spectral gap if $\mathcal{L}$ has a simple eigenvalue of maximum modulus and all other eigenvalues have strictly smaller modulus. The above theorem implies in particular that if $\mathcal{L}_0$ has a spectral gap, then so does $\mathcal{L}_\varepsilon$ for ε sufficiently small. In addition, the related statistical properties (for instance, invariant measures, rates of decay of correlations, variance of the Central Limit Theorem) are stable and vary Hölder continuously as a function of $\rho(\varepsilon)$. This is the framework into which we will place our perturbations of the Lorentz gas.

2.2. **An abstract result for a class of maps with uniform properties.** We begin by fixing the phase space M of a billiard map associated with a periodic Lorentz gas. That is, we place finitely many (disjoint) scatterers Γ_i, $i = 1, \dots d$, on $\mathbb{T}^2$ which have $\mathcal{C}^3$ boundaries with strictly positive curvature. The classical billiard flow on the table $\mathbb{T}^2 \setminus \cup_i\{\text{interior } \Gamma_i\}$ is induced by a particle traveling at unit speed and undergoing elastic collisions at the boundaries. In what follows, we also consider particles whose motion between collisions follows slightly curved trajectories (due to external forces) as well as certain types of collisions which do not obey the usual law of reflection.

In all cases, the billiard map associated with the flow is the Poincaré map corresponding to collisions with the scatterers. Its phase space is $M = \cup_{i=1}^d I_i \times [-\pi/2, \pi/2]$, where each I_i is an interval with endpoints identified and $|I_i| = |\partial\Gamma_i|$, i.e. the length of I_i equals the arclength of $\partial\Gamma_i$, $i = 1, \dots d$. M is parametrized by the canonical coordinates (r, φ) where r represents the arclength parameter on the boundaries of the scatterers (oriented clockwise) and φ represents the angle an outgoing (postcollisional) trajectory makes with the unit normal to the boundary at the point of collision.

The phase space M and coordinates so defined are fixed for all classes of perturbations we consider; however, the configuration space (the billiard table on which the particles flow) and the laws which govern the motion of the particles may vary as long as all variations give rise to the same phase space M, i.e. the number of Γ_i and the arclengths of their boundaries do not change. See Remark 2.9 for a way to relax this requirement on the arclength. For any $x = (r, \varphi) \in M$, we define $\tau(x)$ to be the first collision of the trajectory starting at x under the billiard flow. The billiard map is defined wherever $\tau(x) < \infty$. We say that the billiard has finite horizon if there is an upper bound on the function τ. Otherwise, we say the billiard has infinite horizon. Notice that the function τ depends on the placement of the scatterers in $\mathbb{T}^2$, while M is independent of their placement.

We assume there exists a class of maps $\mathcal{F}$ on M satisfying properties **(H1)**-**(H5)** of Section 3.1 with uniform constants. For each $T \in \mathcal{F}$, in Section 3.2 we define the transfer operator $\mathcal{L}_T$ associated with T on an appropriate class of distributions h by

$$\mathcal{L}_T h(\psi) = h(\psi \circ T), \quad \text{for suitable test functions } \psi.$$

In Section 3.3, we define Banach spaces of distributions $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}_w, |\cdot|_w)$, preserved under the action of $\mathcal{L}_T$, $T \in \mathcal{F}$, such that the unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.

Theorem 2.2. *Fix M as above and suppose there exists a class of maps $\mathcal{F}$ satisfying* **(H1)**-**(H5)** *of Section 3.1. Then $\mathcal{L}_T$ is well defined as a bounded linear operator on $\mathcal{B}$ for each $T \in \mathcal{F}$. In addition, there exist $C > 0$, $\sigma < 1$ such that for any $T \in \mathcal{F}$ and $n \geq 0$,*

$$\begin{aligned} |\mathcal{L}_T^n h|_w &\leq C\eta^n |h|_w \qquad \text{for all } h \in \mathcal{B}_w, \\ \|\mathcal{L}_T^n h\|_{\mathcal{B}} &\leq C\sigma^n \|h\|_{\mathcal{B}} + C\eta^n |h|_w \qquad \text{for all } h \in \mathcal{B}, \end{aligned} \tag{2.3}$$

where $\eta \geq 1$ is from **(H5)**. *This, plus the compactness of $\mathcal{B}$ in $\mathcal{B}_w$, implies that all the operators $\mathcal{L}_T$, $T \in \mathcal{F}$, are quasi-compact with essential spectral radius bounded by σ: i.e., outside of any disk of radius greater than σ, their spectra contain finitely many eigenvalues of finite multiplicity. Moreover, for each $T \in \mathcal{F}$,*

- (i) *the spectral radius of $\mathcal{L}_T$ is 1 and the elements of the peripheral spectrum are measures absolutely continuous with respect to $\overline{\mu} := \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathcal{L}_T^i 1$;*
- (ii) *an ergodic, invariant probability measure ν for T is in $\mathcal{B}$ if and only if ν is a physical measure*[2] *for T;*
- (iii) *there exist a finite number of $q_\ell \in \mathbb{N}$ such that the spectrum of $\mathcal{L}$ on the unit circle is $\cup_\ell\{e^{2\pi i \frac{k}{q_\ell}} : 0 \leq k < q_\ell,\ k \in \mathbb{N}\}$. The peripheral spectrum contains no Jordan blocks.*
- (iv) *Let $\mathcal{S}^{\mathbb{H}}_{\pm n,\varepsilon}$ denote the ε-neighborhood of $\mathcal{S}^{\mathbb{H}}_{\pm n}$, the singularity set for $T^{\pm n}$ (with homogeneity strips). Then for each ν in the peripheral spectrum and $n \in \mathbb{N}$, we have $\nu(\mathcal{S}^{\mathbb{H}}_{\pm n,\varepsilon}) \leq C_n \varepsilon^\alpha$, for some constants $C_n > 0$.*
- (v) *If $(T\overline{\mu})$ is ergodic, then 1 is a simple eigenvalue. If $(T^n, \overline{\mu})$ is ergodic for all $n \in \mathbb{N}$, then 1 is the only eigenvalue of modulus 1, $(T, \overline{\mu})$ is mixing and enjoys exponential decay of correlations for Hölder observables.*

Theorem 2.2 is proved in Section 4. In Section 3.4, we define a distance $d_{\mathcal{F}}(\cdot,\cdot)$ between maps in $\mathcal{F}$. Our next result shows that this distance controls the size of perturbations in the spectra of the associated transfer operators.

Theorem 2.3. *Let $\beta > 0$ be from the definition of $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ in Section 3.3. There exists $C > 0$ such that if $T_1, T_2 \in \mathcal{F}$ with $d_{\mathcal{F}}(T_1, T_2) \leq \varepsilon$, then*

$$|||\mathcal{L}_{T_1} - \mathcal{L}_{T_2}||| \leq C\varepsilon^{\beta/2}, \quad \text{where } |||\cdot||| \text{ is from (2.2).}$$

We prove Theorem 2.3 in Section 5. According to Theorem 2.1, an immediate consequence of Theorems 2.2 and 2.3 is the following.

Corollary 2.4. *If $T_0 \in \mathcal{F}$ has a spectral gap, then all $T \in X_\varepsilon(T_0) = \{T \in \mathcal{F} : d_{\mathcal{F}}(T, T_0) < \varepsilon\}$ have a spectral gap for ε sufficiently small. In particular, the maps in X_ε enjoy the following limit theorems (among others), which follow from the existence of a spectral gap.*

Fix $T \in \mathcal{F}$ with a spectral gap. Let $\gamma = \max\{p, 2\beta + \delta\}$ for some $\delta > 0$, where p and β are from Sect. 3.3. Let $\mathcal{P}$ be a (mod 0) partition of M into countably many open, simply connected

[2]Recall that a physical measure for T is an ergodic, invariant probability measure ν such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n-1} f(T^i x) = \int f\, d\nu$ for a positive Lebesgue measure set of $x \in M$.

sets whose boundaries satisfy the assumptions of Lemma 5.3 and let g be a bounded function on M such that $\sup_{P\in\mathcal{P}} |g|_{C^\gamma(P)} < \infty$. Define $S_n g = \sum_{k=0}^{n-1} g \circ T^k$.

(a) *(Local large deviation estimate) For any (not necessarily invariant) probability measure $\nu \in \mathcal{B}$,*
$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n}\log \nu\Big(x \in M : \frac{1}{n}S_n g(x) \in [t-\varepsilon, t+\varepsilon]\Big) = -I(t)$$
where the rate function $I(t)$ is independent of $\nu \in \mathcal{B}$ (but may depend on T), and t is in a neighborhood of the mean $\overline{\mu}(g)$.

(b) *(Almost-sure invariance principle). Suppose $\overline{\mu}(g) = 0$ and distribute $(g\circ T^j)_{j\in\mathbb{N}}$ according to a probability measure $\nu \in \mathcal{B}$. Then there exist a probability space Ω with random variables $\{X_n\}$ satisfying $S_n g \overset{dist.}{=} X_n$, and a Brownian motion W with variance $\varsigma^2 \geq 0$ such that for any $\lambda > 1/4$,*
$$X_n = W(n) + o(n^\lambda) \qquad \text{as } n\to\infty \text{ almost-surely in } \Omega.$$

The proof of the corollary is given in Section 5.2.

Remark 2.5. *When $T \in \mathcal{F}$ is a map derived from a billiard flow Φ^t (for example, corresponding to any of the applications described in Section 2.4), one is often interested in observables of the type $g(x) = \int_0^{\tau(x)} \tilde{g}(\Phi^t x)\, dt$. An important example is given by $\tilde{g} \equiv 1$ so that $g(x) = \tau(x)$.*

In the finite horizon case, when $\tilde{g}$ is smooth in the phase space of the flow, g is piecewise smooth with singularities curves coinciding with those of T. Since these singularities satisfy the assumptions of Lemma 5.3, the results of Corollary 2.4 apply to such observables, including the free flight function τ.

In the infinite horizon case, g will not in general satisfy the assumptions of Corollary 2.4 even when $\tilde{g}$ is smooth. In particular, the important example $g = \tau$ is unlikely to follow an exponential law in large deviations due to the slow mixing on the level of the flow (see [BM]*).*

2.3. **Smooth random perturbations.** We follow the expositions in [GL, DL]. Suppose $\mathcal{F}$ is a class of maps satisfying **(H1)-(H5)** and let $d_{\mathcal{F}}(\cdot,\cdot)$ be the distance in $\mathcal{F}$ defined in Section 3.4. For $T_0 \in \mathcal{F}$, $\varepsilon > 0$, define
$$X_\varepsilon(T_0) = \{T \in \mathcal{F} : d_{\mathcal{F}}(T, T_0) < \varepsilon\},$$
to be the ε-neighborhood of T_0 in $\mathcal{F}$.

Let (Ω, ν) be a probability space and let $g : \Omega \times M \to \mathbb{R}^+$ be a measurable function satisfying: There exist constants $a, A > 0$ such that

(i) $g(\omega,\cdot) \in \mathcal{C}^1(M, \mathbb{R}^+)$ and $|g(\omega,\cdot)|_{\mathcal{C}^1(M)} \leq A$ for each $\omega \in \Omega$;
(ii) $\int_\Omega g(\omega, x) d\nu(\omega) = 1$ for each $x \in M$;
(iii) $g(\omega, x) \geq a$ for all $\omega \in \Omega$, $x \in M$.

We define a random walk on M by assigning to each $\omega \in \Omega$, a map $T \in X_\varepsilon(T_0)$. Starting at $x \in M$, we choose $T_\omega \in X_\varepsilon(T_0)$ according to the distribution $g(\omega, x)d\nu$. We apply T_ω to x and repeat this process starting at $T_\omega x$. We say the process defined in this way has size $\Delta(\nu, g) \leq \varepsilon$.

Notice that if ν is the Dirac measure centered at ω_0, then this process corresponds to the deterministic perturbation T_{ω_0} of T_0. If $g \equiv 1$, then the choice of T_ω is independent of the position x, while in general this formulation allows the choice of the next map to depend on the previous step taken.

The transfer operator $\mathcal{L}_{(\nu,g)}$ associated with the random process is defined by
$$\mathcal{L}_{(\nu,g)} h(x) = \int_\Omega \mathcal{L}_{T_\omega} h(x)\, g(\omega, T_\omega^{-1}x)\, d\nu(\omega)$$
for all $h \in L^1(M, m)$, where m is Lebesgue measure on M.

Theorem 2.6. *The transfer operator $\mathcal{L}_{(\nu,g)}$ satisfies the uniform Lasota-Yorke inequalities given by Theorem 2.2. Let ε_0 be given by (3.13) and let $\varepsilon \le \varepsilon_0$. If $\Delta(\nu, g) \le \varepsilon$, then there exists a constant $C > 0$ depending only on* **(H1)**-**(H5)**, *such that $|||\mathcal{L}_{(\nu,g)} - \mathcal{L}_{T_0}||| \le CA\varepsilon^{\beta/2}$.*

It follows that all the operators $\mathcal{L}_{(\nu,g)}$ enjoy a spectral gap for ε sufficiently small if $\mathcal{L}_{T_0}$ has a spectral gap and the limit theorems of Corollary 2.4 apply to $\mathcal{L}_{(\nu,g)}$.

Theorem 2.6 is proved in Section 5.3.

2.4. **Applications to concrete classes of deterministic perturbations.** In this section we describe precisely several types of perturbations of the Lorentz gas which fall under the abstract framework we have outlined above. In light of Theorems 2.2 and 2.3, it suffices to check two things for each class of perturbations we will introduce: (1) **(H1)**-**(H5)** hold uniformly in each class; (2) the perturbations are small in the sense of the distance $d_{\mathcal{F}}(\cdot,\cdot)$.

A. Movements and Deformations of Scatterers.
We fix the phase space $M = \cup_{i=1}^d I_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ associated with a billiard map corresponding to a periodic Lorentz gas with d scatterers as described above. We assume that the billiard particle moves along straight lines and undergoes elastic reflections at the boundaries.

For given $I_1, \dots, I_d$, we use the notation $Q = Q(\{\Gamma_i\}_{i=1}^d; \{I_i\}_{i=1}^d)$ to denote the configuration of scatterers $\Gamma_1, \dots, \Gamma_d$ placed on the billiard table such that $|\partial\Gamma_i| = \ell(I_i)$, $i = 1, \dots, d$. Since we have fixed $I_1, \dots, I_d$, M remains the same for all configurations Q that we consider. For each such configuration, we define

$$\tau_{\min}(Q) = \inf\{\tau(x) : \tau(x) \text{ is defined for the configuration } Q\}.$$

Similarly, $\mathcal{K}_{\min}(Q)$ and $\mathcal{K}_{\max}(Q)$ denote the minimum and maximum curvatures respectively of the Γ_i in the configuration Q. The constant $E_{\max}(Q)$ denotes the maximum C^3 norm of the $\partial\Gamma_i$ in Q.

For each fixed $\tau_*, \mathcal{K}_*, E_* > 0$, define $\mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$ to be the collection of all configurations Q such that $\tau_{\min}(Q) \ge \tau_*$, $\mathcal{K}_* \le \mathcal{K}_{\min}(Q) \le K_{\max}(Q) \le \mathcal{K}_*^{-1}$, and $E_{\max}(Q) \le E_*$. The horizon for $Q \in \mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$ is allowed to be finite or infinite. Let $\mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ be the corresponding set of billiard maps induced by the configurations in $\mathcal{Q}_1$. It follows from [DZ] that for any $T \in \mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$, $\mathcal{L}_T$ has a spectral gap in $\mathcal{B}$. We prove the following theorems in Section 6.

Theorem 2.7. *Fix $I_1, \dots, I_d$ and let $\tau_*, \mathcal{K}_*, E_* > 0$. The family $\mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ satisfies* **(H1)**-**(H5)** *with uniform constants depending only on τ_*, $\mathcal{K}_*$ and E_*. As a consequence of Theorem 2.2, $\mathcal{L}_T$ is quasi-compact as an operator on $\mathcal{B}$ for each $T \in \mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ with uniform bounds on its essential spectral radius.*

We fix an initial configuration of scatterers $Q_0 \in \mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$ and consider configurations Q which alter each $\partial\Gamma_i$ in Q_0 to a curve $\partial\tilde{\Gamma}_i$ having the same arclength as $\partial\Gamma_i$. We consider each $\partial\Gamma_i$ as a parametrized curve $u_i : I_i \to M$ and each $\partial\tilde{\Gamma}_i$ as parametrized by $\tilde{u}_i$. Define $\Delta(Q, Q_0) = \sum_{i=1}^d |u_i - \tilde{u}_i|_{C^2(I_i,M)}$.

Theorem 2.8. *Choose $\gamma \le \min\{\tau_*/2, \mathcal{K}_*/2\}$ and let $\mathcal{F}_A(Q_0, E_*; \gamma)$ be the set of all billiard maps corresponding to configurations Q such that $\Delta(Q, Q_0) \le \gamma$ and $E_{\max}(Q) \le E_*$.*

Then $\mathcal{F}_A(Q_0, E_*; \gamma) \subset \mathcal{F}_1(\tau_*/2, \mathcal{K}_*/2, E_*)$ and $d_{\mathcal{F}}(T_1, T_2) \le C|\gamma|^{2/15}$ for any $T_1, T_2 \in \mathcal{F}_A(Q_0, E_*; \gamma)$. If all $T_i \in \mathcal{F}_A(Q_0, E_*; \gamma)$ have uniformly bounded finite horizon, then $d_{\mathcal{F}}(T_1, T_2) \le C|\gamma|^{1/3}$.

As a consequence, the eigenvalues outside a disk of radius $\sigma < 1$ and the corresponding spectral projectors of $\mathcal{L}_T$ vary Hölder continuously for all $T \in \mathcal{F}_A(Q_0, E_*; \gamma)$ and all γ sufficiently small.

Remark 2.9. *(a) A remarkable aspect of this result is that it allows us to move configurations from finite to infinite horizon without interrupting Hölder continuity of the statistical properties such as the rate of decay of correlations and the variance in the CLT, among others.*

(b) The requirement that all deformations of the initial configuration Q_0 maintain the same arclength can be relaxed. The purpose of this requirement is to define the corresponding transfer operators on fixed spaces $\mathcal{B}$ and $\mathcal{B}_w$. If a scatterer Γ_i is deformed into Γ_i' with a slight change in arclength, we can reparametrize Γ_i' (no longer according to arclength) using the same interval I_i as for Γ_i. This will change the derivatives of maps in the class $\mathcal{F}_B(Q_0, E_*; \gamma)$ slightly, but since properties **(H1)**-**(H5)** *have some leeway built into the uniform constants, for small enough reparametrizations the same properties will hold with slightly weakened constants.*

B. Billiards Under Small External Forces with Kicks and Slips.
As in part A, we fix $\tau_*, \mathcal{K}_*$ and E_* and choose a fixed $Q_0 \in \mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$. In this section, we consider the dynamics of the billiard map on the table Q_0, but subject to external forces both during flight and at collisions.

Let $\mathbf{q} = (x, y)$ be the position of a particle in a billiard table Q_0 and $\mathbf{p}$ be the velocity vector. For a C^2 stationary external force, $\mathbf{F} : \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, the perturbed billiard flow Φ^t satisfies the following differential equation between collisions:

$$\frac{d\mathbf{q}}{dt} = \mathbf{p}(t), \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{q}, \mathbf{p}). \tag{2.4}$$

At collision, the trajectory experiences possibly nonelastic reflections with slipping along the boundary:

$$(\mathbf{q}^+(t_i), \mathbf{p}^+(t_i)) = (\mathbf{q}^-(t_i), \mathcal{R}\mathbf{p}^-(t_i)) + \mathbf{G}(\mathbf{q}^-(t_i), \mathbf{p}^-(t_i)) \tag{2.5}$$

where $\mathcal{R}\mathbf{p}^-(t_i) = \mathbf{p}^-(t_i) + 2(n(\mathbf{q}^-) \cdot \mathbf{p}^-)n(\mathbf{q}^-))$ is the usual reflection operator, $n(\mathbf{q})$ is the unit normal vector to the billiard wall ∂Q_0 at $\mathbf{q}$ pointing inside the table Q_0, and $\mathbf{q}^-(t_i), \mathbf{p}^-(t_i)$, $\mathbf{q}^+(t_i)$ and $\mathbf{p}^+(t_i)$ refer to the incoming and outgoing position and velocity vectors, respectively. $\mathbf{G}$ is an external force acting on the incoming trajectories. Note that we allow $\mathbf{G}$ to change both the position and velocity of the particle at the moment of collision. The change in velocity can be thought of as a kick or twist while a change in position can model a slip along the boundary at collision.

In [Ch2, Ch4], Chernov considered billiards under small external forces $\mathbf{F}$ with $\mathbf{G} = 0$, and $\mathbf{F}$ to be stationary. In [Z] a twist force was considered assuming $\mathbf{F} = 0$ and $\mathbf{G}$ depending on and affecting only the velocity, not the position. Here we consider a combination of these two cases for systems under more general forces $\mathbf{F}$ and $\mathbf{G}$. We make four assumptions, combining those in [Ch2, Z].

(A1) (**Invariant space**) *Assume the dynamics preserve a smooth function $\mathcal{E}(\mathbf{q}, \mathbf{p})$. Its level surface $\Omega_c := \mathcal{E}^{-1}(c)$, for any $c > 0$, is a compact 3-d manifold such that $\|\mathbf{p}\| > 0$ on Ω_c and for each $\mathbf{q} \in Q$ and $\mathbf{p} \in S^1$ the ray $\{(\mathbf{q}, t\mathbf{p}), t > 0\}$ intersects the manifold Ω_c in exactly one point.*

Assumption (**A1**) specifies an additional integral of motion, so that we only consider restricted systems on a compact phase space. In particular, (**A1**) implies that the speed $p = \|\mathbf{p}\|$ of the billiard along any typical trajectory at time t satisfies

$$0 < p_{\min} \le p(t) \le p_{\max} < \infty$$

for some constants $p_{\min} \le p_{\max}$. Under this assumption the particle will not become overheated, and its speed will remain bounded. For any phase point $\mathbf{x} = (\mathbf{q}, \mathbf{p}) \in \Omega$ for the flow, let $\tau(\mathbf{x})$ be the length of the trajectory between $\mathbf{x}$ and its next non-tangential collision.

(**A2**) (**Finite horizon**) *There exist $\tau_{\max} > \tau_{\min} > 0$ such that free paths between successive reflections are uniformly bounded, $\tau_*/2 \le \tau_{\min} \le \tau(\mathbf{x}) \le \tau_{\max} \le \tau_*^{-1}$, $\forall \mathbf{x} \in \Omega$. Since $Q_0 \in \mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$, the curvature $\mathcal{K}(r)$ of the boundary is also uniformly bounded for all $r \in \partial Q_0$.*

(**A3**) (**Smallness of the perturbation**). *We assume there exists $\varepsilon_1 > 0$ small enough, such that*

$$\|\mathbf{F}\|_{C^1} < \varepsilon_1, \|\mathbf{G}\|_{C^1} < \varepsilon_1.$$

Let $\mathbf{v} = (\cos\theta, \sin\theta)$ denote the unit velocity vector with $\theta \in [0, 2\pi]$, and $\mathcal{M}$ be a level surface Ω_c with coordinates $(\mathbf{q}, \theta)$, for some fixed $c > 0$. Denote $T_{\mathbf{F},\mathbf{G}} : M \to M$ as the billiard map associated to the flow on $\mathcal{M}$, where M is the collision space containing all post-collision vectors based at the boundary of the billiard table Q_0.

(**A4**) *We assume both forces* $\mathbf{F}$ *and* $\mathbf{G}$ *are stationary and that* $\mathbf{G}$ *preserves tangential collisions. In addition, we assume that the singularity set of $T^{-1}_{\mathbf{F},\mathbf{G}}$ is the same as that of $T^{-1}_{\mathbf{F},\mathbf{0}}$.*[3]

The case $\mathbf{F} = \mathbf{G} = 0$ corresponds to the classical billiard dynamics. It preserves the kinetic energy $\mathcal{E} = \frac{1}{2}\|\mathbf{p}\|^2$. We denote by $\mathcal{F}_B(Q_0, \tau_*, \varepsilon_1)$ the class of all perturbed billiard maps defined by the dynamics (2.4) and (2.5) under forces $\mathbf{F}$ and $\mathbf{G}$, satisfying assumptions **(A1)**-**(A4)**.

Theorem 2.10. *For any $T \in \mathcal{F}(Q_0, \tau_*, \varepsilon_1)$, the perturbed system T satisfies* **(H1)**-**(H5)** *with uniform constants depending only on ε_1, τ_*, $\mathcal{K}_*$ and E_*.*

Theorem 2.11. *Within the class $\mathcal{F}_B(Q_0, \tau_*, \varepsilon_1)$, the change of either the force* $\mathbf{F}$ *or* $\mathbf{G}$ *by a small amount δ yields a perturbation of size $\mathcal{O}(|\delta|^{1/3})$ in the distance $d_{\mathcal{F}}(\cdot,\cdot)$.*

As a consequence, the spectral gap enjoyed by the classical billiard $T_{\mathbf{0},\mathbf{0}}$ persists for all $T_{\mathbf{F},\mathbf{G}} \in \mathcal{F}(Q_0, \tau_*, \varepsilon_1)$ for ε_1 sufficiently small so that we may apply the limit theorems of Corollary 2.4 to any such $T_{\mathbf{F},\mathbf{G}}$.

The limit theorems implied by Theorem 2.11 are new even for the simplified maps $T_{\mathbf{F},\mathbf{0}}$ and $T_{\mathbf{0},\mathbf{G}}$. We provide the proofs of Theorems 2.10 and 2.11 in Section 7.

2.5. **Large perturbations: Large translations, rotations and deformations of scatterers.** If we fix $\tau_*, \mathcal{K}_* > 0$ and $E_* < \infty$, then Theorems 2.2 and 2.7 imply that the transfer operator $\mathcal{L}_T$ corresponding to any $T \in \mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ is quasi-compact with essential spectral radius bounded by $\sigma < 1$. In fact, [DZ, Theorem 2.5] implies that $\mathcal{L}_T$ has a spectral gap.

Now choose a compact interval $J \subset \mathbb{R}$ and parametrize a continuous path in $\mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ according to the distance $d_{\mathcal{F}}(\cdot,\cdot)$. To each point $s \in J$ is assigned a map $T_s \in \mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ and a corresponding transfer operator $\mathcal{L}_s$. Fix $\sigma_1 > \sigma$. Due to Theorem 2.3, there exists $\varepsilon_s > 0$ such that the spectra and spectral projectors of $\mathcal{L}_{s'}$ outside the disk of radius σ_1 vary Hölder continuously for $s' \in B(s, \varepsilon_s) := (s - \varepsilon_s, s + \varepsilon_s)$.

The balls $B(s, \varepsilon_s)$, $s \in J$ form an open cover of J and since J is compact, there is a finite subcover $\{B(s_i, \varepsilon_{s_i})\}_{i=1}^n$. Because these intervals overlap, as we move along the entire path from one end of J to the other, the spectra and spectral projectors of $\mathcal{L}_s$ vary Hölder continuously in s. We have proved the following.

Theorem 2.12. *Let $J \subset \mathbb{R}$ be a compact interval and let $\{T_s\}_{s\in J} \subset \mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ be a continuously parametrized path according to the distance $d_{\mathcal{F}}(\cdot,\cdot)$. Then the spectra and spectral projectors of the associated transfer operators $\mathcal{L}_s$ vary Hölder continuously in the distance $d_{\mathcal{F}}(\cdot,\cdot)$. Moreover, the related statistical properties of T_s, such as the rate of decay of correlations and variance in the Central Limit Theorem, vary Hölder continuously in $\{T_s\}$.*

The related dynamical properties of T_s vary Hölder continuously even across large movements and deformations of scatterers as long as the resulting maps remain in $\mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$. Indeed, since J is compact, the continuity of the spectral data implies that the spectral gap is uniform along such

[3]The assumption on the singularity set of $T^{-1}_{\mathbf{F},\mathbf{G}}$ is not essential to our approach, but is made to simplify the proofs in Section 7, since the paper is already quite long and we include a number of distinct applications.

paths even when the resulting configurations are no longer close to the original. This regularity also holds as we move scatterers in such a way that the table changes from finite to infinite horizon.

Remark 2.13. *One could just as well apply the above large movements of scatterers to billiards under external forces in the uniform families $\mathcal{F}_B(Q_s, \tau_*, \varepsilon_1)$ and allow the configurations Q_s, $s \in J$, to change over a continuously parametrized path in $\mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ as long as the horizon along the path remains bounded uniformly above by τ_*^{-1}. Theorem 2.12 applies to such families of maps as well since they all possess spectral gaps by Theorem 2.11.*

3. Abstract Framework

In this section, we describe the abstract framework into which we will place each class of perturbations that we consider. We begin by formulating general conditions **(H1)-(H5)** under which the perturbations of a billiard map will satisfy the Lasota-Yorke inequalities (2.1) with uniform constants. We also introduce general conditions **(C1)-(C4)** to verify that a perturbation is small in the sense of (2.2). Theorems 2.2 and 2.3 show that these conditions are sufficient to establish the framework of [KL]. Once this is accomplished, we only need to check that these conditions are satisfied for each class of perturbations described above.

3.1. **A class of maps with uniform properties.** We fix the phase space $M = \cup_{i=1}^d I_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ of a billiard map associated with a periodic Lorentz gas as in Section 2.2. We will denote (normalized) Lebesgue measure on M by m, i.e., $dm = \frac{1}{\pi L} dr d\varphi$, where $L = \sum_{i=1}^d |I_i|$.

We define the set $\mathcal{S}_0 = \{\varphi = \pm\frac{\pi}{2}\}$ and for a fixed $k_0 \in \mathbb{N}$, we define for $k \geq k_0$, the homogeneity strips,

$$\mathbb{H}_k = \{(r, \varphi) : \pi/2 - k^{-2} < \varphi < \pi/2 - (k+1)^{-2}\}. \tag{3.1}$$

The strips $\mathbb{H}_{-k}$ are defined similarly near $\varphi = -\pi/2$. We also define $\mathbb{H}_0 = \{(r, \varphi) : -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\}$. The set $\mathcal{S}_{0,H} = \mathcal{S}_0 \cup (\cup_{|k| \geq k_0} \partial \mathbb{H}_{\pm k})$ is therefore fixed and will give rise to the singularity sets for the maps that we define below, i.e. for any map T that we consider, we define $\mathcal{S}_{\pm n}^T = \cup_{i=0}^n T^{\mp i} \mathcal{S}_{0,H}$ to be the singularity sets for $T^{\pm n}$, $n \geq 0$.

Suppose there exists a class of invertible maps $\mathcal{F}$ such that for each $T \in \mathcal{F}$, $T : M \setminus \mathcal{S}_1^T \to M \setminus \mathcal{S}_{-1}^T$ is a C^2 diffeomorphism on each connected component of $M \setminus \mathcal{S}_1^T$. We assume that elements of $\mathcal{F}$ enjoy the following uniform properties.

(H1) Hyperbolicity and singularities.

(H1.A) Hyperbolicity. There exist continuous families of stable and unstable cones $C^s(x)$ and $C^u(x)$, defined on all of M, which are strictly invariant for the class $\mathcal{F}$, i.e., $DT(x)C^u(x) \subset C^u(Tx)$ and $DT^{-1}(x)C^s(x) \subset C^s(T^{-1}x)$ for all $T \in \mathcal{F}$ wherever DT and DT^{-1} are defined.

The cones $C^s(x)$ and $C^u(x)$ are uniformly transverse on M and $\mathcal{S}_{-n}^T$ is uniformly transverse to $C^s(x)$ for each $n \in \mathbb{N}$ and all $T \in \mathcal{F}$. We assume in addition that $C^s(x)$ is uniformly transverse to the horizontal and vertical directions on all of M.[4]

Moreover, there exist constants $C_e > 0$ and $\Lambda > 1$ such that for all $T \in \mathcal{F}$,

$$\|DT^n(x)v\| \geq C_e^{-1} \Lambda^n \|v\|, \forall v \in C^u(x), \quad \text{and} \quad \|DT^{-n}(x)v\| \geq C_e^{-1} \Lambda^n \|v\|, \forall v \in C^s(x), \tag{3.2}$$

for all $n \geq 0$, where $\|\cdot\|$ is the Euclidean norm on the tangent space $\mathcal{T}_x M$.

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[4]This is not a restrictive assumption for perturbations of the Lorentz gas since the standard cones $\hat{C}^s$ and $\hat{C}^u$ for the billiard map satisfy this property (see for example [CM, Section 4.5]); the common cones $C^s(x)$ and $C^u(x)$ shared by all maps in the class $\mathcal{F}$ must therefore lie inside $\hat{C}^s(x)$ and $\hat{C}^u(x)$ and therefore satisfy this property. In any case, a weaker formulation of this assumption is necessary: we use in the compactness argument that the lengths of stable curves in the homogeneity strips $\mathbb{H}_k, k \geq k_0$, are proportional to the width of the strips. This is only true if stable curves are transverse to the horizontal direction in such strips.

(H1.B) Accumulation of singularities. The set of curves comprising $T\mathcal{S}_0$ accumulate on no more than K points in M, where K is uniform for the class $\mathcal{F}$. Moreover, there exists an indexing scheme for the singularity curves in $T\mathcal{S}_0 = \{S_n\}_{n\in\mathbb{N}}$ and a uniform constant $N > 0$ such that if $n \geq N$ and $S_n \cap \mathbb{H}_k \neq \emptyset$, then

$$k \geq c_s n^{\upsilon_0}, \tag{3.3}$$

for some uniform constants $c_s, \upsilon_0 > 0$.

(H1.C) Expansion near singularities. For any stable curve $W \in \widehat{\mathcal{W}}^s$ (see **(H2)** below), let W_n denote the part of W between S_n and S_{n+1}, where S_n are singularity curves given by the indexing scheme above. We assume there exists $C_a > 0$ such that

$$C_a n[\cos\varphi(T^{-1}x)]^{-1}\|v\| \leq \|DT^{-1}(x)v\| \leq C_a^{-1} n[\cos\varphi(T^{-1}x)]^{-1}\|v\|, \quad \forall x \in W_n, \forall v \in C^s(x), \tag{3.4}$$

where $\varphi(y)$ denotes the angle at the point $y = (r,\varphi) \in M$. Let $\exp_x$ denote the exponential map from $\mathcal{T}_x M$ to M. We require the following bound on the second derivative,

$$C_a n^2[\cos\varphi(T^{-1}x)]^{-3} \leq \|D^2T^{-1}(x)v\| \leq C_a^{-1} n^2[\cos\varphi(T^{-1}x)]^{-3}, \quad \forall x \in W_n, \tag{3.5}$$

for all $v \in \mathcal{T}_x M$ such that $T^{-1}(\exp_x(v))$ and $T^{-1}x$ lie in the same homogeneity strip.

Remark: *Note that by bounded distortion* **(H4)***, the expansion factors on each component of* $(M \setminus \mathcal{S}_{-1}) \cap \mathbb{H}_k$*,* $k \geq k_0$*, satisfy* (3.4) *and* (3.5) *even when* $W \cap T\mathcal{S}_0 = \emptyset$*.*

(H2) Families of stable and unstable curves. We call W a *stable curve* for a map $T \in \mathcal{F}$ if the tangent line to W, $\mathcal{T}_x W$ lies in $C^s(x)$ for all $x \in W$. We call W *homogeneous* if W is contained in one homogeneity strip $\mathbb{H}_k$. Unstable curves are defined similarly.

Let $\widehat{\mathcal{W}}^s$ denote the set of $\mathcal{C}^2$ homogeneous stable curves in M whose curvature is bounded above by a uniform constant $B > 0$. We assume there exists a choice of B such that $\widehat{\mathcal{W}}^s$ is invariant under $\mathcal{F}$ in the following sense: For any $W \in \widehat{\mathcal{W}}^s$ and $T \in \mathcal{F}$, the connected components of $T^{-1}W$ are again elements of $\widehat{\mathcal{W}}^s$. A family of unstable curves $\widehat{\mathcal{W}}^u$ is defined analogously, with obvious modifications: For example, we require the connected components of TW to be elements of $\widehat{\mathcal{W}}^u$ for all $W \in \widehat{\mathcal{W}}^u$ and $T \in \mathcal{F}$.

(H3) Complexity bounds.

(H3.A) One-step expansion. We assume that there exists an adapted norm $\|\cdot\|_*$, uniformly equivalent to $\|\cdot\|$, in which the constant C_e in (3.2) can be taken to be 1, i.e. we have expansion and contraction in one step in the adapted norm for all maps in the class $\mathcal{F}$.

Let $W \in \widehat{W}^s$. For any $T \in \mathcal{F}$, we partition the connected components of $T^{-1}W$ into maximal pieces $V_i = V_i(T)$ such that each V_i is a homogeneous stable curve in some $\mathbb{H}_k$, $k \geq k_0$, or $\mathbb{H}_0$. Let $|J_{V_i}T|_*$ denote the minimum contraction on V_i under T in the metric induced by the adapted norm $\|\cdot\|_*$. We assume that for some choice of k_0,

$$\limsup_{\delta\to 0} \sup_{T\in\mathcal{F}} \sup_{|W|<\delta} \sum_i |J_{V_i}T|_* < 1, \tag{3.6}$$

where $|W|$ denotes the arclength of W.

(H3.B) One-step expansion with weakened exponent. There exists $\varsigma_0 < 1$ such that for all $\varsigma > \varsigma_0$, there exists $C_\varsigma = C_\varsigma(\varsigma, \delta)$ such that for all $T \in \mathcal{F}$ and any $W \in \widehat{\mathcal{W}}^s$ with $|W| < \delta$,

$$\sum_i |J_{V_i}T|^\varsigma_{\mathcal{C}^0(V_i)} < C_\varsigma, \tag{3.7}$$

where $J_{V_i}T$ denotes the stable Jacobian of T along the curve V_i with respect to arc length.

Remark: *Note that comparing to* **(H3.A)***, the above sum converges even when the expansion on each piece is weakened slightly. We formulate* (3.7) *in terms of the usual Euclidean norm since we do not need* $C_\varsigma < 1$*, i.e. we only need the above sum to be finite in some uniform sense.*

(H4) Bounded distortion. There exists a constant $C_d > 0$ with the following properties. Let $W' \in \widehat{\mathcal{W}}^s$ and for any $T \in \mathcal{F}$, $n \in \mathbb{N}$, let $x, y \in W$ for some connected component $W \subset T^{-n}W'$ such that T^iW is a homogeneous stable curve for each $0 \le i \le n$. Then,

$$\left|\frac{J_\mu T^n(x)}{J_\mu T^n(y)} - 1\right| \le C_d d_W(x,y)^{1/3} \text{ and } \left|\frac{J_W T^n(x)}{J_W T^n(y)} - 1\right| \le C_d d_W(x,y)^{1/3}, \tag{3.8}$$

where $J_\mu T^n$ is the Jacobian of T^n with respect to the smooth measure $d\mu = \frac{\pi}{2}\cos\varphi dm$.

We assume the analogous bound along unstable leaves: If $W \in \widehat{\mathcal{W}}^u$ is an unstable curve such that T^iW is a homogeneous unstable curve for $0 \le i \le n$, then for any $x, y \in W$,

$$\left|\frac{J_\mu T^n(x)}{J_\mu T^n(y)} - 1\right| \le C_d d(T^n x, T^n y)^{1/3}. \tag{3.9}$$

(H5) Control of Jacobian. Let $\beta, q < 1$ be from the definition of the norms in Section 3.3 and let $\theta_* < 1$ be from (3.10). Assume there exists a constant $\eta < \min\{\Lambda^\beta, \Lambda^q, \theta_*^{\alpha-1}\}$ such that for any $T \in \mathcal{F}$,

$$(J_\mu T(x))^{-1} \le \eta, \quad \text{wherever } J_\mu T(x) \text{ is defined.}$$

In what follows, we will work exclusively with maps in a class $\mathcal{F}$ which satisfy **(H1)**-**(H5)** with uniform constants. This will allow us to establish our abstract framework in the remainder of Section 3 and in Sections 4 and 5. We will then turn to our applications in Sections 6 and 7.

3.2. **Transfer operator.** Recall the family of stable curves $\widehat{\mathcal{W}}^s$ defined by **(H2)**. We define a subset $\mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ as follows. By **(H3)** we may choose $\delta_0 > 0$ for which there exists $\theta_* < 1$ such that

$$\sup_{T\in\mathcal{F}} \sup_{|W|\le\delta_0} \sum_i |J_{V_i}T|_* \le \theta_*. \tag{3.10}$$

We shrink δ_0 further if necessary so that the graph transform argument in Lemma 3.3(a) holds. The set $\mathcal{W}^s$ comprises all those stable curves $W \in \widehat{\mathcal{W}}^s$ such that $|W| \le \delta_0$.

For any $T \in \mathcal{F}$, we define scales of spaces using the set of stable curves $\mathcal{W}^s$ on which the *transfer operator* $\mathcal{L}_T$ associated with T will act. Define $T^{-n}\mathcal{W}^s$ to be the set of homogeneous stable curves W such that T^n is smooth on W and $T^iW \in \mathcal{W}^s$ for $0 \le i \le n$. It follows from **(H2)** that $T^{-n}\mathcal{W}^s \subset \mathcal{W}^s$.

For $W \in T^{-n}\mathcal{W}^s$, a complex-valued test function $\psi : M \to \mathbb{C}$, and $0 < p \le 1$ define $H^p_W(\psi)$ to be the Hölder constant of ψ on W with exponent p measured in the Euclidean metric. Define $H^p_n(\psi) = \sup_{W\in T^{-n}\mathcal{W}^s} H^p_W(\psi)$ and let $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s) = \{\psi : M \to \mathbb{C} \mid H^p_n(\psi) < \infty\}$, denote the set of complex-valued functions which are Hölder continuous on elements of $T^{-n}\mathcal{W}^s$. The set $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$ equipped with the norm $|\psi|_{\mathcal{C}^p(T^{-n}\mathcal{W}^s)} = |\psi|_\infty + H^p_n(\psi)$ is a Banach space. Similarly, we define $\tilde{\mathcal{C}}^p(\widehat{\mathcal{W}}^u)$, the set of functions which are Hölder continuous with exponent p on unstable curves $\widehat{\mathcal{W}}^u$.

It follows from (4.6) that if $\psi \in \tilde{\mathcal{C}}^p(T^{-(n-1)}\mathcal{W}^s)$, then $\psi \circ T \in \tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$. Thus if $h \in (\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s))'$, is an element of the dual of $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$, then $\mathcal{L}_T : (\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s))' \to (\tilde{\mathcal{C}}^p(T^{-(n-1)}\mathcal{W}^s))'$ acts on h by

$$\mathcal{L}_T h(\psi) = h(\psi \circ T) \quad \forall \psi \in \tilde{\mathcal{C}}^p(T^{-(n-1)}\mathcal{W}^s).$$

Recall that $d\mu = \frac{\pi}{2}\cos\varphi dm$ denotes the smooth invariant measure for the unperturbed Lorentz gas. If $h \in L^1(M, \mu)$, then h is canonically identified with a signed measure absolutely continuous with respect to μ, which we shall also call h, i.e., $h(\psi) = \int_M \psi h \, d\mu$. With the above identification, we write $L^1(M,\mu) \subset (\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s))'$ for each $n \in \mathbb{N}$. Then restricted to $L^1(M,\mu)$, $\mathcal{L}_T$ acts according to the familiar expression

$$\mathcal{L}_T^n h = h \circ T^{-n} \, (J_\mu T^n(T^{-n}))^{-1} \quad \text{for any } n \geq 0 \text{ and } h \in L^1(M,\mu).$$

Remark 3.1. *In* [DZ]*, we used Lebesgue measure as a reference measure to show that the functional analytic framework developed there did not need to assume the existence of a smooth invariant measure. Now that μ has been established in our function space $\mathcal{B}$ (defined in Sect. 3.3), however, we find it more convenient to use it as a starting point in our study of the classes of perturbations considered here. It also simplifies our norms and estimates slightly since for example, it eliminates the need for the* $\cos W$ *weight in our test functions that was used in* [DZ]*. We do not assume that μ is an invariant measure for $T \in \mathcal{F}$; indeed, the SRB measures for such T are in general singular with respect to Lebesgue measure.*

3.3. **Definition of the Norms.** The norms are defined via integration on the set of stable curves $\mathcal{W}^s$. Before defining the norms, we define the notion of a distance $d_{\mathcal{W}^s}(\cdot,\cdot)$ between such curves as well as a distance $d_q(\cdot,\cdot)$ defined among functions supported on these curves.

Due to the transversality condition on the stable cones $C^s(x)$ given by **(H1)**, each stable curve W can be viewed as the graph of a function $\varphi_W(r)$ of the arc length parameter r. For each $W \in \mathcal{W}^s$, let I_W denote the interval on which φ_W is defined and set $G_W(r) = (r, \varphi_W(r))$ to be its graph so that $W = \{G_W(r) : r \in I_W\}$. We let m_W denote the unnormalized arclength measure on W, defined using the Euclidean metric.

Let $W_1, W_2 \in \mathcal{W}^s$ and identify them with the graphs G_{W_i} of their functions φ_{W_i}, $i = 1,2$. Suppose W_1, W_2 lie in the same component of M and let I_{W_i} be the r-interval on which each curve is defined. Denote by $\ell(I_{W_1} \triangle I_{W_2})$ the length of the symmetric difference between I_{W_1} and I_{W_2}. Let $\mathbb{H}_{k_i}$ be the homogeneity strip containing W_i. We define the distance between W_1 and W_2 to be,

$$d_{\mathcal{W}^s}(W_1,W_2) = \eta(k_1,k_2) + \ell(I_{W_1} \triangle I_{W_2}) + |\varphi_{W_1} - \varphi_{W_2}|_{\mathcal{C}^1(I_{W_1} \cap I_{W_2})}$$

where $\eta(k_1,k_2) = 0$ if $k_1 = k_2$ and $\eta(k_1,k_2) = \infty$ otherwise, i.e., we only compare curves which lie in the same homogeneity strip.

For $0 \leq p \leq 1$, denote by $\tilde{\mathcal{C}}^p(W)$ the set of continuous complex-valued functions on W with Hölder exponent p, measured in the Euclidean metric, which we denote by $d_W(\cdot,\cdot)$. We then denote by $\mathcal{C}^p(W)$ the closure of $\mathcal{C}^\infty(W)$ in the $\tilde{\mathcal{C}}^p$-norm[5]: $|\psi|_{\mathcal{C}^p(W)} = |\psi|_{\mathcal{C}^0(W)} + H^p_W(\psi)$, where $H^p_W(\psi)$ is the Hölder constant of ψ along W. Notice that with this definition, $|\psi_1\psi_2|_{\mathcal{C}^p(W)} \leq |\psi_1|_{\mathcal{C}^p(W)} |\psi_2|_{\mathcal{C}^p(W)}$. We define $\tilde{\mathcal{C}}^p(M)$ and $\mathcal{C}^p(M)$ similarly.

Given two functions $\psi_i \in \mathcal{C}^q(W_i, \mathbb{C})$, $q > 0$, we define the distance between ψ_1, ψ_2 as

$$d_q(\psi_1,\psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathcal{C}^q(I_{W_1} \cap I_{W_2})}.$$

We will define the required Banach spaces by closing $\mathcal{C}^1(M)$ with respect to the following set of norms. For $s, p \geq 0$, define the following norms for test functions,

$$|\psi|_{W,s,p} := |W|^s \cdot |\psi|_{\mathcal{C}^p(W)}.$$

Now fix $0 < p \leq \frac{1}{3}$. Given a function $h \in \mathcal{C}^1(M)$, define the *weak norm* of h by

$$|h|_w := \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^p(W) \\ |\psi|_{W,0,p} \leq 1}} \int_W h\psi \, dm_W. \tag{3.11}$$

[5]While $\mathcal{C}^p(W)$ may not contain all of $\tilde{\mathcal{C}}^p(W)$, it does contain $\mathcal{C}^{p'}(W)$ for all $p' > p$.

Choose[6] α, β, $q > 0$ such that $\alpha < 1 - \varsigma_0$, $q < p$ and $\beta \le \min\{\alpha, p - q\}$. We define the *strong stable norm* of h as

$$\|h\|_s := \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in \mathcal{C}^q(W) \\ |\psi|_{W,\alpha,q} \le 1}} \int_W h\psi \, dm_W \tag{3.12}$$

and the *strong unstable norm* as

$$\|h\|_u := \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon}} \sup_{\substack{\psi_i \in \mathcal{C}^p(W_i) \\ |\psi_i|_{W_i,0,p} \le 1 \\ d_q(\psi_1,\psi_2) \le \varepsilon}} \frac{1}{\varepsilon^\beta} \left| \int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W \right| \tag{3.13}$$

where $\varepsilon_0 > 0$ is chosen less than δ_0, the maximum length of $W \in \mathcal{W}^s$ which is determined by (3.10). We then define the *strong norm* of h by

$$\|h\|_{\mathcal{B}} = \|h\|_s + b\|h\|_u$$

where b is a small constant chosen in Section 4.

We define $\mathcal{B}$ to be the completion of $\mathcal{C}^1(M)$ in the strong norm[7] and $\mathcal{B}_w$ to be the completion of $\mathcal{C}^1(M)$ in the weak norm.

3.4. **Distance in $\mathcal{F}$.** We define a distance in $\mathcal{F}$ as follows. Let ε_0 be from (3.13). For $T_1, T_2 \in \mathcal{F}$ and $\varepsilon \le \varepsilon_0$, let $N_\varepsilon(\mathcal{S}^i_{-1})$ denote the ε-neighborhood in M of the singularity set $\mathcal{S}^i_{-1}$ of T_i^{-1}, $i = 1, 2$. We say $d_{\mathcal{F}}(T_1, T_2) \le \varepsilon$ if the maps are close away from their singularity sets in the following sense: For $x \notin N_\varepsilon(\mathcal{S}^1_{-1} \cup \mathcal{S}^2_{-1})$,

(C1) $d(T_1^{-1}(x), T_2^{-1}(x)) \le \varepsilon$;

(C2) $\left| \dfrac{J_\mu T_i(x)}{J_\mu T_j(x)} - 1 \right| \le \varepsilon$, $i, j = 1, 2$;

(C3) $\left| \dfrac{J_W T_i(x)}{J_W T_j(x)} - 1 \right| \le \varepsilon$, for any $W \in \mathcal{W}^s$, $i, j = 1, 2$, and $x \in W$;

(C4) $\|DT_1^{-1}(x)v - DT_2^{-1}(x)v\| \le \sqrt{\varepsilon}$, for any unit vector $v \in \mathcal{T}_x W$, $W \in \mathcal{W}^s$.

We do not assume that the sets $\mathcal{S}^1_{-1}$ and $\mathcal{S}^2_{-1}$ are close together in any sense.

3.5. **Preliminary estimates.** Before proving the Lasota-Yorke inequalities, we show how **(H1)**-**(H5)** imply several other uniform properties for our class of maps $\mathcal{F}$. In particular, we will be interested in iterating the one-step expansion relations given by **(H3)**. We recall the estimates we need from [DZ, Section 3.2].

Let $T \in \mathcal{F}$ and $W \in \mathcal{W}^s$. Let V_i denote the maximal connected components of $T^{-1}W$ after cutting due to singularities and the boundaries of the homogeneity strips. To ensure that each component of $T^{-1}W$ is in $\mathcal{W}^s$, we subdivide any of the long pieces V_i whose length is $> \delta_0$, where δ_0 is chosen in (3.10). This process is then iterated so that given $W \in \mathcal{W}^s$, we construct the components of $T^{-n}W$, which we call the n^{th} generation $\mathcal{G}_n(W)$, inductively as follows. Let $\mathcal{G}_0(W) = \{W\}$ and suppose we have defined $\mathcal{G}_{n-1}(W) \subset \mathcal{W}^s$. First, for any $W' \in \mathcal{G}_{n-1}(W)$, we partition $T^{-1}W'$ into at most countably many pieces W'_i so that T is smooth on each W'_i and each W'_i is a homogeneous stable curve. If any W'_i have length greater than δ_0, we subdivide those

[6] The restrictions on the constants are placed according to the dynamical properties of T. For example, $p \le 1/3$ due to the distortion bounds in **(H4)**, while $\alpha < 1 - \varsigma_0$ so that Lemma 3.2(d) can be applied with $\varsigma = 1 - \alpha > \varsigma_0$.

[7] As a measure, $h \in \mathcal{C}^1(M)$ is identified with $h d\mu$ according to our earlier convention. As a consequence, Lebesgue measure $dm = (\cos\varphi)^{-1} d\mu$ is not automatically included in $\mathcal{B}$ since $(\cos\varphi)^{-1} \notin \mathcal{C}^1(M)$. We will prove in Lemma 3.5 that in fact, $m \in \mathcal{B}$ (and $\mathcal{B}_w$).

pieces into pieces of length between $\delta_0/2$ and δ_0. We define $\mathcal{G}_n(W)$ to be the collection of all pieces $W_i^n \subset T^{-n}W$ obtained in this way. Note that each W_i^n is in $\mathcal{W}^s$ by **(H2)**.

At each iterate of T^{-1}, typical curves in $\mathcal{G}_n(W)$ grow in size, but there exist a portion of curves which are trapped in tiny homogeneity strips and in the infinite horizon case, stay too close to the infinite horizon points. In Lemma 3.2, we make precise the sense in which the proportion of curves that never grow to a fixed length decays exponentially fast.

For $W \in \mathcal{W}^s$, $n \geq 0$, and $0 \leq k \leq n$, let $\mathcal{G}_k(W) = \{W_i^k\}$ denote the k^{th} generation pieces in $T^{-k}W$. Let $B_k(W) = \{i : |W_i^k| < \delta_0/3\}$ and $L_k(W) = \{i : |W_i^k| \geq \delta_0/3\}$ denote the index of the short and long elements of $\mathcal{G}_k(W)$, respectively. We consider $\{\mathcal{G}_k\}_{k=0}^n$ as a tree with W as its root and $\mathcal{G}_k$ as the k^{th} level.

At level n, we group the pieces as follows. Let $W_{i_0}^n \in \mathcal{G}_n(W)$ and let $W_j^k \in L_k(W)$ denote the most recent long "ancestor" of $W_{i_0}^n$, i.e. $k = \max\{0 \leq \ell \leq n : T^{n-\ell}(W_{i_0}^n) \subset W_j^\ell \text{ and } j \in L_\ell\}$. If no such ancestor exists, set $k = 0$ and $W_j^k = W$. Note that if $W_{i_0}^n$ is long, then $W_j^k = W_{i_0}^n$. Let

$$\mathcal{I}_n(W_j^k) = \{i : W_j^k \in L_k(W) \text{ is the most recent long ancestor of } W_i^n \in \mathcal{G}_n(W)\}.$$

The set $\mathcal{I}_n(W)$ represents those curves W_i^n that belong to short pieces in $\mathcal{G}_k(W)$ at each time step $1 \leq k \leq n$, i.e. such W_i^n are never part of a piece that has grown to length $\geq \delta_0/3$.

We collect the results of [DZ, Section 3.2] in the following lemma.

Lemma 3.2. *([DZ]) Let $W \in \mathcal{W}^s$, $T \in \mathcal{F}$ and for $n \geq 0$, let $\mathcal{I}_n(W)$ and $\mathcal{G}_n(W)$ be defined as above. There exist constants $C_1, C_2, C_3 > 0$, independent of W and T, such that for any $n \geq 0$,*

(a) $\displaystyle\sum_{i \in \mathcal{I}_n(W)} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \leq C_1 \theta_*^n$;

(b) $\displaystyle\sum_{W_i^n \in \mathcal{G}_n(W)} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \leq C_2$;

(c) *for any* $0 \leq \varsigma \leq 1$, $\displaystyle\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|W_i^n|^\varsigma}{|W|^\varsigma} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \leq C_2^{1-\varsigma}$;

(d) *for* $\varsigma > \varsigma_0$, $\displaystyle\sum_{W_i^n \in \mathcal{G}_n(W)} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)}^\varsigma \leq C_3^n$, *where C_3 depends on ς.*

Proof. The proofs of these items are combinatorial and require no more specific information about the maps than the uniform properties given by **(H2)**, **(H3)** and **(H4)**.

(a) This is Lemma 3.1 of [DZ]. The constant C_1 depends only on the constant relating the Euclidean norm $\|\cdot\|$ to the adapted norm $\|\cdot\|_*$. As such, C_1 is independent of $T \in \mathcal{F}$, $W \in \mathcal{W}^s$ and $n \in \mathbb{N}$.

(b) This statement is [DZ, Lemma 3.2]. The constant $C_2 = C_2(\delta_0, \theta_*, C_1, C_d)$.

(c) This is [DZ, Lemma 3.3]. It follows from (b) by an application of Jensen's inequality.

(d) This follows from (3.7) and is proved in [DZ, Lemma 3.4]. The constant $C_3 = \delta_0^{-1} C_\varsigma (1 + C_d)^{2\varsigma}$ is uniform for $T \in \mathcal{F}$, but depends on ς. □

Next we prove a distortion bound for the stable Jacobian of T along different stable curves in the following context. Let $W^1, W^2 \in \mathcal{W}^s$ and suppose there exist $U^k \subset T^{-n}W^k$, $k = 1, 2$, such that for $0 \leq i \leq n$,

(i) $T^i U^k \in \mathcal{W}^s$ and the curves $T^i U^1$ and $T^i U^2$ lie in the same homogeneity strip;
(ii) U^1 and U^2 can be put into a 1-1 correspondence by a smooth foliation $\{\gamma_x\}_{x \in U^1}$ of curves $\gamma_x \in \widehat{\mathcal{W}}^u$ such that $\{T^n \gamma_x\} \subset \widehat{\mathcal{W}}^u$ creates a 1-1 correspondence between $T^n U^1$ and $T^n U^2$;
(iii) $|T^i \gamma_x| \leq 2 \max\{|T^i U^1|, |T^i U^2|\}$, for all $x \in U^1$.

Let $J_{U^k} T^n$ denote the stable Jacobian of T^n along the curve U^k with respect to arclength.

Lemma 3.3. *In the setting above, for $x \in U^1$, define $x^* \in \gamma_x \cap U^2$. There exists $C_0 > 0$, independent of $T \in \mathcal{F}$, $W \in \mathcal{W}^s$ and $n \geq 0$ such that*

(a) $d_{\mathcal{W}^s}(U^1, U^2) \leq C_0 \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$;

(b) $\left| \frac{J_{U^1} T^n(x)}{J_{U^2} T^n(x^*)} - 1 \right| \leq C_0 [d(T^n x, T^n x^*)^{1/3} + \theta(T^n x, T^n x^*)]$,

where $\theta(T^n x, T^n x^*)$ is the angle formed by the tangent lines of $T^n U^1$ and $T^n U_2$ at $T^n x$ and $T^n x^*$, respectively.

Proof. (a) This is essentially a graph transform argument adapted for this class of maps satisfying **(H1)**. What we need to show here is that we do not need to cut curves lying in homogeneity strips any further in order to get the required contraction and control on distortion.

First notice that due to the uniform expansion of γ_x under T^n given by (3.2) of **(H1)**, we have $|\gamma_x| \leq C_e C_t \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$, where C_t is a constant depending only on the minimum angle between $C^u(x)$ and $C^s(x)$ and between $C^u(x)$ and the horizontal direction. Again by the transversality of γ_x with U^1 and U^2, the r-intervals on which the functions φ_{U^1}, φ_{U^2} describing the curves U^1, U^2 are defined can differ by no more than $C_e C_t^2 \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$. Letting I denote the intersection of intervals on which both functions are defined and recalling the definition of $d_{\mathcal{W}^s}(\cdot, \cdot)$ from Section 3.3, it remains to estimate $|\varphi_{U^1} - \varphi_{U^2}|_{\mathcal{C}^1(I)}$.

By the same observation as above, we have $|\varphi_{U^1} - \varphi_{U^2}|_{\mathcal{C}^0(I)} \leq C_t^2 C_e \Lambda^{-n} d_{W^s}(W^1, W^2)$. In order to show that the slopes of these curves also contract exponentially, we make the usual graph transform argument using charts in the adapted norm $\| \cdot \|_*$ from **(H3)**.

Fix $x \in U^1$ and define charts along the orbit of x so that $x_i := T^i x$, $0 \leq i \leq n$, corresponds to the origin in each chart with the stable direction at x_i given by the horizontal axis and the unstable direction by the vertical axis in the charts. Let $\vartheta < 1$ denote the maximum absolute value of slopes of stable curves in the chart. Due to property (iii) before the statement of the lemma, we may choose the size of the charts to have stable and unstable diameters $\leq C|T^i U1|$ for each i, for some uniform constant C. The dynamics induced by T^{-1} on these charts is defined by

$$\tilde{T}_{x_i}^{-1} = \chi_{x_{i-1}}^{-1} \circ T^{-1} \circ \chi_{x_i}$$

where χ_{x_i} are smooth maps with $|\chi_{x_i}|_{\mathcal{C}^2}, |\chi_{x_i}^{-1}|_{\mathcal{C}^2} \leq C$ for some uniform constant C.

Note that $D\tilde{T}_{x_i}^{-1}$ and $D^2\tilde{T}_{x_i}^{-1}$ satisfy **(H1)** with possibly larger C_a and $C_e = 1$. In the chart coordinates, since $\tilde{T}_{x_i}^{-1}(0) = 0$, we have

$$\tilde{T}_{x_i}^{-1}(s,t) = (A_i s + \alpha_i(s,t), B_i t + \beta_i(s,t))$$

where A_i is the expansion at x_i in the stable direction and B_i is the contraction at x_i in the unstable direction given by $DT_{x_i}^{-1}(0)$. The nonlinear functions α_i, β_i satisfy $\alpha_i(0,0) = \beta_i(0,0) = 0$ and their Lipschitz constants are bounded by the maximum of

$$\|D\tilde{T}_{x_i}^{-1}(u) - D\tilde{T}_{x_i}^{-1}(v)\| \leq \|D^2\tilde{T}_{x_i}(z)\| \|u - v\| \tag{3.14}$$

where u, v, z range over the chart at x_i.

We fix i and let φ_1, φ_2 denote two Lipschitz functions whose graphs lie in the stable cone of the chart at x_i and satisfy $\varphi_j(0) = 0$, $j = 1, 2$. Define $L(\varphi_1, \varphi_2) = \sup_{s \neq 0} \frac{|\varphi_1(s) - \varphi_2(s)|}{|s|}$. Let $\varphi_1' = \tilde{T}_*^{-1}\varphi_1$ and $\varphi_2' = \tilde{T}_*^{-1}\varphi_2$ denote the graphs of the images of these two curves in the chart at x_{i-1}. We wish to estimate $L(\varphi_1', \varphi_2')$. For s on the horizontal axis in the chart at x_i, we write,

$$\begin{aligned}
|\varphi_1'(A_i s + \alpha_i(s, \varphi_1(s))) - \varphi_2'(A_i s + \alpha_i(s, \varphi_1(s)))| &\leq |\varphi_1'(A_i s + \alpha_i(s, \varphi_1(s))) - \varphi_2'(A_i s + \alpha_i(s, \varphi_2(s)))| \\
&\quad + |\varphi_2'(A_i s + \alpha_i(s, \varphi_2(s))) - \varphi_2'(A_i s + \alpha_i(s, \varphi_1(s)))| \\
&\leq |B_i||\varphi_1(s) - \varphi_2(s)| + |\beta_i(s, \varphi_1(s)) - \beta_i(s, \varphi_2(s))| + \vartheta|\alpha_i(s, \varphi_1(s)) - \alpha_i(s, \varphi_2(s))| \\
&\leq (|B_i| + \mathrm{Lip}(\beta_i) + \vartheta \mathrm{Lip}(\alpha_i))|\varphi_1(s) - \varphi_2(s)|
\end{aligned}$$

On the other hand, by (3.4),

$$|A_i s + \alpha_i(s, \varphi_1(s))| \geq (|A_i| - \mathrm{Lip}(\alpha_i)(1+\vartheta))|s|.$$

Putting these together, we see that,

$$L(\varphi_1', \varphi_2') \leq \sup_{s\neq 0} \frac{(|B_i| + \mathrm{Lip}(\beta_i) + \vartheta \mathrm{Lip}(\alpha_i))|\varphi_1(s) - \varphi_2(s)|}{(|A_i| - \mathrm{Lip}(\alpha_i)(1+\vartheta))|s|} \leq \frac{|B_i| + \mathrm{Lip}(\beta_i) + \vartheta \mathrm{Lip}(\alpha_i)}{|A_i| - \mathrm{Lip}(\alpha_i)(1+\vartheta)} L(\varphi_1, \varphi_2). \tag{3.15}$$

Suppose that x_{i-1} lies in the homogeneity strip $\mathbb{H}_k$ and x_i lies on a curve with index n according to the index given by **(H1)**. Then by (3.14) and (3.4) and (3.5) of **(H1)**, the Lipshitz constants of α_i and β_i are bounded by $C_a^{-1} n^2 k^6 (C_a^{-1} n^{-1} k^{-5}) = C_a^{-2} nk$ since the size of the chart is taken to be on the order of the length of the curve $T^i U^1$ by property (iii) of the matching. Thus,

$$L(\varphi_1', \varphi_2') \leq \frac{\Lambda^{-1} + C_a^{-2} nk(1+\vartheta)}{C_a nk^2 - C_a^{-2} nk(1+\vartheta)} L(\varphi_1, \varphi_2) \leq \frac{4C_a^{-3}}{k} L(\varphi_1, \varphi_2),$$

for large k, which can be made smaller than Λ^{-1}. Note that since $k \geq c_s n^{v_0}$ by **(H1)**, this bound is also small for large n. Thus we may choose $N_0, K_0 > 0$ such that the contraction is less than Λ^{-1} on all curves with index $n \geq N_0$ or landing in homogeneity strip $\mathbb{H}_k$, $k \geq K_0$. On the remainder of M, the first and second derivatives of T^{-1} are uniformly bounded by constants depending on N_0 and K_0. For curves in this part of M, we choose δ_0, the maximum length of stable curves in $\mathcal{W}^s$, sufficiently small that the distortion given by (3.14) is less than $\frac{1}{2}(\Lambda^{-1/2} - \Lambda^{-1})$. Then by (3.15), since $\vartheta < 1$, the contraction on these pieces is less than Λ^{-1} as well.

If φ_1 and φ_2 do not pass through the origin, the exponential contraction in the C^0 norm coupled with the above argument yields the required contraction.

(b) It is equivalent to estimate the ratio $\log \frac{J_{T^n U_1} T^{-n}(T^n x)}{J_{T^n U_2} T^{-n}(T^n x^*)}$. We write

$$\log \frac{J_{T^n U_1} T^{-n}(T^n x)}{J_{T^n U_2} T^{-n}(T^n x^*)} \leq \sum_{i=1}^{n} \frac{1}{A_i} |J_{T^i U_1} T^{-1}(T^i x) - J_{T^i U_2} T^{-1}(T^i x^*)| \tag{3.16}$$

where $A_i = \min\{J_{T^i U_1} T^{-1}(T^i x), J_{T^i U_2} T^{-1}(T^i x^*)\}$.

We estimate the differences one term at a time and assume without loss of generality that the minimum for A_i is attained at $T^i x$. Set $x_i = T^i x$, $x_i^* = T^i x^*$. Let $\vec{u}_1(x_i)$ denote the unit tangent vector to $T^i U^1$ at x_i and notice that $J_{T^i U_1} T^{-1}(x_i) = \|DT^{-1}(x_i)\vec{u}_1\|$. Define $\vec{u}_2(x_i^*)$ similarly. Then

$$\begin{aligned} |\,\|DT^{-1}(x_i)\vec{u}_1\| - \|DT^{-1}(x_i^*)\vec{u}_2\|\,| &\leq |\,\|DT^{-1}(x_i)\vec{u}_1\| - \|DT^{-1}(x_i)\vec{u}_2\|\,| \\ &\quad + |\,\|DT^{-1}(x_i)\vec{u}_2\| - \|DT^{-1}(x_i^*)\vec{u}_2\|\,| \\ &\leq \|DT^{-1}(x_i)\|\,\|\vec{u}_1 - \vec{u}_2\| + \|D^2T^{-1}(z_i)\| d(x_i, x_i^*), \end{aligned}$$

where z_i is some point on $T^i \gamma_x$.

Suppose $T^{-1}x_i$ lies in the homogeneity strip $\mathbb{H}_k$ and x_i lies on some curve W_n according to the index given in **(H1)**. Then $\|DT^{-1}(x_i)\|/\|DT^{-1}(x_i)\vec{u}\| \leq C$ where C is some uniform constant for all unit vectors $\vec{u} \in C^s(x_i)$. Also by **(H1)**, we have $|T^i U^j| \leq C_a/nk^5$, $j = 1, 2$, so that by property (iii) before the statement of the lemma, $d(x_i, x_i^*) \leq 2C_a/(nk^5)$. Thus

$$\frac{\|D^2 T^{-1}(z_i)\| d(x_i, x_i^*)}{\|DT^{-1}(x_i)\vec{u}_1\|} \leq \frac{(C_a n^2 k^6)(2C_a/(nk^5))}{C_a^{-1} nk^2} \leq \frac{2C_a^3}{k} \leq 2C_a^3 d(x_{i-1}, x_{i-1}^*)^{1/3}.$$

Using these estimates in (3.16), we have

$$\log \frac{J_{T^n U_1} T^{-n}(T^n x)}{J_{T^n_U 2} T^{-n}(T^n x^*)} \leq C \sum_{i=1}^{n} \|\vec{u}_1(x_i) - \vec{u}_2(x_i^*)\| + d(x_{i-1}, x_{i-1}^*)^{1/3}.$$

Now $\|\vec{u}_1(x_i)-\vec{u}_2(x_i^*)\| \le \theta(x_i, x_i^*) \le C_0\Lambda^{i-n}\theta(T^n x, T^n x^*)$ by part (a) of the lemma together with the fact that curves in $\mathcal{W}^s$ have $\mathcal{C}^2$ norm uniformly bounded above. Finally, by **(H1)**, $d(x_{i-1}, x_{i-1}^*) \le C_e\Lambda^{i-n-1}d(T^n x, T^n x^*)$, which completes the proof of the lemma. □

3.6. **Properties of the Banach spaces.** We first prove that the weak and strong norms dominate distributional norms on M in the following sense.

Lemma 3.4. *There exists $C > 0$ such that for any $h \in \mathcal{B}_w$, $T \in \mathcal{F}$, $n \ge 0$ and $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$,*

$$|h(\psi)| \le C|h|_w(|\psi|_\infty + H_n^p(\psi)).$$

This is the analogue of Lemma 3.9 of [DZ], but it does not follow from the argument given there since condition **(H5)** and the weakened Lasota-Yorke inequalities in Theorem 2.2 suggest that the spectral radius of $\mathcal{L}_T$ can be as much as $\eta > 1$. It is a consequence of Lemma 3.4 that the spectral radius is in fact 1 (see Section 4, proof of Theorem 2.2).

Proof of Lemma 3.4. Define $M_\ell := I_\ell \times [-\pi/2, \pi/2]$. We partition the set $\mathbb{H}_0$ into finitely many boxes B_j whose boundary curves are elements of $\mathcal{W}^s$ and $\mathcal{W}^u$ as well as the horizontal lines $\pm\pi/2 \mp 1/k_0^2$. We construct the boxes so that each B_j has diameter $\le \delta_0$ and is foliated by a smooth family of stable curves $\{W_\xi\}_{\xi \in E_j} \subset \mathcal{W}^s$, each of whose elements completely crosses B_j in the approximate stable direction.

We decompose the smooth measure $d\mu = \cos\varphi dm$ on B_j into $d\mu = \hat{\mu}(d\xi)d\mu_\xi$, where μ_ξ is the conditional measure of μ on W_ξ and $\hat{\mu}$ is the transverse measure on E_j. We normalize the measures so that $\mu_\xi(W_\xi) = \int_{W_\xi} \cos\varphi \, dm_{W_\xi}$. Since the foliation is smooth, $d\mu_\xi = \rho_\xi \cos\varphi dm_{W_\xi}$ where $|\rho_\xi|_{\mathcal{C}^1(W_\xi)} \le C$ for some constant C independent of ξ. Note that $\hat{\mu}(E_j) \le C\delta_0$ due to the transversality of curves in $\mathcal{W}^s$ and $\mathcal{W}^u$. Next we choose on each homogeneity strip $\mathbb{H}_t$, $t \ge k_0$, a smooth foliation $\{W_\xi\}_{\xi\in E_t} \subset \mathcal{W}^s$ whose elements all have endpoints lying in the two boundary curves of $\mathbb{H}_t$. We again decompose μ on $\mathbb{H}_t$ into $d\mu = \hat{\mu}(d\xi)d\mu_\xi$, $\xi \in E_t$, and $d\mu_\xi = \rho_\xi \cos\varphi dm_{W_\xi}$ is normalized as above. By construction, $\hat{\mu}(E_t) = \mathcal{O}(1)$. Given $h \in \mathcal{C}^1(M)$, $\psi \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$, since $T^{-n}M = M$ *(mod 0)*, we have $h(\psi) = \int_M h\psi \, d\mu = \int_M \mathcal{L}^n h \, \psi \circ T^{-n} \, d\mu$. We split $M = \cup_\ell M_\ell$ and integrate one ℓ at a time.

$$\begin{aligned}
\int_{M_\ell} \mathcal{L}^n h \, \psi \circ T^{-n} \, d\mu &= \sum_j \int_{B_j} \mathcal{L}^n h \, \psi \circ T^{-n} \, d\mu + \sum_{|t| \ge k_0} \int_{\mathbb{H}_t} \mathcal{L}^n h \, \psi \circ T^{-n} \, d\mu \\
&= \sum_j \int_{E_j} \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, d\mu_W d\hat{\mu}(\xi) + \sum_{|t|\ge k_0} \int_{E_t} \int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, d\mu_W d\hat{\mu}(\xi)
\end{aligned}$$

We change variables and estimate the integrals on one W_ξ at a time. Letting $W_{\xi,i}^n$ denote the components of $\mathcal{G}_n(W_\xi)$ defined in Section 3.5 and recalling that $J_{W_{\xi,i}^n}T^n$ denotes the stable Jacobian of T^n along the curve $W_{\xi,i}^n$, we write,

$$\begin{aligned}
\int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, d\mu_W &= \sum_i \int_{W_{\xi,i}^n} h\psi (J_\mu T^n)^{-1} J_{W_{\xi,i}^n} T^n \rho_\xi \circ T^n \cos\varphi \circ T^n \, dm_W \\
&\le \sum_i |h|_w |\psi|_{\mathcal{C}^p(W_{\xi,i}^n)} |(J_\mu T^n)^{-1} J_{W_{\xi,i}^n} T^n|_{\mathcal{C}^p(W_{\xi,i}^n)} |\rho_\xi \circ T^n|_{\mathcal{C}^p(W_{\xi,i}^n)} |\cos\varphi \circ T^n|_{\mathcal{C}^p(W_{\xi,i}^n)}.
\end{aligned}$$

By (4.6), we have $|\rho_\xi \circ T^n|_{\mathcal{C}^p(W_{\xi,i}^n)} \le C|\rho_\xi|_{\mathcal{C}^p(W_\xi)} \le C$ for some uniform constant C. The disortion bounds given by **(H4)**, equation (3.8), imply that

$$|(J_\mu T^n)^{-1} J_{W_{\xi,i}^n} T^n|_{\mathcal{C}^p(W_{\xi,i}^n)} \le (1 + 2C_d)|(J_\mu T^n)^{-1} J_{W_{\xi,i}^n} T^n|_{\mathcal{C}^0(W_{\xi,i}^n)}. \tag{3.17}$$

For $W \in \mathcal{W}^s$, let $\cos W$ denote the average value of $\cos\varphi$ on W, i.e.

$$\cos W := |W|^{-1} \int_W \cos\varphi \, dm_W.$$

Note that there exists $C_c > 0$, depending only on k_0 and the uniform transversality of $C^s(x)$ with the horizontal direction, such that

$$C_c^{-1} \cos W \le \cos \varphi(x) \le C_c \cos W \qquad \text{for all } x \in W. \tag{3.18}$$

Thus $|\cos \varphi \circ T^n|_{\mathcal{C}^0(W^n_{\xi,i})} \le C_c \cos W_\xi$. Then for $x, y \in W^n_{\xi,i}$, we have using (3.2) of **(H1)**,

$$\frac{|\cos \varphi \circ T^n(x) - \cos \varphi \circ T^n(y)|}{d_{W^n_{\xi,i}}(x,y)^p} \le \frac{d_{W_\xi}(T^n x, T^n y)}{d_{W^n_{\xi,i}}(x,y)^p} \le C_e^p \Lambda^{-np} |W_\xi|^{1-p}.$$

Thus we have $H^p_{W^n_{\xi,i}}(\cos \varphi \circ T^n) \le C_e^p |W_\xi|^{1-p}$. If $W_\xi \subset \mathbb{H}_t$, then $\cos W_\xi \ge c t^{-2}$ while $|W_\xi| \le C' t^{-3}$ for uniform constants $c, C' > 0$, depending on the minimum angle between $C^s(x)$ and the horizontal. Thus since $p \le 1/3$, we have $|\cos \varphi \circ T^n|_{\mathcal{C}^p(W^n_{\xi,i})} \le C \cos W_\xi$ for some uniform constant C.

Gathering these estimates together, we have

$$\int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, d\mu_W \le C |h|_w (|\psi|_\infty + H^p_n(\psi)) \cos(W_\xi) \sum_i |(J_\mu T^n)^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})}, \tag{3.19}$$

where C is uniform in T and n. We group the pieces $W^n_{\xi,i} \in \mathcal{G}_n(W_\xi)$ according to most recent long ancestor $W^k_{\xi,j} \in \mathcal{G}_k(W_\xi)$ as described in Section 3.5. Then splitting up the Jacobians according to times k and $n-k$ and using **(H5)**, we have

$$\begin{aligned}
\sum_i |(J_\mu T^n)^{-1} J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})} &\le \sum_{i \in \mathcal{I}_n(W)} \eta^n |J_{W^n_{\xi,i}} T^n|_{\mathcal{C}^0(W^n_{\xi,i})} \\
&\quad + \sum_{k=1}^n \sum_{j \in L_k(W_\xi)} |(J_\mu T^k)^{-1} J_{W^k_{\xi,j}} T^k|_{\mathcal{C}^0(W^k_{\xi,j})} \left(\sum_{i \in \mathcal{I}_n(W^k_j)} \eta^{n-k} |J_{W^n_{\xi,i}} T^{n-k}|_{\mathcal{C}^0(W^n_{\xi,i})} \right) \\
&\le C_1 (\eta\theta_*)^n + \sum_{k=1}^n \sum_{j \in L_k(W_\xi)} |(J_\mu T^k)^{-1} J_{W^k_{\xi,j}} T^k|_{\mathcal{C}^0(W^k_{\xi,j})} C_1 (\eta\theta_*)^{n-k}
\end{aligned} \tag{3.20}$$

where we have used Lemma 3.2(a) on each of the terms involving $\mathcal{I}_n(W^k_{\xi,j})$ from time k to time n.

For each k, since $|W^k_{\xi,j}| \ge \delta_0/3$, we have by bounded distortion **(H4)**,

$$\begin{aligned}
\sum_{j \in L_k(W_\xi)} |(J_\mu T^k)^{-1} J_{W^k_{\xi,j}} T^k|_{\mathcal{C}^0(W^k_{\xi,j})} &\le (1 + C_d)^2 3\delta_0^{-1} \sum_{j \in L_k(W_\xi)} \int_{W^k_{\xi,j}} (J_\mu T^k)^{-1} J_{W^k_{\xi,j}} T^k \, dm_W \\
&\le C \delta_0^{-1} \int_{W_\xi} J_\mu T^{-k} \, dm_W.
\end{aligned}$$

Putting this estimate together with (3.19) and (3.20) and bringing $\cos W_\xi$ into the integral,

$$\int_{W_\xi} \mathcal{L}^n h \, \psi \circ T^{-n} \, \rho_\xi \, d\mu_W \le C |h|_w (|\psi|_\infty + H^p_n(\psi)) \Big(\cos W_\xi + \sum_{k=1}^n (\eta\theta_*)^{n-k} \int_{W_\xi} J_\mu T^{-k} \, d\mu_W \Big)$$

for some uniform constant C. Thus

$$\Big| \int_{M_\ell} \mathcal{L}^n h \, \psi \circ T^{-n} \, dm \Big| \le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \Big(\sum_j \int_{E_j} \cos W_\xi \, \hat\mu(d\xi) + \sum_{|t| \ge k_0} \int_{E_t} \cos W_\xi \, \hat\mu(d\xi)$$

$$+ \sum_j \sum_{k=1}^n (\eta\theta_*)^{n-k} \int_{B_j} J_\mu T^{-k} \, d\mu + \sum_{|t|\ge k_0} \sum_{k=1}^n (\eta\theta_*)^{n-k} \int_{\mathbb{H}_t} J_\mu T^{-k} \, d\mu \Big)$$

$$\le C|h|_w(|\psi|_\infty + H^p_n(\psi)) \Big(\sum_j \hat\mu(E_j) + \sum_{|t|\ge k_0} t^{-2} \hat\mu(E_t) + \sum_{k=1}^n (\eta\theta_*)^{n-k} \int_{M_\ell} J_\mu T^{-k} d\mu \Big)$$

where in the last line we have used the fact that $\cos W \le Ct^{-2}$ for $W \subset \mathbb{H}_t$. The first two sums are finite since there are only finitely many E_j and $\hat\mu(E_t)$ is of order 1 for each t. Since there are only finitely many M_ℓ, the first two sums remain finite when we sum over ℓ. For the third sum, we sum over ℓ and use the fact that $\int_M J_\mu T^{-k} \, d\mu = 1$ for each $k \ge 1$. Thus the contribution from the third sum is uniformly bounded in n using the fact that $\eta\theta_* < 1$ by **(H5)**. □

Several other properties of the spaces $\mathcal{B}$ and $\mathcal{B}_w$ proved in [DZ] do not need to be reproved since their proofs remain essentially unchanged. They are as follows.

(i) ([DZ, Lemma 3.7]) $\mathcal{B}$ contains piecewise Hölder continuous functions h with exponent greater than 2β provided the discontinuities of h are uniformly transverse to the stable cones $C^s(x)$.

(ii) ([DZ, Lemma 2.1]) $\mathcal{L}$ is well-defined as a continuous linear operator on both $\mathcal{B}$ and $\mathcal{B}_w$. Moreover, there is a sequence of embeddings $\mathcal{C}^\gamma(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^p(M))'$, for all $\gamma > 2\beta$.

(iii) ([DZ, Lemma 3.10]) The unit ball of $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is compactly embedded in $(\mathcal{B}_w, |\cdot|_w)$.

Lemma 3.4 and items (i) and (ii) characterize the spaces $\mathcal{B}$ and $\mathcal{B}_w$ as spaces of distributions containing all Hölder continuous and certain classes of piecewise Hölder continuous functions. The last item is necessary in order to deduce the quasi-compactness of $\mathcal{L}_T$ from the Lasota-Yorke inequalities given by Theorem 2.2.

There remains one final fact to establish. As mentioned earlier, since we identify $h \in \mathcal{C}^1(M)$ with the measure $h\mu$ as an element of $\mathcal{B}$, *a priori* Lebesgue measure may not be in $\mathcal{B}$. The following Lemma shows that Lebesgue measure is in fact in $\mathcal{B}$ and therefore so is hdm for any $h \in \mathcal{C}^1(M)$.

Lemma 3.5. *The function $(\cos\varphi)^{-1}$ is in $\mathcal{B}$. Therefore, Lebesgue measure $m = (\cos\varphi)^{-1}\mu$ is also in $\mathcal{B}$ and so is hm for any $h \in \mathcal{C}^1(M)$. Indeed, any piecewise Hölder continuous function as in item (i) above times Lebesgue belongs to $\mathcal{B}$.*

Proof. In order to show $(\cos\varphi)^{-1} \in \mathcal{B}$, we must show that $(\cos\varphi)^{-1}$ can be approximated by functions $h \in \mathcal{C}^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm. Since $\|f\|_{\mathcal{B}} = \sup_k \|f|_{\mathbb{H}_k}\|_{\mathcal{B}}$, our strategy will be to show that $\|(\cos\varphi)^{-1}|_{\mathbb{H}_k}\|_{\mathcal{B}} \le Ck^{-1/2}$ for some uniform constant C. We can then approximate $(\cos\varphi)^{-1}$ by 0 in homogeneity strips of sufficiently high index. More precisely, given $\varepsilon > 0$, we choose K such that $CK^{-1/2} < \varepsilon$. Then on the remaining strips $k < K$, $(\cos\varphi)^{-1}$ has finite $\mathcal{C}^1$-norm and satisfies the assumptions of [DZ, Lemma 3.7]. Thus we may find $f_\varepsilon \in C^1(M)$ as in the proof of that lemma such that

$$\|(\cos\varphi)^{-1} - f_\varepsilon\|_{\mathcal{B}} \le \sup_{k\ge K} \|(\cos\varphi)^{-1}|_{\mathbb{H}_k}\|_{\mathcal{B}} + \sup_{k<K} \|((\cos\varphi)^{-1} - f_\varepsilon)|_{\mathbb{H}_k}\|_B < 2\varepsilon,$$

proving that $(\cos\varphi)^{-1} \in \mathcal{B}$.

It remains to prove the claim $\|(\cos\varphi)^{-1}|_{\mathbb{H}_k}\|_{\mathcal{B}} \le Ck^{-1/2}$. Choose $W \in \mathcal{W}^s$, $W \subset \mathbb{H}_k$, and let $\psi \in \mathcal{C}^q(W)$ with $|\psi|_{W,\alpha,q} \le 1$. Then,

$$\int_W (\cos\varphi)^{-1}\,\psi\,dm_W \le |(\cos\varphi)^{-1}|_{\mathcal{C}^0(W)}|\psi|_{\mathcal{C}^0(W)}|W| \le |(\cos\varphi)^{-1}|_{\mathcal{C}^0(W)}|W|^{1-\alpha}$$

since $|\psi|_{\mathcal{C}^p(W)} \le |W|^{-\alpha}$. On $\mathbb{H}_k$, we have $|W| \le ck^{-3}$ and $|\cos\varphi|^{-1} \le Ck^2$ for some uniform constants c and C which depend only on the minimum angle of $C^s(x)$ with the horizontal. Then, since $\alpha < 1/6$,

$$|(\cos\varphi)^{-1}|_{\mathcal{C}^0(W)}|W|^{1-\alpha} \le cCk^2k^{-3+3\alpha} \le C'k^{-1/2}. \tag{3.21}$$

Taking the suprema over $W \subset \mathbb{H}_k$ and ψ with $|\psi|_{W,\alpha,q} \le 1$, we have $\|(\cos\varphi)^{-1}|_{\mathbb{H}_k}\|_s \le C'k^{-1/2}$, completing the estimate on the strong stable norm.

To estimate the strong unstable norm, let $\varepsilon \le \varepsilon_0$ and choose two curves in $\mathbb{H}_k$, $W^1, W^2 \in \mathcal{W}^s$, such that $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$. For $i = 1, 2$, let $\psi_i \in \mathcal{C}^p(W^i)$ with $|\psi_i|_{\mathcal{C}^p(W^i)} \le 1$ and $d_q(\psi_1, \psi_2) \le \varepsilon$.

Recalling the notation of Section 3.3, denote $W^i = \{G_{W^i}(r) = (r, \varphi_{W^i}(r)) : r \in I_{W^i}\}$, $i = 1, 2$ and note that by definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$, W^1 and W^2 can be put into one-to-one correspondence by a foliation of vertical line segments of length at most ε, except possibly near their endpoints. Denote by U^i the single matched connected component of W^i and by V^i_j the at most 2 unmatched components of W^i. We let $\Theta : U^1 \to U^2$ denote the holonomy map along the vertical foliation. We estimate,

$$\begin{aligned}\int_{W^1}(\cos\varphi)^{-1}\psi_1\,dm_W - \int_{W^2}(\cos\varphi)^{-1}\psi_2\,dm_W &= \sum_{i,j}\int_{V^i_j}(\cos\varphi)^{-1}\psi_i\,dm_W \\ &\quad + \int_{U^1}(\cos\varphi)^{-1}\psi_1\,dm_W - \int_{U^2}(\cos\varphi)^{-1}\psi_2\,dm_W.\end{aligned} \tag{3.22}$$

We first estimate over the unmatched pieces V^i_j. Note that $|V^i_j| \le C\varepsilon$ where C depends only on the minimum angle of $C^s(x)$ with the vertical. Recalling that $|\psi_i|_{\mathcal{C}^0(W^i)} \le 1$ and using (3.21) since $\beta \le \alpha$, we estimate

$$\left|\sum_{i,j}\int_{V^i_j}(\cos\varphi)^{-1}\psi_i\,dm_W\right| \le \sum_{i,j}|V^i_j|^\beta|V^i_j|^{1-\beta}|(\cos\varphi)^{-1}|_{\mathcal{C}^0(V^i_j)} \le C\varepsilon^\beta k^{-1/2}. \tag{3.23}$$

To estimate the difference on the matched pieces U^i, we change variables to U^1 using Θ,

$$\begin{aligned}\left|\int_{U^1}(\cos\varphi)^{-1}\psi_1\,dm_W - \int_{U^2}(\cos\varphi)^{-1}\psi_2\,dm_W\right| &= \left|\int_{U^1}(\cos\varphi)^{-1}\psi_1 - [(\cos\varphi)^{-1}\psi_2]\circ\Theta\,J\Theta\,dm_W\right| \\ &\le |U^1||(\cos\varphi)^{-1}\psi_1 - [(\cos\varphi)^{-1}\psi_2]\circ\Theta\,J\Theta|_{\mathcal{C}^0(U^1)}.\end{aligned}$$

To estimate the $\mathcal{C}^0$ norm of the test function, we split the difference into 3 terms and use the fact that $|\psi_i|_{\mathcal{C}^0} \le 1$,

$$\begin{aligned}|(\cos\varphi)^{-1}\psi_1 - [(\cos\varphi)^{-1}\psi_2]\circ\Theta\,J\Theta|_{\mathcal{C}^0(U^1)} &\le |(\cos\varphi)^{-1} - (\cos\varphi)^{-1}\circ\Theta|_{\mathcal{C}^0(U^1)} \\ &\quad + |(\cos\varphi)^{-1}|_{\mathcal{C}^0(U^2)}|\psi_1 - \psi_2\circ\Theta|_{\mathcal{C}^0(U^1)} + |(\cos\varphi)^{-1}|_{\mathcal{C}^0(U^2)}|1 - J\Theta|_{\mathcal{C}^0(U^1)}.\end{aligned} \tag{3.24}$$

For the first term above, note that for $x \in U^1$, $|\cos\varphi(x) - \cos\varphi\circ\Theta(x)| \le d(x, \Theta(x)) \le \min\{\varepsilon, Ck^{-3}\}$ for some uniform constant $C > 0$. Thus

$$|(\cos\varphi)^{-1}(x) - (\cos\varphi)^{-1}\circ\Theta(x)| \le \frac{d(x,\Theta(x))}{\cos\varphi(x)\cos\varphi\circ\Theta(x)} \le C'\frac{\varepsilon^\beta k^{-3(1-\beta)}}{k^{-4}} \le C'\varepsilon^\beta k^{3/2},$$

since $\beta \le 1/6$. To estimate the second term in (3.24), denote $x \in U^1$ by $x = G_{W^1}(r)$ for some $r \in I_{W^1} \cap I_{W^2} =: I$. Then $|\psi_1(x) - \psi_2 \circ \Theta(x)| = |\psi_1 \circ G_{W^1}(r) - \psi_2 \circ G_{W^2(r)}| \le \varepsilon$ by definition of $d_q(\cdot,\cdot)$. Thus

$$|(\cos\varphi)^{-1}|_{\mathcal{C}^0(U^2)} |\psi_1 - \psi_2 \circ \Theta|_{\mathcal{C}^0(U^1)} \le Ck^2 \varepsilon.$$

Finally, we estimate the third term of (3.24) by noting that

$$|1 - J\Theta| = \left| 1 - \frac{\sqrt{1 + (\varphi'_{W^1})^2}}{\sqrt{1 + (\varphi'_{W^2})^2}} \right| \le |\varphi'_{W^1} - \varphi'_{W^2}| \le \varepsilon,$$

where we have used the fact that the derivative of $\sqrt{1+t^2}$, $\frac{t}{\sqrt{1+t^2}}$, is bounded by 1 for $t \ge 0$.

Putting these 3 estimates together in (3.24), we estimate the norm on the matched pieces by

$$\left| \int_{U^1} (\cos\varphi)^{-1} \psi_1 \, dm_W - \int_{U^2} (\cos\varphi)^{-1} \psi_2 \, dm_W \right| \le |U^1| C(\varepsilon^\beta k^{3/2} + \varepsilon k^2 + \varepsilon k^2) \le C' \varepsilon^\beta k^{-1},$$

using the fact that $|U^1| \le Ck^{-3}$. This, combined with (3.23), yields the required estimate on the strong unstable norm. □

4. Proof of Theorem 2.2

The proof of Theorem 2.2 relies on the following proposition.

Proposition 4.1. *There exists $C > 0$, depending only on* **(H1)-(H5)**, *such that for any $T \in \mathcal{F}$, $h \in \mathcal{B}$ and $n \ge 0$,*

$$|\mathcal{L}_T^n h|_w \le C\eta^n |h|_w \tag{4.1}$$

$$\|\mathcal{L}_T^n h\|_s \le C(\theta_1^{(1-\alpha)n} + \Lambda^{-qn})\eta^n \|h\|_s + C\eta^n \delta_0^{-\alpha} |h|_w \tag{4.2}$$

$$\|\mathcal{L}_T^n h\|_u \le C\eta^n \Lambda^{-\beta n} \|h\|_u + C\eta^n C_3^n \|h\|_s \tag{4.3}$$

Proof of Theorem 2.2 given Proposition 4.1. Choose $1 > \sigma > \eta \max\{\theta_1^{1-\alpha}, \Lambda^{-q}, \Lambda^{-\beta}\}$ and choose $N \ge 0$ such that

$$\begin{aligned}\|\mathcal{L}_T^N h\|_{\mathcal{B}} = \|\mathcal{L}_T^N h\|_s + b\|\mathcal{L}_T^N h\|_u &\le \frac{\sigma^N}{2}\|h\|_s + C\delta_0^{-\alpha}\eta^N |h|_w + b\sigma^N \|h\|_u + bC\eta^N C_3^N \|h\|_s \\ &\le \sigma^N \|h\|_{\mathcal{B}} + C_{\delta_0} \eta^N |h|_w\end{aligned}$$

providing b is chosen sufficiently small with respect to N. This is the required inequality (2.3) for Theorem 2.2 which implies the essential spectrum of $\mathcal{L}_T$ is less than σ. Outside the disk of radius σ, the spectrum of $\mathcal{L}_T$ has finitely many eigenvalues, each with finite multiplicity. This follows using the compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ [DZ, Lemma 3.10].

Despite the fact that η may be greater than 1, the spectral radius of $\mathcal{L}_T$ equals 1. To see this, suppose $z \in \mathbb{C}$, $|z| > 1$, satisfies $\mathcal{L}_T h = zh$ for some $h \in \mathcal{B}$, $h \ne 0$. For $\psi \in \mathcal{C}^p(M)$, Lemma 3.4 implies that,

$$|h(\psi)| = |z^{-n} \mathcal{L}_T^n h(\psi)| = |z^{-n} h(\psi \circ T^n)| \le |z|^{-n} C |h|_w (|\psi|_\infty + H_n^p(\psi \circ T^n)) \xrightarrow[n\to\infty]{} 0$$

since $H_n^p(\psi \circ T^n) \le C_e \Lambda^{-pn} |\psi|_{\mathcal{C}^p(M)}$ by (4.6). Thus $h = 0$, contradicting the assumption on z.

The characterization of the peripheral spectrum follows from Lemmas 5.1 and 5.2 of [DZ]. □

To prove Proposition 4.1, we fix $T \in \mathcal{F}$ and prove the required Lasota-Yorke inequalities (4.1)-(4.3). It is shown in [DZ, Section 4] that $\mathcal{L}_T$ is a continuous operator on both $\mathcal{B}$ and $\mathcal{B}_w$ so that it suffices to prove the inequalities for $h \in \mathcal{C}^1(M)$. They extend to the completions by continuity. Since these estimates are similar to those in [DZ], our purpose in repeating them is to show how they depend explicitly on the uniform constants given by **(H1)-(H5)** and do not require additional information.

4.1. **Estimating the weak norm.** Let $h \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ such that $|\psi|_{W,0,p} \le 1$. For $n \ge 0$, we write,

$$\int_W \mathcal{L}^n h\, \psi\, dm_W = \sum_{W_i^n \in \mathcal{G}_n(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{J_\mu T^n} \psi \circ T^n dm_W \tag{4.4}$$

where $J_{W_i^n} T^n$ denotes the Jacobian of T^n along W_i^n.

Using the definition of the weak norm on each W_i^n, we estimate (4.4) by

$$\int_W \mathcal{L}^n h\, \psi\, dm_W \ \le \sum_{W_i^n \in \mathcal{G}_n} |h|_w |(J_\mu T^n)^{-1} J_{W_i^n} T^n|_{\mathcal{C}^p(W_i^n)} |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)}. \tag{4.5}$$

For $x, y \in W_i^n$, we use **(H1)** to estimate,

$$\frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(T^n x, T^n y)^p} \cdot \frac{d_W(T^n x, T^n y)^p}{d_W(x,y)^p} \le |\psi|_{\mathcal{C}^p(W)} |J_{W_i^n} T^n|^p_{\mathcal{C}^0(W_i^n)} \le C_e \Lambda^{-pn} |\psi|_{\mathcal{C}^p(W)}, \tag{4.6}$$

so that $|\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} \le C_e |\psi|_{\mathcal{C}^p(W)} \le C_e$. We use this estimate together with **(H5)** and (3.17) to bound (4.5) by

$$\int_W \mathcal{L}^n h\, \psi\, dm_W \le C_e(1 + 2C_d)\eta^n |h|_w \sum_{W_i^n \in \mathcal{G}_n} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \le C' \eta^n |h|_w,$$

where $C' = C_e(1 + 2C_d)C_2$ and we have used Lemma 3.2(b) for the last inequality. Taking the supremum over all $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,0,p} \le 1$ yields (4.1) expressed with uniform constants given by **(H1)-(H5)**.

4.2. **Estimating the strong stable norm.** Let $W \in \mathcal{W}^s$ and let W_i^n denote the elements of $\mathcal{G}_n(W)$ as defined above. For $\psi \in \mathcal{C}^q(W)$, $|\psi|_{W,\alpha,q} \le 1$, define $\overline{\psi}_i = |W_i^n|^{-1} \int_{W_i^n} \psi \circ T^n\, dm_W$. Using equation (4.4), we write

$$\int_W \mathcal{L}^n h\, \psi\, dm_W = \sum_i \int_{W_i^n} h\, \frac{J_{W_i^n} T^n}{J_\mu T^n} \,(\psi \circ T^n - \overline{\psi}_i)\, dm_W + \overline{\psi}_i \int_{W_i^n} h\, \frac{J_{W_i^n} T^n}{J_\mu T^n}\, dm_W. \tag{4.7}$$

To estimate the first term of (4.7), we first estimate $|\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W_i^n)}$. If $H_W^q(\psi)$ denotes the Hölder constant of ψ along W, then equation (4.6) implies

$$\frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(x,y)^q} \le C_e \Lambda^{-nq} H_W^q(\psi) \tag{4.8}$$

for any $x, y \in W_i^n$. Since $\overline{\psi}_i$ is constant on W_i^n, we have $H^q_{W_i^n}(\psi \circ T^n - \overline{\psi}_i) \le C_e \Lambda^{-qn} H^q_W(\psi)$. To estimate the $\mathcal{C}^0$ norm, note that $\overline{\psi}_i = \psi \circ T^n(y_i)$ for some $y_i \in W_i^n$. Thus for each $x \in W_i^n$,

$$|\psi \circ T^n(x) - \overline{\psi}_i| = |\psi \circ T^n(x) - \psi \circ T^n(y_i)| \le H^q_{W_i^n}(\psi \circ T^n)|W_i^n|^q \le C_e H^q_W(\psi)\Lambda^{-nq}.$$

This estimate together with (4.8) and the fact that $|\varphi|_{W,\alpha,q} \le 1$, implies

$$|\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W_i^n)} \le C_e \Lambda^{-nq} |\psi|_{\mathcal{C}^q(W)} \le C_e \Lambda^{-qn} |W|^{-\alpha}. \tag{4.9}$$

We apply (3.17), (4.9) and the definition of the strong stable norm to the first term of (4.7),

$$\begin{aligned} \sum_i \int_{W_i^n} h \frac{J_{W_i^n} T^n}{J_\mu T^n} \,(\psi \circ T^n - \overline{\psi}_i)\, dm_W &\le (1 + 2C_d) C_e \sum_i \|h\|_s \frac{|W_i^n|^\alpha}{|W|^\alpha} \left| \frac{J_{W_i^n} T^n}{J_\mu T^n} \right|_{C^0(W_i^n)} \Lambda^{-qn} \\ &\le \ \eta^n (1 + 2C_d) C_e \Lambda^{-qn} \|h\|_s \sum_i \frac{|W_i^n|^\alpha}{|W|^\alpha} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \ \le \ C_4 \eta^n \Lambda^{-qn} \|h\|_s, \end{aligned} \tag{4.10}$$

where $C_4 = (1 + 2C_d) C_e C_2^{1-\alpha}$ and in the second line we have used **(H5)** and Lemma 3.2(c) with $\varsigma = \alpha$.

For the second term of (4.7), we use the fact that $|\overline{\psi}_i| \le |W|^{-\alpha}$ since $|\psi|_{W,\alpha,q} \le 1$. Recall the notation introduced before the statement of Lemma 3.2. Grouping the pieces $W_i^n \in \mathcal{G}_n(W)$ according to most recent long ancestors $W_j^k \in L_k(W)$, we have

$$\sum_i |W|^{-\alpha} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{J_\mu T^n} \, dm_W = \sum_{k=1}^n \sum_{j \in L_k(W)} \sum_{i \in \mathcal{I}_n(W_j^k)} |W|^{-\alpha} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{J_\mu T^n} \, dm_W + \sum_{i \in \mathcal{I}_n(W)} |W|^{-\alpha} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{J_\mu T^n} \, dm_W$$

where we have split up the terms involving $k = 0$ and $k \ge 1$. We estimate the terms with $k \ge 1$ by the weak norm and the terms with $k = 0$ by the strong stable norm. Using again (3.17) and **(H5)**,

$$\sum_i |W|^{-\alpha} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{J_\mu T^n} \, dm_W \le \eta^n (1 + 2C_d) \sum_{k=1}^n \sum_{j \in L_k(W)} \sum_{i \in \mathcal{I}_n(W_j^k)} |W|^{-\alpha} |h|_w |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} + \eta^n (1 + 2C_d) \sum_{i \in \mathcal{I}_n(W)} \frac{|W_i^n|^\alpha}{|W|^\alpha} \|h\|_s |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)}.$$

In the first sum above corresponding to $k \ge 1$, we write

$$|J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \le |J_{W_i^n} T^{n-k}|_{\mathcal{C}^0(W_i^n)} |J_{W_j^k} T^k|_{\mathcal{C}^0(W_j^k)}.$$

Thus using Lemma 3.2(a) from time k to time n,

$$\begin{aligned} \sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{I}_n(W_j^k)} |W|^{-\alpha} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} &\le \sum_{k=1}^n \sum_{j \in L_k(W)} |J_{W_j^k} T^k|_{\mathcal{C}^0(W_j^k)} |W|^{-\alpha} \sum_{i \in \mathcal{I}_n(W_j^k)} |J_{W_i^n} T^{n-k}|_{\mathcal{C}^0(W_i^n)} \\ &\le 3\delta_0^{-\alpha} \sum_{k=1}^n \sum_{j \in L_k(W)} |J_{W_j^k} T^k|_{\mathcal{C}^0(W_j^k)} \frac{|W_j^k|^\alpha}{|W|^\alpha} C_1 \theta_*^{n-k}, \end{aligned}$$

since $|W_j^k| \ge \delta_0/3$. The inner sum is bounded by $C_2^{1-\alpha}$ for each k by Lemma 3.2(c) while the outer sum is bounded by $C_1/(1-\theta_*)$ independently of n.

Finally, for the sum corresponding to $k = 0$, since

$$|J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \le (1 + C_d) |T^n W_i^n| |W_i^n|^{-1} \le (1 + C_d) |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)},$$

we use Jensen's inequality and Lemma 3.2(a) to estimate,

$$\sum_{i \in \mathcal{I}_n(W)} \frac{|W_i^n|^\alpha}{|W|^\alpha} |J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \le (1 + C_d) \left(\sum_{i \in \mathcal{I}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|} \right)^{1-\alpha} \le (1 + C_d) C_1 \theta_*^{n(1-\alpha)}.$$

Gathering these estimates together, we have

$$\sum_i |W|^{-\alpha} \left| \int_{W_i^n} h (J_\mu T^n)^{-1} J_{W_i^n} T^n \, dm_W \right| \le C_5 \eta^n \delta_0^{-\alpha} |h|_w + C_6 \|h\|_s \eta^n \theta_*^{n(1-\alpha)}, \tag{4.11}$$

where $C_5 = 3(1 + 2C_d) C_1 C_2^{1-\alpha}/(1 - \theta_*)$ and $C_6 = (1 + 2C_d)^2 C_1$. Putting together (4.10) and (4.11) proves (4.2),

$$\|\mathcal{L}^n h\|_s \le C' \eta^n \left(\Lambda^{-qn} + \theta_*^{n(1-\alpha)} \right) \|h\|_s + C' \eta^n \delta_0^{-\alpha} |h|_w,$$

with $C' = \max\{C_4, C_5, C_6\}$, a uniform constant depending only on **(H1)**-**(H5)**.

4.3. **Estimating the strong unstable norm.** Fix $\varepsilon \le \varepsilon_0$ and consider two curves $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$. For $n \ge 1$, we describe how to partition $T^{-n}W^\ell$ into "matched" pieces U_j^ℓ and "unmatched" pieces V_k^ℓ, $\ell = 1, 2$. In the what follows, we use C_t to denote a transversality constant which depends only on the minimum angle between various transverse directions: the minimum angle between $C^s(x)$ and $C^u(x)$, between S_{-n}^T and $C^s(x)$, and between $C^s(x)$ and the vertical and horizontal directions.

Let ω be a connected component of $W^1 \setminus \mathcal{S}_{-n}^T$ such that $T^{-n}\omega \in \mathcal{G}_n(W)$. We define a smooth local foliation $\{\gamma_x\}_{x \in T^{-n}\omega}$ about $T^{-n}\omega$ such that for each $x \in T^{-n}\omega$: (1) γ_x is centered at x, (2) $\gamma_x \in \widehat{W}^u$; (3) $|\gamma_x| \le 2BC_tC_e\Lambda^{-n}\varepsilon$ such that its image $T^n\gamma_x$, if not cut by a singularity or the boundary of a homogeneity strip, will have a projection on the vertical direction of length 2ε. By item (3) and the definition of $d_{\mathcal{W}^s}(W^1, W^2)$, it follows that any curve $T^n\gamma_x$ that is not cut by a singularity or the boundary of a homogeneity strip must necessarily intersect W^2, except possibly if $T^n\gamma_x$ lies near the endpoints of W^1. By **(H2)**, $T^i\gamma_x \in \widehat{\mathcal{W}}^u$ for each $i \ge 0$.

Doing this for each connected component of $W^1 \setminus \mathcal{S}_{-n}^T$, we subdivide $W^1 \setminus \mathcal{S}_{-n}^T$ into a countable collection of subintervals of points for which $T^n\gamma_x$ intersects $W^2 \setminus \mathcal{S}_{-n}^T$ and subintervals for which this is not the case. This in turn induces a corresponding partition on $W^2 \setminus \mathcal{S}_{-n}^T$.

We denote by V_k^ℓ the pieces in $T^{-n}W^\ell$ which are not matched up by this process and note that the images $T^nV_k^\ell$ occur either at the endpoints of W^ℓ or because the curve γ_x has been cut by a singularity or the boundary of a homogeneity strip. In both cases, the length of the curves $T^nV_k^\ell$ can be at most $C_t\varepsilon$ due to the uniform transversality of $\mathcal{S}_{-n}^T$ with $C^s(x)$, of $C^s(x)$ with $C^u(x)$ and of $C^s(x)$ with the horizontal.

In the remaining pieces the foliation $\{T^n\gamma_x\}_{x \in T^{-n}W^1}$ provides a one to one correspondence between points in W^1 and W^2. We partition these pieces in such a way that the lengths of their images under T^{-i} are less than δ_0 for each $0 \le i \le n$ and the pieces are pairwise matched by the foliation $\{\gamma_x\}$. We call these matched pieces $\widetilde{U}_j^\ell$ and note that $T^i\widetilde{U}_j^\ell \in \mathcal{G}_{n-i}(W^\ell)$ for each $i = 0, 1, \dots n$. For convenience, we further trim the $\widetilde{U}_j^\ell$ to pieces U_j^ℓ so that U_j^1 and U_j^2 are both defined on the same arclength interval I_j. The at most two components of $T^n(\widetilde{U}_j^\ell \setminus U_j^\ell)$ have length less than $C_t\varepsilon$ due to the uniform transversality of $C^s(x)$ with the vertical direction. We attach these trimmed pieces to the adjacent U_i^ℓ or V_k^ℓ as appropriate so as not to create any additional components in the partition.

We further relabel any pieces U_j^ℓ as V_j^ℓ and consider them unmatched if for some i, $0 \le i \le n$, $|T^i\gamma_x| > 2|T^iU_j^\ell|$. i.e. we only consider pieces matched if at each intermediate step, the distance between them is at most of the same order as their length. We do this in order to be able to apply Lemma 3.3 to the matched pieces. Notice that since the distance between the curves at each intermediate step is at most $C_tC_e\varepsilon$ and due to the uniform contraction of stable curves going forward, we have $|T^nV_k^\ell| \le C_tC_e^2\varepsilon$ for all such pieces considered unmatched by this last criterion.

In this way we write $W^\ell = (\cup_j T^nU_j^\ell) \cup (\cup_k T^nV_k^\ell)$. Note that the images $T^nV_k^\ell$ of the unmatched pieces must have length $\le C_v\varepsilon$ for some uniform constant C_v while the images of the matched pieces U_j^ℓ may be long or short.

Recalling the notation of Section 3.3, we have arranged a pairing of the pieces U_j^ℓ with the following property:

$$\text{If } U_j^1 = G_{U_j^1}(I_j) = \{(r, \varphi_{U_j^1}(r)) : r \in I_j\}, \text{ then } U_j^2 = G_{U_j^2}(I_j) = \{(r, \varphi_{U_j^2}(r)) : r \in I_j\}, \tag{4.12}$$

so that the point $x = (r, \varphi_{U_j^1}(r)) \in U_j^1$ can associated with the point $\bar{x} = (r, \varphi_{U_j^2}(r)) \in U_j^2$ by the vertical line $\{(r, s)\}_{s \in [-\pi/2, \pi/2]}$, for each $r \in I_j$. In addition, the U_j^ℓ satisfy the assumptions of Lemma 3.3.

Given ψ_ℓ on W^ℓ with $|\psi_\ell|_{W^\ell,0,p} \le 1$ and $d_q(\psi_1, \psi_2) \le \varepsilon$, with the above construction we must estimate

$$\begin{aligned}\left| \int_{W^1} \mathcal{L}^n h \, \psi_1 \, dm_W - \int_{W^2} \mathcal{L}^n h \, \psi_2 \, dm_W \right| \; &\le \; \sum_{\ell,k} \left| \int_{V_k^\ell} h (J_\mu T^n)^{-1} J_{V_k^\ell} T^n \psi_\ell \circ T^n \, dm_W \right| \\ &+ \sum_j \left| \int_{U_j^1} h (J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \circ T^n \, dm_W - \int_{U_j^2} h (J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \circ T^n \, dm_W \right| \end{aligned} \tag{4.13}$$

We do the estimate over the unmatched pieces V_k^ℓ first using the strong stable norm. Note that by (4.6), $|\psi_\ell \circ T^n|_{\mathcal{C}^q(V_k^\ell)} \le C_e |\psi_\ell|_{\mathcal{C}^p(W^\ell)} \le C_e$. We estimate as in Section 4.2, using the fact that $|T^n V_k^\ell| \le C_v \varepsilon$, as noted above,

$$\begin{aligned}\sum_{\ell,k} \Big| \int_{V_k^\ell} h (J_\mu T^n)^{-1} J_{V_k^\ell} T^n \psi_\ell \circ T^n \, dm_W \Big| &\le C_e \sum_{\ell,k} \|h\|_s |V_k^\ell|^\alpha |(J_\mu T^n)^{-1} J_{V_k^\ell} T^n|_{\mathcal{C}^q(V^\ell,k)} \\ &\le C_e (1 + 2C_d) \eta^n \|h\|_s \sum_{\ell,k} |V_k^\ell|^\alpha |J_{V_k^\ell} T^n|_{\mathcal{C}^0(V_k^\ell)} \\ &\le C' \varepsilon^\alpha \eta^n \|h\|_s \sum_{\ell,k} |J_{V_k^\ell} T^n|^{1-\alpha}_{\mathcal{C}^0(V_k^\ell)} \le 2C' \varepsilon^\alpha \eta^n \|h\|_s C_3^n, \end{aligned} \tag{4.14}$$

with $C' = C_e (1 + 2C_d)^2 C_v^\alpha$, where we have applied Lemma 3.2(d) with $\varsigma = 1 - \alpha > \varsigma_0$ since there are at most two V_k^ℓ corresponding to each element $W_i^{\ell,n} \in \mathcal{G}_n(W^\ell)$ as defined in Section 3.5 and $|J_{V_k^\ell} T^n|_{\mathcal{C}^0(V_k^\ell)} \le |J_{W_i^{\ell,n}} T^n|_{\mathcal{C}^0(W_i^{\ell,n})}$ whenever $V_k^\ell \subseteq W_i^{\ell,n}$.

Next, we must estimate

$$\sum_j \left| \int_{U_j^1} h (J_\mu T^n)^{-1} J_{U_j^1} T^n \, \psi_1 \circ T^n \, dm_W - \int_{U_j^2} h (J_\mu T^n)^{-1} J_{U_j^2} T^n \, \psi_2 \circ T^n \, dm_W \right|.$$

We fix j and estimate the difference. Define

$$\phi_j = ((J_\mu T^n)^{-1} J_{U_j^1} T^n \, \psi_1 \circ T^n) \circ G_{U_j^1} \circ G_{U_j^2}^{-1}.$$

The function ϕ_j is well-defined on U_j^2 and we can write,

$$\begin{aligned}&\left| \int_{U_j^1} h (J_\mu T^{-1} J_{U_j^1} T^n \, \psi_1 \circ T^n - \int_{U_j^2} h (J_\mu T^n)^{-1} J_{U_j^2} T^n \, \psi_2 \circ T^n \right| \\ &\le \left| \int_{U_j^1} h (J_\mu T^n)^{-1} J_{U_j^1} T^n \, \psi_1 \circ T^n - \int_{U_j^2} h \, \phi_j \right| + \left| \int_{U_j^2} h (\phi_j - (J_\mu T^n)^{-1} J_{U_j^2} T^n \, \psi_2 \circ T^n) \right|. \end{aligned} \tag{4.15}$$

We estimate the first term on the right hand side of (4.15) using the strong unstable norm. Using **(H5)**, (3.17) and (4.6),

$$|(J_\mu T^n)^{-1} J_{U_j^1} T^n \cdot \psi_1 \circ T^n|_{\mathcal{C}^p(U_j^1)} \le C_e (1 + 2C_d) \eta^n |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)}. \tag{4.16}$$

Notice that

$$|G_{U_j^1} \circ G_{U_j^2}^{-1}|_{\mathcal{C}^1(U_j^2)} \le \sup_{r \in U_j^2} \frac{\sqrt{1 + (d\varphi_{U_j^1}/dr)^2}}{\sqrt{1 + (d\varphi_{U_j^2}/dr)^2}} \le \sqrt{1 + \Gamma^2} =: C_g, \tag{4.17}$$

where Γ is the maximum slope of curves in $\mathcal{W}^s$ given by **(H1)**. Using this, we estimate as in (4.16),

$$|\phi_j|_{\mathcal{C}^p(U_j^2)} \le C_g C_e (1 + 2C_d) \eta^n |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)}.$$

By the definition of ϕ_j and $d_q(\cdot,\cdot)$,

$$d_q((J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \circ T^n, \phi_j) = \left| \left[(J_\mu T^n)^{-1} J_{U_j^1} T^n \psi_1 \circ T^n \right] \circ G_{U_j^1} - \phi_j \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} = 0.$$

By Lemma 3.3(a), we have $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le C_0 \Lambda^{-n} \varepsilon =: \varepsilon_1$. In view of (4.16) and following, we renormalize the test functions by $R_j = C_7 \eta^n |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)}$ where $C_7 = C_g C_e (1 + 2C_d)$. Then we apply the definition of the strong unstable norm with ε_1 in place of ε. Thus,

$$\sum_j \left| \int_{U_j^1} h (J_\mu T^n)^{-1} J_{U_j^1} T^n \, \psi_1 \circ T^n - \int_{U_j^2} h \, \phi_j \right| \le C_7 C_0^\beta \varepsilon^\beta \Lambda^{-\beta n} \eta^n \|h\|_u \sum_j |J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \tag{4.18}$$

where the sum is $\le C_2$ by Lemma 3.2(b) since there is at most one matched piece U_j^1 corresponding to each element $W_i^{1,n} \in \mathcal{G}_n(W^1)$ and $|J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \le |J_{W_i^{1,n}} T^n|_{\mathcal{C}^0(W_i^{1,n})}$ whenever $U_j^1 \subseteq W_i^{1,n}$.

It remains to estimate the second term in (4.15) using the strong stable norm.

$$\left| \int_{U_j^2} h (\phi_j - (J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \circ T^n) \right| \le \|h\|_s |U_j^2|^\alpha \left| \phi_j - (J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \circ T^n \right|_{\mathcal{C}^q(U_j^2)}. \tag{4.19}$$

In order to estimate the $\mathcal{C}^q$-norm of the function in (4.19), we split it up into two differences. Since $|G_{U_j^\ell}|_{\mathcal{C}^1} \le C_g$ and $|G_{U_j^\ell}^{-1}|_{\mathcal{C}^1} \le 1$, $\ell = 1, 2$, we write

$$\begin{aligned}
&|\phi_j - ((J_\mu T^n)^{-1} J_{U_j^2} T^n) \cdot \psi_2 \circ T^n|_{\mathcal{C}^q(U_j^2)} \\
&\le \left| \left[((J_\mu T^n)^{-1} J_{U_j^1} T^n) \cdot \psi_1 \circ T^n \right] \circ G_{U_j^1} - \left[((J_\mu T^n)^{-1} J_{U_j^2} T^n) \cdot \psi_2 \circ T^n \right] \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} \\
&\le \left| ((J_\mu T^n)^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} \left[\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2} \right] \right|_{\mathcal{C}^q(I_j)} \\
&\quad + \left| \left[((J_\mu T^n)^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} - ((J_\mu T^n)^{-1} J_{U_j^2} T^n) \circ G_{U_j^2} \right] \psi_2 \circ T^n \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} \\
&\le C_g (1 + 2C_d) |(J_\mu T^n)^{-1} J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \left| \psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} \\
&\quad + C_g C_e \left| ((J_\mu T^n)^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} - ((J_\mu T^n)^{-1} J_{U_j^2} T^n) \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)}
\end{aligned} \tag{4.20}$$

To bound the two differences above, we need the following lemma.

Lemma 4.2. *There exist constants $C_8, C_9 > 0$, depending only on* **(H1)**-**(H5)**, *such that,*

(a) $|((J_\mu T^n)^{-1} J_{U_j^1} T^n) \circ G_{U_j^1} - (J_\mu T^n)^{-1} J_{U_j^2} T^n) \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le C_8 |(J_\mu T^n)^{-1} J_{U_j^2} T^n|_{\mathcal{C}^0(U_j^2)} \varepsilon^{1/3 - q}$;

(b) $|\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{\mathcal{C}^q(I_{r_j})} \le C_9 \varepsilon^{p-q}$.

We postpone the proof of the lemma to Section 4.3.1 and show how this completes the estimate on the strong unstable norm. It follows from Lemma 4.2(a) that

$$|(J_\mu T^n)^{-1} J_{U_j^1} T^n|_{\mathcal{C}^0(U_j^1)} \le (1 + C_8 \varepsilon^{1/3 - q}) |(J_\mu T^n)^{-1} J_{U_j^2} T^n|_{\mathcal{C}^0(U_j^2)}$$

which we will use to simplify (4.20). Starting from (4.19), we apply Lemma 4.2 to (4.20) to obtain,

$$\begin{aligned}
&\sum_j \Big| \int_{U_j^2} h (\phi_j - (J_\mu T^n)^{-1} J_{U_j^2} T^n \psi_2 \circ T^n) \, dm_W \Big| \\
&\le \bar{C} \|h\|_s \sum_j |U_j^2|^\alpha |(J_\mu T^n)^{-1} J_{U_j^2} T^n|_{\mathcal{C}^0(U_j^2)} \, \varepsilon^{p-q} \le \bar{C} \eta^n \|h\|_s \varepsilon^{p-q} \sum_j |J_{U_j^2} T^n|_{\mathcal{C}^0(U_j^2)},
\end{aligned} \tag{4.21}$$

for some uniform constant $\bar{C}$ where again the sum is finite as in (4.18). This completes the estimate on the second term in (4.15). Now we use this bound, together with (4.14) and (4.18) to estimate (4.13)

$$\left| \int_{W^1} \mathcal{L}^n h \, \psi_1 \, dm_W - \int_{W^2} \mathcal{L}^n h \, \psi_2 \, dm_W \right| \le CC_3^n \eta^n \|h\|_s \varepsilon^\alpha + C\|h\|_u \Lambda^{-\beta n} \eta^n \varepsilon^\beta + C\eta^n \|h\|_s \varepsilon^{p-q},$$

where again C depends only on **(H1)**-**(H5)** through the estimates above. Since $p - q \ge \beta$ and $\alpha \ge \beta$, we divide through by ε^β and take the appropriate suprema to complete the proof of (4.3).

4.3.1. *Proof of Lemma 4.2.* First we prove the following general fact and then use it to prove Lemma 4.2.

Lemma 4.3. *Let (N, d) be a metric space and let $0 < r < s \le 1$. Suppose $g_1, g_2 \in \mathcal{C}^s(N, \mathbb{R})$ satisfy $|g_1 - g_2|_{\mathcal{C}^0(N)} \le D_1 \varepsilon^s$ for some constant $D_1 > 0$. Then $|g_1 - g_2|_{\mathcal{C}^r(N)} \le 3\varepsilon^{s-r} \max\{D_1, H^s(g_1) + H^s(g_2)\}$, where $H^s(\cdot)$ denotes the Hölder constant with exponent s on N.*

Proof. Since $|\cdot|_{\mathcal{C}^r(N)} = |\cdot|_{\mathcal{C}^0(N)} + H^r(\cdot)$, we must estimate $H^r(g_1 - g_2)$. Let $x, y \in N$. Then on the one hand, since $|g_1 - g_2| \le D_1 \varepsilon^s$, we have

$$\frac{|(g_1(x) - g_2(x)) - (g_1(y) - g_2(y))|}{d(x,y)^r} \le 2D_1 \varepsilon^s d(x,y)^{-r}$$

On the other hand, using the fact that $g_1, g_2 \in \mathcal{C}^s(N)$, we have

$$\frac{|(g_1(x) - g_2(x)) - (g_1(y) - g_2(y))|}{d(x,y)^r} \le (H^s(g_1) + H^s(g_2)) d(x,y)^{s-r}.$$

These two estimates together imply that the Hölder constant of $g_1 - g_2$ is bounded by

$$H^r(g_1 - g_2) \le \sup_{x,y \in N} \min\{2D_1 \varepsilon^s d(x,y)^{-r}, (H^s(g_1) + H^s(g_2)) d(x,y)^{s-r}\}.$$

This expression is maximized when $2D_1 \varepsilon^s d(x,y)^{-r} = (H^s(g_1) + H^s(g_2)) d(x,y)^{s-r}$, i.e., when $d(x,y) = \varepsilon \left(\frac{2D_1}{H^s(g_1) + H^s(g_2)} \right)^{1/s}$. Thus the Hölder constant of $g_1 - g_2$ satisfies,

$$H^r(g_1 - g_2) \le \varepsilon^{s-r} (2D_1)^{1 - \frac{r}{s}} (H^s(g_1) + H^s(g_2))^{\frac{r}{s}}.$$

□

Proof of Lemma 4.2(a). Throughout the proof, for ease of notation we write J_ℓ^n for $(J_\mu T^n)^{-1} J_{U_j^\ell} T^n$.

For any $r \in I_j$, $x = G_{U_j^1}(r)$ and $\bar{x} = G_{U_j^2}(r)$ lie on a common vertical segment. By the construction at the beginning of Section 4.3, U_j^1, U_j^2 lie in two homogeneous stable curves $\widetilde{U}_j^1$ and $\widetilde{U}_j^2$ which are connected by the foliation $\{\gamma_x\}$. Thus $x^* := \gamma_x \cap \widetilde{U}_j^2$ is uniquely defined for all $x \in U_j^1$. Then $T^n(x)$ and $T^n(x^*)$ lie on the element $T^n \gamma_x \in \mathcal{W}^u$ which intersects W^1 and W^2 and has length at most $C_t \varepsilon$. By (3.9) and Lemma 3.3(b),

$$|J_1^n(x) - J_2^n(x^*)| \le C_d C_0 |J_2^n|_{\mathcal{C}^0(U_j^2)} (d_W(T^n x, T^n x^*)^{1/3} + \theta(T^n x, T^n x^*)),$$

where $\theta(T^n x, T^n x^*)$ is the angle between the tangent line to W^1 at $T^n x$ and the tangent line to W^2 at $T^n x^*$. Let $y \in W^2$ be the unique point in W^2 which lies on the same vertical segment as $T^n x$. Since by assumption $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$, we have $\theta(T^n x, y) \le \varepsilon$. Due to the uniform transversality of curves in $\mathcal{W}^u$ and $\mathcal{W}^s$ and the fact that W^2 is the graph of a $\mathcal{C}^2$ function with $\mathcal{C}^2$ norm bounded by B from **(H2)**, we have $\theta(y, T^n x^*) \le BC_t \varepsilon$ and so $\theta(T^n x, T^n x^*) \le (1 + BC_t)\varepsilon$. Thus

$$|J_1^n(x) - J_2^n(x^*)| \le C_d C_0 (C_t + 1 + BC_t) \varepsilon^{1/3} |J_2^n|_{\mathcal{C}^0(U_j^2)}. \tag{4.22}$$

Also, by (3.8), since x^* and $\bar{x}$ are both on $\widetilde{U}_j^2$, we have $|J_2^n(x^*) - J_2^n(\bar{x})| \le C_d^2 |J_2^n|_{\mathcal{C}^0(U_j^2)} d_W(x^*, \bar{x})^{1/3}$. Putting this together with (4.22) and using the fact that $d_W(x^*, \bar{x}) \le C_t \varepsilon$ by the transversality of γ_x with $\mathcal{W}^s$ yields,

$$|J_1^n(x) - J_2^n(\bar{x})| \le C' \varepsilon^{1/3} |J_2^n|_{\mathcal{C}^0(U_j^2)}, \tag{4.23}$$

where $C' = C_d C_0 (2C_t + 1 + BC_t)$.

Now using the fact that $|G_{U_j^\ell}|_{\mathcal{C}^1(I_j)} \le C_g$ from (4.17), we apply Lemma 4.3 with $D_1 = C_1 |J_2^n|_{\mathcal{C}^0(U_j^2)}$ and $g_i = J_i^n \circ G_{U_j^i}$, $i = 1, 2$. By (4.23), we have

$$|J_1^n|_{\mathcal{C}^0(U_j^1)} \le (1 + C' \varepsilon^{1/3}) |J_2^n|_{\mathcal{C}^0(U_j^2)}, \tag{4.24}$$

and invoking (3.8), we complete the proof of (a). □

Proof of (b). Let φ_{W^ℓ} be the function whose graph is W^ℓ, defined for $r \in I_{W^\ell}$, and set $f_j^\ell := G_{W^\ell}^{-1} \circ T^n \circ G_{U_j^\ell}$, $k = 1, 2$. Notice that since $|G_{W^\ell}^{-1}|_{\mathcal{C}^1} \le 1$ and $|G_{U_j^\ell}|_{\mathcal{C}^1} \le C_g$, and due to the uniform contraction along stable curves, we have $\mathrm{Lip}(f_j^\ell) \le C_f$, where C_f is independent of W^ℓ, T and j. We may assume that $f_j^\ell(I_j) \subset I_{W^1} \cap I_{W^2}$ since if not, by the transversality of $C^u(x)$ and $C^s(x)$, we must be in a neighborhood of one of the endpoints of W^ℓ of length at most $C_t \varepsilon$; such short pieces may be estimated as in (4.14) using the strong stable norm. Thus

$$\begin{aligned} |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le{} & |\psi_1 \circ G_{W^1} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^1|_{\mathcal{C}^q(I_j)} \\ & + |\psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2|_{\mathcal{C}^q(I_j)}. \end{aligned} \tag{4.25}$$

Using the above observation about f_j^1, we estimate the first term of (4.25) by

$$|\psi_1 \circ G_{W^1} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^1|_{\mathcal{C}^q(I_j)} \le C_f |\psi_1 \circ G_{W^1} - \psi_2 \circ G_{W^2}|_{\mathcal{C}^q(f_j^1(I_j))} \le C_f \varepsilon, \tag{4.26}$$

since $d_q(\psi_1, \psi_2) \le \varepsilon$. To estimate the second term of (4.25), notice that since U_j^1 and U_j^2 are joined by the transverse foliation $\{\gamma_x\} \subset \widehat{\mathcal{W}}^u$ and using the uniform contraction along stable curves under T^n, we have $|f_j^1 - f_j^2|_{\mathcal{C}^0(I_j)} \le \tilde{C} \varepsilon$ for a constant $\tilde{C}$ depending only on the uniform hyperbolicity of **(H1)** and the uniform transversality conditions in **(H2)**. Thus for $r \in I_j$,

$$|\psi_2 \circ G_{W^2} \circ f_j^1(r) - \psi_2 \circ G_{W^2} \circ f_j^2(r)| \le C_g |\psi_2|_{\mathcal{C}^p} |f_j^1(r) - f_j^2(r)|^p \le C_g \tilde{C} |\psi_2|_{\mathcal{C}^p} \varepsilon^p. \tag{4.27}$$

Now we again apply Lemma 4.3 to obtain

$$|\psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2|_{\mathcal{C}^q(I_j)} \le C |\psi_2|_{\mathcal{C}^p} \varepsilon^{p-q},$$

for a uniform constant C. This estimate combined with (4.26) proves part (b) since $|\psi_2|_{\mathcal{C}^p(W^2)} \le 1$. □

5. Proof of Theorem 2.3

Fix $\varepsilon < \varepsilon_0$ and suppose $T_1, T_2 \in \mathcal{F}$ with $d_{\mathcal{F}}(T_1, T_2) \le \varepsilon$. We denote by $\mathcal{S}_{-n}^\ell$ the singularity sets for T_ℓ, $\ell = 1, 2$. Let $h \in \mathcal{C}^1(M)$, $\|h\|_{\mathcal{B}} \le 1$, and $W \in \mathcal{W}^s$. Let $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,0,p} \le 1$. We must estimate

$$\begin{aligned} \int_W (\mathcal{L}_1 h - \mathcal{L}_2 h) \psi \, dm_W &= \int_W \mathcal{L}_1 h \psi \, dm_W - \int_W \mathcal{L}_2 h \psi \, dm_W \\ &= \int_{T_1^{-1} W} h \, \psi \circ T_1 (J_\mu T_1)^{-1} J_{T_1^{-1} W} T \, dm_W - \int_{T_2^{-1} W} h \, \psi \circ T_2 (J_\mu T_2)^{-1} J_{T_2^{-1} W} T_2 \, dm_W. \end{aligned} \tag{5.1}$$

Notice that the estimate required is similar to that done in Section 4.3, except that instead of two close stable curves iterated under the same map, we have one stable curve iterated under two different maps.

We partition $T_1^{-1}W$ and $T_2^{-1}W$ into matched and unmatched pieces as in the beginning of Section 4.3. Let $\mathcal{G}_1^\ell(W)$, $\ell = 1, 2$, denote the elements of $T_\ell^{-1}W$ as described in Section 3.5. Let $\omega \in \mathcal{G}_1^1(W)$. Due to **(C1)**, to each point $x \in \omega$, we associate a curve $\gamma_x \in \widehat{W}^u$ of length at most $C_t\varepsilon$ which terminates on a piece of $T_2^{-1}W$ that lies in the same homogeneity strip, if one exists. We also require that γ_x is not cut by $\mathcal{S}_1^1 \cup \mathcal{S}_1^2$.

We denote by V_k^ℓ those components of $T_\ell^{-1}W$ not matched by this process. We also include in the set of V_k^ℓ all images of connected components of $W \cap N_\varepsilon(\mathcal{S}_{-1}^1 \cup \mathcal{S}_{-1}^2)$ under T_ℓ^{-1}. Note that the $T_\ell V_k^\ell$ occur either at the endpoints of W or near a singularity or the boundary of $N_\varepsilon(\mathcal{S}_{-1}^1 \cup \mathcal{S}_{-1}^2)$. In all cases, the length of the curves $T_\ell V_k^\ell$ can be at most $C_tC_e\varepsilon$ due to the uniform transversality of $\mathcal{S}_{-1}^\ell$ with C^s and of C^s with C^u.

In the remaining pieces the foliation $\{\gamma_x\}$ provides a one-to-one correspondence between points in $T_1^{-1}W$ and $T_2^{-1}W$. We further partition these pieces in such a way that their lengths are between $\delta_0/2$ and δ_0 and the pieces are pairwise matched by the foliation $\{\gamma_x\}$. We call these matched pieces $\widetilde{U}_j^\ell$. As in Section 4.3, we trim the $\widetilde{U}_j^\ell$ to pieces U_j^ℓ so that U_j^1 and U_j^2 are defined on the same arclength interval I_j. The at most two components of $T_\ell(\widetilde{U}_j^\ell \setminus U_j^\ell)$ have length at most $C_tC_e\Lambda^{-1}\varepsilon$. We adjoin these trimmed pieces to the adjacent U_i^ℓ or V_k^ℓ as appropriate so as not to create more pieces in the partition of $T_\ell^{-1}W$.

As one final step in the construction (to be used in the proof of Lemma 5.1), we require that $T_2^{-1} \circ T_1(x) \in \widetilde{U}_j^2$ for each $x \in U_j^1$. If this is not the case, then we are once again in a $C_t\varepsilon$ neighborhood of the endpoints of U_j^1 and so such points may be treated as unmatched pieces V_k^ℓ as above.

In this way, we write $T_\ell^{-1}W = (\cup_j U_j^\ell) \cup (\cup_k V_k^\ell)$ and note that the images $T_\ell V_k^\ell$ have length at most $C_v\varepsilon$ for some uniform constant C_v, $\ell = 1, 2$.

Now using (5.1), we have

$$\begin{aligned}\int_W (\mathcal{L}_1 h - \mathcal{L}_2 h)\psi\, dm_W &= \sum_{\ell,k} \int_{V_k^\ell} h\,\psi\circ T_\ell\,(J_\mu T_\ell)^{-1} J_{V_k^\ell}T_\ell\, dm_W \\ &\quad + \sum_j \int_{U_j^1} h\,\psi\circ T_1\,(J_\mu T_1)^{-1} J_{U_j^1}T_1\, dm_W - \int_{U_j^2} h\,\psi\circ T_2\,(J_\mu T_2)^{-1} J_{U_j^2}T_2\, dm_W.\end{aligned} \tag{5.2}$$

We estimate the integral on short pieces V_k^ℓ first using the strong stable norm. By (4.6), we have $|\psi\circ T_\ell|_{\mathcal{C}^q(V_k^\ell)} \le C_e|\psi|_{\mathcal{C}^p(W)} \le C_e$. Following the estimate in (4.14), we have

$$\sum_{\ell,k}\left|\int_{V_k^\ell} h(J_\mu T_\ell)^{-1}J_{V_k^\ell}T_\ell\,\psi\circ T_\ell\, dm\right| \le C\varepsilon^\alpha\|h\|_s \sum_{\ell,k}|J_{V_k^\ell}T_\ell|_{\mathcal{C}^0(V_k^\ell)}^{1-\alpha}. \tag{5.3}$$

The sum is finite by (3.7) of **(H3)** with $\varsigma = 1-\alpha$ since there are at most two V_k^ℓ corresponding to each element $W_i^{\ell,1} \in \mathcal{G}_1^\ell(W)$ as defined in Section 3.5 and $|J_{V_k^\ell}T_\ell|_{\mathcal{C}^0(V_k^\ell)} \le |J_{W_i^{\ell,1}}T_\ell|_{\mathcal{C}^0(W_i^{\ell,1})}$ whenever $V_j^\ell \subseteq W_i^{\ell,1}$. The constant C above depends only on properties **(H1)**-**(H5)**, but for brevity we do not write out the explicit dependence since these estimates are similar to those done in Section 4.3 and the constants are the same.

Next, we must estimate

$$\sum_j\left|\int_{U_j^1} h\,(J_\mu T_1)^{-1}J_{U_j^1}T_1\,\psi\circ T_1\, dm_W - \int_{U_j^2} h\,(J_\mu T_2)^{-1}J_{U_j^2}T_2\,\psi\circ T_2\, dm_W\right|.$$

Using notation analogous to (4.12), we fix j and estimate the difference. Define

$$\phi_j = ((J_\mu T_1)^{-1}J_{U_j^1}T_1\,\psi\circ T_1)\circ G_{U_j^1}\circ G_{U_j^2}^{-1}.$$

The function ϕ_j is well-defined on U_j^2 and we can write,

$$
\begin{aligned}
&\left| \int_{U_j^1} h \,(J_\mu T_1)^{-1} J_{U_j^1} T_1 \,\psi \circ T_1 - \int_{U_j^2} h\,(J_\mu T_2)^{-1} J_{U_j^2} T_2 \,\psi \circ T_2 \right| \\
&\le \left| \int_{U_j^1} h\,(J_\mu T_1)^{-1} J_{U_j^1} T_1 \,\psi \circ T_1 - \int_{U_j^2} h\,\phi_j \right| + \left| \int_{U_j^2} h(\phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \,\psi \circ T_2) \right|.
\end{aligned} \tag{5.4}
$$

To estimate the two terms above, we need the following adaptation of Lemma 4.2.

Lemma 5.1. *There exists $\bar C > 0$, independent of $W \in \mathcal{W}^s$ and $T_1, T_2 \in \mathcal{F}$, such that for each j,*

(a) $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le \bar C \varepsilon^{1/2}$;

(b) $|((J_\mu T_1)^{-1} J_{U_j^1} T_1) \circ G_{U_j^1} - ((J_\mu T_2)^{-1} J_{U_j^2} T_2) \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le \bar C |(J_\mu T_2)^{-1} J_{U_j^2} T_2|_{\mathcal{C}^0(U_j^2)} \varepsilon^{1/3-q}$;

(c) $|\psi \circ T_1 \circ G_{U_j^1} - \psi \circ T_2 \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le \bar C \varepsilon^{p-q}$.

We estimate the first term in equation (5.4) using the strong unstable norm. The estimates (3.17) and (4.6) and property **(H5)** imply that

$$
|(J_\mu T_1)^{-1} J_{U_j^1} T_1 \cdot \psi \circ T_1|_{U_j^1,0,p} \le \eta C_e |J_{U_j^1} T_1|_{\mathcal{C}^0(U_j^1)}. \tag{5.5}
$$

Similarly, since by (4.17), $|G_{U_j^1} \circ G_{U_j^2}^{-1}|_{\mathcal{C}^1} \le C_g$, we have $|\phi_j|_{U_j^2,0,p} \le C_g \eta C_e |J_{U_j^1} T_1|_{\mathcal{C}^0(U_j^1)}$. By the definition of ϕ_j and $d_q(\cdot,\cdot)$,

$$
d_q((J_\mu T_1)^{-1} J_{U_j^1} T_1 \psi \circ T_1, \phi_j) = \left| \left[(J_\mu T_1)^{-1} J_{U_j^1} T_1 \psi \circ T_1 \right] \circ G_{U_j^1} - \phi_j \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} = 0.
$$

In view of (5.5) and following, we renormalize the test functions by $R_j = \eta C_g C_e |J_{U_j^1} T_1|_{\mathcal{C}^0(U_j^1)}$. Then we apply the definition of the strong unstable norm using Lemma 5.1(a) to obtain,

$$
\sum_j \left| \int_{U_j^1} h\,(J_\mu T_1)^{-1} J_{U_j^1} T_1 \,\psi \circ T_1 - \int_{U_j^2} h\,\phi_j \right| \le C \varepsilon^{\beta/2} \|h\|_u \sum_j |J_{U_j^1} T_1|_{\mathcal{C}^0(U_j^1)}, \tag{5.6}
$$

where the sum is $\le C_2$ by Lemma 3.2(b) since there is at most one matched piece U_j^1 corresponding to each curve $W_i^1 \in \mathcal{G}_1^1(W)$.

We estimate the second term in (5.4) using the strong stable norm.

$$
\left| \int_{U_j^2} h(\phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \psi \circ T) \right| \le C \|h\|_s |U_j^2|^\alpha \left| \phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \psi \circ T_2 \right|_{\mathcal{C}^q(U_j^2)}. \tag{5.7}
$$

In order to estimate the $\mathcal{C}^q$-norm of the function in (5.7), we split it up into two differences. Following (4.20) line by line, we obtain

$$
\begin{aligned}
&|\phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \cdot \psi) \circ T_2 \,|_{\mathcal{C}^q(U_j^2)} \\
&\le C |(J_\mu T_1)^{-1} J_{U_j^1} T_1|_{\mathcal{C}^0(U_j^1)} \left| \psi \circ T_1 \circ G_{U_j^1} - \psi \circ T_2 \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)} \\
&\quad + C \left| ((J_\mu T_1)^{-1} J_{U_j^1} T_1) \circ G_{U_j^1} - ((J_\mu T_2)^{-1} J_{U_j^2} T_2) \circ G_{U_j^2} \right|_{\mathcal{C}^q(I_j)}
\end{aligned} \tag{5.8}
$$

Notice that Lemma 5.1(b) implies that

$$
|(J_\mu T_1)^{-1} J_{U_j^1} T_1|_{\mathcal{C}^0(U_j^1)} \le \left(1 + \bar C \varepsilon^{1/3-q}\right) |(J_\mu T_2)^{-1} J_{U_j^2} T_2|_{\mathcal{C}^0(U_j^2)}.
$$

Then using Lemma 5.1(b) and (c) together with (5.8) yields by (5.7)

$$
\sum_j \Big| \int_{U_j^2} h(\phi_j - (J_\mu T_2)^{-1} J_{U_j^2} T_2 \psi \circ T_2)\, dm_W \Big| \le C \|h\|_s \varepsilon^{p-q} \sum_j |J_{U_j^2} T_2|_{\mathcal{C}^0(U_j^2)},
$$

where again the sum is finite by Lemma 3.2(b). This completes the estimate on the second term in (5.4). Now we use this bound, together with (5.3) and (5.6) to estimate (5.2)

$$\left|\int_W \mathcal{L}_1 h\,\psi\,dm_W - \int_W \mathcal{L}_2 h\,\psi\,dm_W\right| \;\le\; C\|h\|_s\varepsilon^\alpha + C\|h\|_u\varepsilon^{\beta/2} + C\|h\|_s\varepsilon^{p-q}. \tag{5.9}$$

Since $p-q \ge \beta$ and $\alpha \ge \beta$, the theorem is proved.

5.1. **Proof of Lemma 5.1.**

Proof of (a). Note that by construction U_j^1 and U_j^2 lie in the same homogeneity strip. Also, they are both defined on the same interval I_j so the length of the symmetric difference of their r-intervals is 0. Recalling the definition of $d_{\mathcal{W}^s}(U_j^1, U_j^2)$, we see that it remains only to estimate $|\varphi_{U_j^1} - \varphi_{U_j^2}|_{\mathcal{C}^1(I_j)}$ for their defining functions $\varphi_{U_j^k}$.

For $x = (r, \varphi_{U_j^1}(r))$, define $\bar{x} = (r, \varphi_{U_j^2}(r))$ and $x_\varepsilon = T_2^{-1} \circ T_1(x)$. By the construction of U_j^1 at the beginning of this section, $x_\varepsilon \in \widetilde{U}_j^2$. Since x and x_ε are images of the same point $u \in W$ under T_1^{-1} and T_2^{-1} respectively, it follows from **(C1)** that x and x_ε are at most ε apart. Then since all vectors in the stable cone have slope bounded away from $\pm\infty$, it follows that x and $\bar{x}$ are at most $C\varepsilon$ apart (and so by the triangle inequality, also $\bar{x}$ and x_ε are at most $C\varepsilon$ apart).

This proves that $|\varphi_{U_j^1} - \varphi_{U_j^2}|_{\mathcal{C}^0(I_j)} \le C\varepsilon$. It remains to estimate $|\varphi'_{U_j^1} - \varphi'_{U_j^2}|$, where $\varphi'_{U_j^\ell}$ denotes the derivative of $\varphi_{U_j^\ell}$ with respect to r.

Let $\vec{v}_W(u)$ be the unit tangent vector to W at $u := T_1(x) = T_2(x)$, as before. The tangent vector to U_j^ℓ is given by $DT_\ell^{-1}(u)\vec{v}_W(u)$, $\ell = 1, 2$. By **(C4)**,

$$\|DT_1^{-1}(u)\vec{v}_W(u) - DT_2^{-1}(u)\vec{v}_W(u)\| \le \varepsilon^{1/2}. \tag{5.10}$$

Then since $\|DT_\ell^{-1}(u)\vec{v}_W(u)\| \ge C_e^{-1}$ by **(H1)**, we have $\theta(x, x_\varepsilon) \le C_e\varepsilon^{1/2}$, where $\theta(x, x_\varepsilon)$ is the angle between the tangent vectors to U_j^1 and U_j^2 at x and x_ε respectively.

For $y \in U_j^\ell$, let $\phi(y)$ denote the angle that $G_{U_j^\ell}$ makes with the positive r-axis at y. Then

$$|\varphi'_{U_j^1}(x) - \varphi'_{U_j^2}(\bar{x})| = |\tan\phi(x) - \tan\phi(\bar{x})| \le \Big[\sup_{z \in U_j^\ell} \sec^2\phi(z)\Big]|\phi(x) - \phi(\bar{x})| = \Big[\sup_{z \in U_j^\ell} \sec^2\phi(z)\Big]\theta(x, \bar{x}).$$

Since the slopes of curves in $C^s(x)$ are uniformly bounded away from $\pm\infty$, we have $\sec^2\phi(z)$ uniformly bounded above for any $z \in U_j^k$. The proof of the lemma is completed by writing $\theta(x, \bar{x}) \le \theta(x, x_\varepsilon) + \theta(x_\varepsilon, \bar{x})$. The first term is $\le C\varepsilon^{1/2}$ using (5.10) and the second term is $\le C\varepsilon$ since x_ε and $\bar{x}$ both lie on $\widetilde{U}_j^2$ and stable curves have a uniform $\mathcal{C}^2$ bound by **(H2)**. □

Proof of (b). We prove that the closeness condition **(C3)** implies the existence of a constant $C > 0$, independent of $W \in \mathcal{W}^s$ and $T_1, T_2 \in \mathcal{F}$, such that

$$|J_{U_j^1}T_1 \circ G_{U_j^1} - J_{U_j^2}T_2 \circ G_{U_j^2}|_{\mathcal{C}^q(I_j)} \le C|J_{U_j^2}T_2|_{\mathcal{C}^0(U_j^2)}\varepsilon^{1/3-q}. \tag{5.11}$$

The analogous statement concerning $(J_\mu T_k)^{-1}$ follows from condition **(C2)**. Then since

$$|f_1g_1 - f_2g_2|_{\mathcal{C}^q} \le |f_1|_{\mathcal{C}^q}|g_1 - g_2|_{\mathcal{C}^q} + |g_2|_{\mathcal{C}^q}|f_1 - f_2|_{\mathcal{C}^q},$$

for any $\mathcal{C}^q$ functions f_1, g_1, f_2, g_2, part (b) of the lemma follows from these two estimates using the fact that $|\cdot|_{\mathcal{C}^q} \le (1 + C_d)|\cdot|_{\mathcal{C}^0}$ by bounded distortion for the functions we are estimating. We proceed to prove (5.11).

For any $r \in I_j$, we write

$$\begin{aligned} |J_{U_j^1}T_1 \circ G_{U_j^1}(r) - J_{U_j^2}T_2 \circ G_{U_j^2}(r)| \;&\le\; |J_{U_j^1}T_1 \circ G_{U_j^1}(r) - J_{U_j^1}T_2 \circ G_{U_j^1}(r)| \\ &\quad + |J_{U_j^1}T_2 \circ G_{U_j^1}(r) - J_{U_j^2}T_2 \circ G_{U_j^2}(r)|. \end{aligned} \tag{5.12}$$

The first term above is $\le |J_{U_j^1}T_2|_{\mathcal{C}^0(I_j)}\varepsilon$ by **(C3)**.

Recall that U_j^1, U_j^2 lie inside the longer curves $\widetilde{U}_j^1, \widetilde{U}_j^2$ which are matched by the foliation $\{\gamma_x\}_{x\in\widetilde{U}_j^1} \subset \widehat{\mathcal{W}}^u$. Thus $|J_{U_j^1}T_2|_{\mathcal{C}^0(I_j)} \le C|J_{U_j^2}T_2|_{\mathcal{C}^0(I_j)}$ by the same argument used to prove (4.24), completing the estimate on the first term of (5.12).

The second term of (5.12) is $\le C'\varepsilon^{1/3}|J_{U_j^2}T_2|_{\mathcal{C}^0(I_j)}$ using (4.23) since it involves the Jacobian of a single map in $\mathcal{F}$ evaluated on two stable curves that are matched by a foliation of unstable curves. Thus

$$|J_{U_j^1}T_1 \circ G_{U_j^1}(r) - J_{U_j^2}T_2 \circ G_{U_j^2}(r)| \le C\varepsilon^{1/3}|J_{U_j^2}T_2|_{\mathcal{C}^0(U_j^2)}. \tag{5.13}$$

This implies in particular that $|J_{U_j^1}T_1|_{\mathcal{C}^0(U_j^1)} \le C|J_{U_j^2}T_2|_{\mathcal{C}^0(U_j^2)}$. Now we use (3.8) and the fact that $|G_{U_j^\ell}|_{\mathcal{C}^1(I_j)} \le C_g$ to apply Lemma 4.3 and complete the proof of (5.11). □

Proof of (c). Let $x = (r, \varphi_{U_j^1}(r))$ and as above, define $\bar{x} = (r, \varphi_{U_j^2}(r))$ and $x_\varepsilon = T_2^{-1} \circ T_1(x)$. Since $\bar{x}$ and x_ε are at most $C\varepsilon$ apart and lie on $\widetilde{U}_j^2$, we have $d_W(T_2\bar{x}, T_2x_\varepsilon) \le C\varepsilon$ by the uniform contraction given by **(H1)**. Thus,

$$|\psi \circ T_1 \circ G_{U_j^1}(r) - \psi \circ T_2 \circ G_{U_j^2}(r)| \le |\psi|_{\mathcal{C}^p(W)} d_W(T_1x, T_2\bar{x})^p. \tag{5.14}$$

Since $d_W(T_1x, T_2x_\varepsilon) = 0$, we may use the triangle inequality to conclude that the difference above is bounded by $C|\psi|_{\mathcal{C}^p(W)}\varepsilon^p$.

Again applying Lemma 4.3 with $|\psi|_{\mathcal{C}^p(W)} \le 1$ completes the proof of part (c). □

5.2. **Proof of Corollary 2.4.** We follow the proof of [DZ, Theorem 2.6] and remark on the essential differences. The strategy of the proof will be to show that for $T \in \mathcal{F}$ and a suitable observable g, the generalized transfer operator defined for $z \in \mathbb{C}$ by

$$\mathcal{L}_{zg}^n h(\psi) = h(e^{zS_ng}\psi \circ T^n), \quad \text{for all } h \in \mathcal{B}, \psi \in \mathcal{W}^s,$$

is an analytic perturbation of $\mathcal{L} = \mathcal{L}_0$ for small $|z|$.

We shall need the following result from [DZ].

Lemma 5.2. *(*[DZ, Lemma 3.7]*) Let $\mathcal{P}$ be a (mod 0) partition of M into countably many open, simply connected sets such that (1) there is a constant K_1 such that for each $P \in \mathcal{P}$, ∂P comprises at most K smooth curves, each of which is transverse to $C^s(x)$, with a minimum angle uniform for all $P \in \mathcal{P}$; (2) each homogeneity strip $\mathbb{H}_k$ intersects at most finitely many $P \in \mathcal{P}$.*

Let $\gamma > 2\beta$. Suppose h is a function on M such that $\sup_{P\in\mathcal{P}} |h|_{C^\gamma(P)} < \infty$. Then $h \in \mathcal{B}$.

We shall prove the following multiplier property for our Banach spaces which generalizes [DZ, Lemma 6.1] to allow functions with discontinuities.

Lemma 5.3. *Let $\mathcal{P}$ be a countable partition of M that satisfies the conditions of Lemma 5.2 and suppose in addition that there is a uniform upper bound N_1 on the number of $P \in \mathcal{P}$ that each $\mathbb{H}_k$ can intersect.*

Let $\gamma = \max\{p, 2\beta+\varepsilon\}$ for some $\varepsilon > 0$. Suppose f is a function on M such that $\sup_{P\in\mathcal{P}} |f|_{C^\gamma(P)} < \infty$ and let $h \in \mathcal{B}$. Then $hf \in \mathcal{B}$ and $\|hf\|_{\mathcal{B}} \le C\|h\|_{\mathcal{B}} \sup_{P\in\mathcal{P}} |f|_{C^\gamma(P)}$ for some uniform constant C.

Postponing the proof of the lemma, we show how it establishes the analyticity of $\mathcal{L}_{zg}$ for a function g which has discontinuity curves satisfying the conditions of Lemma 5.3.

Define the operator $\mathcal{A}_nh = \mathcal{L}(g^nh)$, for $h \in \mathcal{B}$. Then Lemma 5.3 implies that $g^nh \in \mathcal{B}$ and moreover,

$$\|\mathcal{A}_n(h)\|_{\mathcal{B}} = \|\mathcal{L}(g^nh)\| \le \|\mathcal{L}\|\|g^nh\|_{\mathcal{B}} \le V\|\mathcal{L}\|\|h\|_{\mathcal{B}} \sup_{P\in\mathcal{P}} |g|^n_{C^\gamma(P)}.$$

Thus the operator $\sum_{n=0}^{\infty} \frac{z^n}{n!} \mathcal{A}_n$ is well defined on $\mathcal{B}$ and equals $\mathcal{L}_{zg}$ since once we know the sum converges,

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \mathcal{A}_n h(\psi) = h\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} g^n \cdot \psi \circ T \right) = h(e^{zg} \psi \circ T) = \mathcal{L}_{zg} h(\psi), \quad \text{for } \psi \in \mathcal{C}^p(\mathcal{W}^s).$$

Once the analyticity of $\mathcal{L}_{zg}$ is established, the proof of Corollary 2.4 follows precisely as in [DZ, Theorem 2.6] and will not be repeated here. Note that the error exponent $\lambda > 1/4$ in Corollary 2.4(b) is justified by [G, eq (1.2)] since $g \in L^\infty(M)$. It remains to prove Lemma 5.3.

Proof of Lemma 5.3. Let $\mathcal{P}$ and f be as in the statement of the lemma. By density, it suffices to prove the lemma for $h \in C^1(M)$. By Lemma 5.2, $hf \in \mathcal{B}$. We proceed to estimate its norm. For brevity, we write

$$|f|_{\mathcal{C}^\gamma(\mathcal{P})} = \sup_{P \in \mathcal{P}} |f|_{\mathcal{C}^\gamma(P)}.$$

To estimate the strong stable norm, we fix $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^q(W)$ such that $|\psi|_{W,\alpha,q} \le 1$. For each $P_i \in \mathcal{P}$, set $W_i = W \cap P_i$. Then

$$\int_W hf\psi \, dm_W = \sum_i \int_{W_i} hf\psi \, dm_W \le \sum_i \|h\|_s |W_i|^\alpha |f|_{\mathcal{C}^q(W_i)} |\psi|_{\mathcal{C}^q(W_i)} \le N_1 K_1 \|h\|_s |f|_{\mathcal{C}^\gamma(\mathcal{P})},$$

where we have used the assumptions on ∂P_i to bound the maximum number of W_i by $N_1 K_1$.

Now to estimate the strong unstable norm of hf, we let $\varepsilon \le \varepsilon_0$ and choose $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) < \varepsilon$. For $\ell = 1, 2$, let $\psi_\ell \in \mathcal{C}^p(W^\ell)$ such that $|\psi_\ell|_{\mathcal{C}^p(W^\ell)} \le 1$ and $d_q(\psi_1, \psi_2) \le \varepsilon$.

Recalling the notation of Section 3.3, we write

$$W^\ell = G_{W^\ell}(I_{W^\ell}) = \{(r, \varphi_{W^\ell}(r)) : r \in I_{W^\ell}\}.$$

We subdivide each curve W^ℓ into matched and unmatched pieces, similar to those in Section 4.3. To each point $x \in W^1$, we attach a vertical line segment γ_x, centered at x of length 2ε. We define $U_j^\ell \subset W^\ell$ to be the maximal connected curves for which U_j^ℓ belongs to a single element $P \in \mathcal{P}$ and the family $\{\gamma_x\}_{x \in U_j^\ell}$ intersects W^2 but does not intersect ∂P for any $P \in \mathcal{P}$. We label by $V_i^\ell \subset W^\ell$ the remaining maximal pieces for which there is no matching by the vertical segments γ_x. We also require each V_i^ℓ to be contained in a single $P \in \mathcal{P}$. Note that there are at most $2N_1K_1 + 2$ unmatched pieces and at most $N_1 K_1$ matched pieces by assumption on $\mathcal{P}$. Also, due to the uniform transversality of ∂P with $C^s(x)$, we have $|V_i^\ell| \le C_t \varepsilon$ for each ℓ, j and a uniform constant C_t.

We define $\phi = (f\psi_1) \circ G_{W^1} \circ G_{W^2}^{-1}$ and note that ϕ is well defined on each matched piece U_j^2. We must estimate

$$\begin{aligned}\int_{W^1} hf\psi_1 \, dm_W - \int_{W^2} hf\psi_2 \, dm_W &= \sum_{i,\ell} \int_{V_i^\ell} hf\psi_\ell \, dm_W \\ &+ \sum_j \left(\int_{U_j^1} hf\psi_1 \, dm_W - \int_{U_j^2} h\phi \, dm_W \right) + \int_{U_j^2} h(\phi - f\psi_2) \, dm_W.\end{aligned} \tag{5.15}$$

The first sum on the right hand side of (5.15) over unmatched pieces is estimated by,

$$\sum_{i,\ell} \int_{V_i^\ell} hf\psi_\ell \, dm_W \le \sum_{i,\ell} \|h\|_s |V_i^\ell|^\alpha |f|_{\mathcal{C}^q(V_i^\ell)} |\psi_\ell|_{\mathcal{C}^q(V_i^\ell)} \le (2N_1K_1 + 2)\|h\|_s |f|_{\mathcal{C}^\gamma(\mathcal{P})} C_t \varepsilon^\alpha. \tag{5.16}$$

Next we estimate the difference over matched pieces in (5.15). By construction, $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le \varepsilon$ since $U_j^\ell \subset W^\ell$. Moreover, letting I_j denote the common r-interval over which U_j^1 and U_j^2 are both defined, we have

$$d_q(f\psi_1, \phi) = |(f\psi_1) \circ G_{W^1} - \phi \circ G_{W^2}|_{\mathcal{C}^q(I_j)} = 0.$$

Also, since $G_{W^1} \circ G_{W^2}^{-1}$ has bounded C^1-norm, we have $|f\psi_1|_{\mathcal{C}^p(U_j^1)}, |\phi|_{\mathcal{C}^p(U_j^2)} \le C|f|_{\mathcal{C}^\gamma(\mathcal{P})}$ for some uniform constant C. Renormalizing the test functions, we apply the definition of the strong unstable norm to estimate

$$\sum_j \left| \int_{U_j^1} hf\psi_1 \, dm_W - \int_{U_j^2} h\phi \, dm_W \right| \le N_1 K_1 \varepsilon^\beta \|h\|_u C |f|_{\mathcal{C}^\gamma(\mathcal{P})}. \tag{5.17}$$

Finally, we estimate the third sum on the right hand side of (5.15) using the strong stable norm.

$$\sum_j \left| \int_{U_j^2} h(\phi - f\psi_2) \, dm_W \right| \le \sum_j \|h\|_s |\phi - f\psi_2|_{\mathcal{C}^q(U_j^2)} |U_j^2|^\alpha.$$

Again using that G_{W^2} has bounded C^1-norm, we estimate

$$|\phi - f\psi_2|_{\mathcal{C}^q(U_j^2)} \le C|(f\psi_1) \circ G_{W^1} - (f\psi_2) \circ G_{W^2}|_{\mathcal{C}^q(I_j)}.$$

For $r \in I_j$, we have

$$|(f\psi_1) \circ G_{W^1}(r) - (f\psi_2) \circ G_{W^2}(r)| \le |f|_\infty |\psi_1 \circ G_{W^1}(r) - \psi_2 \circ G_{W^2}(r)| + |\psi_2|_{\mathcal{C}^0(W^2)} |f \circ G_{W^1}(r) - f \circ G_{W^2}(r)|.$$

The first difference above is bounded by ε due to the assumption $d_q(\psi_1, \psi_2) \le \varepsilon$. The second difference is bounded by $|f|_{\mathcal{C}^\gamma(\mathcal{P})} \varepsilon^\gamma$. Now using Lemma 4.3, we conclude

$$|\phi - f\psi_2|_{\mathcal{C}^q(U_j^2)} \le C|f|_{\mathcal{C}^\gamma(\mathcal{P})} \varepsilon^{p-q}. \tag{5.18}$$

Putting together (5.16), (5.17) and (5.18) with (5.15), we have

$$\left| \int_{W^1} hf\psi_1 \, dm_W - \int_{W^2} hf\psi_2 \, dm_W \right| \le C|f|_{\mathcal{C}^\gamma(\mathcal{P})} (\|h\|_s \varepsilon^\alpha + \|h\|_u \varepsilon^\beta + \|h\|_s \varepsilon^{p-q}),$$

for some uniform constant C depending on N_1 and K_1. This completes the estimate on the strong unstable norm since $\beta \le \min\{\alpha, p-q\}$. □

5.3. **Random Perturbations: Proof of Theorem 2.6.** We fix a class of maps $\mathcal{F}$ for which **(H1)**-**(H5)** hold with uniform constants and choose $T_0 \in \mathcal{F}$. Define $X_\varepsilon(T_0) = \{T \in \mathcal{F} : d_{\mathcal{F}}(T, T_0) \le \varepsilon\}$. Recall the transfer operator $\mathcal{L}_{(\nu,g)}$ associated with the random process drawn from $X_\varepsilon(T_0)$ as defined in Section 2.3. Our first lemma is a generalization of Theorem 2.3 which shows that the transfer operator $\mathcal{L}_{(\nu,g)}$ is close to $\mathcal{L}_{T_0}$ in the norms we have defined.

Lemma 5.4. *There exists $C > 0$ such that if $\varepsilon \le \varepsilon_0$, then $|||\mathcal{L}_{(\nu,g)} - \mathcal{L}_{T_0}||| \le CA\varepsilon^\beta$.*

Proof. Let $h \in \mathcal{C}^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in \mathcal{C}^p(W)$ with $|\psi|_{W,0,p} \le 1$. Then using (5.9),

$$\begin{aligned} \left| \int_W \mathcal{L}_{(\nu,g)} h \, \psi \, dm_W - \int_W \mathcal{L}_{T_0} h \, \psi \, dm_W \right| &= \left| \int_\Omega \int_W (\mathcal{L}_{T_\omega} h(x) - \mathcal{L}_{T_0} h(x)) \psi(x) g(\omega, T_\omega^{-1} x) \, dm_W d\nu \right| \\ &\le \int_\Omega C b^{-1} \varepsilon^{\beta/2} \|h\| |g(\omega, \cdot)|_{\mathcal{C}^1(M)} d\nu(\omega) \le C b^{-1} A \varepsilon^{\beta/2} \|h\|, \end{aligned}$$

where we have interchanged order of integration since $\int_W \mathcal{L}_{T_w}(h) \, \psi \, g(\omega, \cdot) \, dm_W$ is uniformly and absolutely integrable for each $\omega \in \Omega$ by Theorem 2.2. □

It remains to prove the uniform Lasota-Yorke inequalities for $\mathcal{L}_{\nu,g}$. Let $\overline{\omega}_n = (\omega_1, \ldots, \omega_n) \in \Omega^n$ and define $T_{\overline{\omega}_n} = T_{\omega_n} \circ \cdots \circ T_{\omega_1}$. We first prove that the random compositions $T_{\overline{\omega}_n}$ have the same properties **(H1)**-**(H5)** as the maps $T_\omega \in \mathcal{F}$, with possibly modified constants.

The singularity sets for $T_{\overline{\omega}_n}$ are $\mathcal{S}_n^{T_{\overline{\omega}_n}} = \cup_{k=1}^n T_{\omega_1}^{-1} \circ \cdots \circ T_{\omega_k}^{-1} \mathcal{S}_0$, for $n \ge 0$, and similarly for $\mathcal{S}_{-n}^{T_{\overline{\omega}_n}}$. Thus the transversality properties **(H1)** of $\mathcal{S}_{-n}^{T_{\overline{\omega}_n}}$ with respect to C^s and C^u hold due to the uniformity of this transversality for all maps in $\mathcal{F}$. The family $\mathcal{W}^s$ is preserved under $T_{\overline{\omega}_n}^{-1}$ since it is preserved by each map in the composition.

The uniform expansion given by (3.2) of **(H1)** also holds since $DT_{\overline{\omega}_n} = \prod_{k=1}^n DT_{\omega_k} \circ T_{\overline{\omega}_{k-1}}$ and in the adapted metric $\| \cdot \|_*$ given by **(H3)**, the expansion holds with $C_e = 1$ for each map in the composition. Translating to the Euclidean norm at the last step, we get **(H1)** with C_e depending only on the uniform constant relating the adapted and Euclidean metrics. Equations (3.4) and (3.5) also hold trivially since they concern only one iterate of a map drawn from $\mathcal{F}$. **(H5)** follows for the same reason.

Due to the uniform expansion along stable and unstable leaves, (3.8) and (3.9) of **(H4)** hold with a possibly larger distortion constant C_d^*, again using the bounded distortion of each map in the composition $T_{\overline{\omega}_n}$.

Finally, we establish that the iteration of the one-step expansion given in **(H3)** holds for random sequences of maps in the class $\mathcal{F}$. As in Section 3.5, for $W \in \mathcal{W}^s$ we define the nth generation $\mathcal{G}_n^{\overline{\omega}_n}(W) \subset \mathcal{W}^s$ of smooth curves in $T_{\overline{\omega}_n}^{-1} W$. The elements of $\mathcal{G}_n^{\overline{\omega}_n}(W)$ are denoted by W_i^n as before and long and short pieces are defined similarly. Analogously, $\mathcal{I}_n^{\overline{\omega}_n}(W_j^k)$ denotes the set of indices i in generation n such that W_j^k is the most recent long ancestor of W_i^n under $T_{\overline{\omega}_n}$. Thus $\mathcal{I}_n^{\overline{\omega}_n}(W)$ denotes the set of curves that are never part of a curve that has grown to length $\delta_0/3$ at each time step $1 \le k \le n$.

Lemma 5.5. *Let $W \in \mathcal{W}^s$ and for $n \ge 0$, let $\mathcal{I}_n^{\overline{\omega}_n}(W)$ and $\mathcal{G}_n^{\overline{\omega}_n}(W)$ be defined as above. There exist constants $C_1, C_2, C_3 > 0$, independent of $W \in \mathcal{W}^s$ and $\overline{\omega}_n \in \Omega^n$, such that for any $n \ge 0$,*

(a) $\displaystyle\sum_{i \in \mathcal{I}_n^{\overline{\omega}_n}(W)} |J_{W_i^n} T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)} \le C_1 \theta_*^n$;

(b) $\displaystyle\sum_{W_i^n \in \mathcal{G}_n^{\overline{\omega}_n}(W)} |J_{W_i^n} T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)} \le C_2$;

(c) *for any* $0 \le \varsigma \le 1$, $\displaystyle\sum_{W_i^n \in \mathcal{G}_n^{\overline{\omega}_n}(W)} \frac{|W_i^n|^\varsigma}{|W|^\varsigma} |J_{W_i^n} T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)} \le C_2^{1-\varsigma}$;

(d) *for* $\varsigma > \varsigma_0$, $\displaystyle\sum_{W_i^n \in \mathcal{G}_n^{\overline{\omega}_n}(W)} |J_{W_i^n} T_{\overline{\omega}_n}|^\varsigma_{\mathcal{C}^0(W_i^n)} \le C_3^n$.

Proof. (a) Fix $W \in \mathcal{W}^s$ and for $\overline{\omega}_n \in \Omega^n$, define $\mathcal{Z}_n(W) = \sum_{i \in \mathcal{I}_n^{\overline{\omega}_n}(W)} |J_{W_i^n} T_{\overline{\omega}_n}|_*$, where $|J_{W_i^n} T_{\overline{\omega}_n}|_*$ denotes the least contraction on W_i^n under $T_{\overline{\omega}_n}$ measured in the metric induced by the adapted norm. We will prove by induction on $n \in \mathbb{N}$ that $\mathcal{Z}_n(W) \le \theta_*^n$. Then, since $\| \cdot \|_*$ is equivalent to $\| \cdot \|$, statement (a) follows.

Note that at each iterate between 1 and n, every piece W_i^n, $i \in \mathcal{I}_n^{\overline{\omega}_n}(W)$, is created by genuine cuts due to singularities and homogeneity strips and not by any artificial subdivisions, since those are only made when a piece has grown to length greater than δ_0. Thus we may apply the one-step expansion (3.6) to conclude,

$$\mathcal{Z}_1(W) \le \theta_*, \quad \forall\, W \in \mathcal{W}^s. \tag{5.19}$$

Assume that $\mathcal{Z}_n(W) \le \theta_*^n$ is proved for some $n \ge 1$ and all $W \in \mathcal{W}^s$. We apply it to each component $W_i^1 \in \mathcal{G}_1^{\omega_1}(W)$ such that $i \in \mathcal{I}_1^{\omega_1}(W)$. Then $\mathcal{Z}_n(W_i^1) \le \theta_*^n$ since $W_i^1 \in \mathcal{W}^s$.

Given $\overline{\omega}_n \in \Omega^n$, we use the notation $\overline{\omega}'_{n-k} = (\omega_n, \dots, \omega_{n-k+1})$ so that we may split up compositions $\overline{\omega}_n = (\overline{\omega}'_{n-k}, \overline{\omega}_k)$ into two pieces. Given a sequence $\overline{\omega}_{n+1}$, we group the components of $W_i^{n+1} \in \mathcal{G}_{n+1}^{\overline{\omega}_{n+1}}(W)$ with $i \in \mathcal{I}_{n+1}^{\overline{\omega}_{n+1}}(W)$ according to elements with index in $\mathcal{I}_1^{\omega_1}(W)$. More precisely, for $j \in \mathcal{I}_1^{\omega_1}(W)$, let $A_j = \{i : W_i^{n+1} \in \mathcal{G}_{n+1}^{\overline{\omega}_{n+1}}(W), T_{\overline{\omega}'_n} W_i^{n+1} \subset W_j^1\}$. Note that $|J_{W_i^{n+1}} T_{\overline{\omega}_{n+1}}|_* \le |J_{W_i^{n+1}} T_{\overline{\omega}'_n}|_* |J_{W_j^1} T_{\omega_1}|_*$ whenever $T_{\overline{\omega}'_n} W_i^{n+1} \subseteq W_j^1$. Combining this and (5.19) with the inductive hypothesis, we get

$$\begin{aligned}\mathcal{Z}_{n+1}(W) &= \sum_{j\in\mathcal{I}_1^{\omega_1}(W)}\sum_{i\in A_j}|J_{W_i^{n+1}}T_{\overline{\omega}_{n+1}}|_* \;\le\; \sum_{j\in\mathcal{I}_1^{\omega_1}(W)}\left(\sum_{i\in A_j}|J_{W_i^{n+1}}T_{\overline{\omega}'_n}|_*\right)|J_{W_j^1}T_{\omega_1}|_* \\ &= \sum_{j\in\mathcal{I}_1^{\omega_1}(W)}\mathcal{Z}_n(W_j^1)\cdot|J_{W_j^1}(T_{\omega_1})|_* \;\le\; \theta_*^{n+1}.\end{aligned}$$

(b) Fix $W\in\mathcal{W}^s$ and $\overline{\omega}_n\in\Omega^n$. For any $0\le k\le n$ and $W_i^n\in\mathcal{G}_n^{\overline{\omega}_n}(W)$, we have

$$|J_{W_i^n}T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)}\le|J_{W_i^n}T_{\overline{\omega}'_{n-k}}|_{\mathcal{C}^0(W_i^n)}|J_{W_j^k}T_{\overline{\omega}_k}|_{\mathcal{C}^0(W_j^k)},\tag{5.20}$$

whenever $T_{\overline{\omega}'_{n-k}}W_i^n\subseteq W_j^k\in\mathcal{G}_k^{\overline{\omega}_k}(W)$.

Now grouping $W_i^n\in\mathcal{G}_n^{\overline{\omega}_n}(W)$ by most recent long ancestor $W_j^k\in L_k^{\overline{\omega}_k}(W)$ as described in Section 3.5 and using (5.20), we have

$$\begin{aligned}\sum_i|J_{W_i^n}T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)} &= \sum_{k=0}^n\sum_{W_j^k\in L_k^{\overline{\omega}_k}(W)}\sum_{i\in\mathcal{I}_n^{\overline{\omega}_n}(W_j^k)}|J_{W_i^n}T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)} \\ &\le\sum_{k=1}^n\sum_{W_j^k\in L_k^{\overline{\omega}_k}(W)}\Big(\sum_{i\in\mathcal{I}_n^{\overline{\omega}_n}(W_j^k)}|J_{W_i^n}T_{\overline{\omega}'_{n-k}}|_{\mathcal{C}^0(W_i^n)}\Big)|J_{W_j^k}T_{\overline{\omega}_k}|_{\mathcal{C}^0(W_j^k)} \;+\; \sum_{i\in\mathcal{I}_n^{\overline{\omega}_n}(W)}|J_{W_i^n}T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)},\end{aligned}$$

where we have split off the terms involving $k=0$ that have no long ancestor. We have

$$|J_{W_j^k}T_{\overline{\omega}_k}|_{\mathcal{C}^0(W_j^k)}\le(1+C_d^*)|T_{\overline{\omega}_k}W_j^k||W_j^k|^{-1}\le3\delta_0^{-1}(1+C_d^*)|T_{\overline{\omega}_k}W_j^k|$$

since $|W_j^k|\ge\delta_0/3$. Since $\mathcal{I}_n^{\overline{\omega}_n}(W_j^k)$ and $\mathcal{I}_{n-k}^{\overline{\omega}'_{n-k}}(W_j^k)$ correspond to the same set of short pieces in the $(n-k)^{\text{th}}$ generation of W_j^k, we apply part (a) of this lemma to each of these sums. Thus,

$$\begin{aligned}\sum_i|J_{W_i^n}T_{\overline{\omega}_n}|_{\mathcal{C}^0(W_i^n)} &\le\sum_{k=0}^{n-1}\sum_{W_j^k\in L_k^{\overline{\omega}_k}(W)}C_1\theta_*^{n-k}|J_{W_j^k}T_{\overline{\omega}_k}|_{\mathcal{C}^0(W_j^k)}\;+\;C_1\theta_*^n \\ &\le C\delta_0^{-1}\sum_{k=0}^{n-1}\sum_{W_j^k\in L_k^{\overline{\omega}_k}(W)}\theta_*^{n-k}|T_{\overline{\omega}_k}W_j^k|+C\theta_*^n\;\le\;C\delta_0^{-1}|W|\sum_{k=0}^{n-1}\theta_*^{n-k}+C\theta_*^n,\end{aligned}$$

which is uniformly bounded in n.

(c) follows from (b) by an application of Jensen's inequality and (d) follows from **(H3)** using an inductive argument similar to the proof of (a). □

We complete the proof of Theorem 2.6 via the following proposition. The uniform Lasota-Yorke inequalities of Theorem 2.2 then follow from the argument given at the beginning of Section 4.

Proposition 5.6. *Choose $\varepsilon\le\varepsilon_0$ sufficiently small that $\sigma(1+\varepsilon)<1$ and let $\Delta(\nu,g)\le\varepsilon$. There exists a constant C, depending on a, A, and* **(H1)**-**(H5)** *such that for $h\in\mathcal{B}$ and $n\ge0$,*

$$|\mathcal{L}_{(\nu,g)}^n h|_w \;\le\; C\eta^n|h|_w \tag{5.21}$$

$$\|\mathcal{L}_{(\nu,g)}^n h\|_s \;\le\; C\eta^n(\theta_*^{(1-\alpha)n}+\Lambda^{-qn})\|h\|_s+C\delta_0^{-\alpha}\eta^n|h|_w \tag{5.22}$$

$$\|\mathcal{L}_{(\nu,g)}^n h\|_u \;\le\; C\eta^n\Lambda^{-\beta n}\|h\|_u+C\eta^nC_3^n\|h\|_s \tag{5.23}$$

Proof. We record for future use,

$$\mathcal{L}^n_{(\nu,g)}h(x) = \int_{\Omega^n} h \circ T^{-1}_{\overline{\omega}_n}(J_\mu T_{\overline{\omega}_n} \circ T^{-1}_{\overline{\omega}_n})^{-1} \prod_{j=1}^n g(\omega_j, T^{-1}_{\omega_j} \circ \cdots \circ T^{-1}_{\omega_n} x) d\nu^n(\overline{\omega}_n).$$

The proofs of the inequalities are the same as in Section 4 except that we have the additional function $g(\omega, x)$. We show how to adapt the estimates of Section 4 to the operator $\mathcal{L}_{(\nu,g)}$ in the case of the strong stable norm. The other estimates are similar.

Estimating the Strong Stable Norm. Following Section 4.2, we write,

(5.24)

$$\int_W \mathcal{L}^n_{(\nu,g)} h\, \psi dm_W = \int_{\Omega^n} \sum_i \Bigg\{ \int_{W^n_i} h(\psi \circ T_{\overline{\omega}_n} - \overline{\psi}_i)(J_\mu T_{\overline{\omega}_n})^{-1} J_{W^n_i} T_{\overline{\omega}_n} \prod_{j=1}^n g(\omega_j, T_{\overline{\omega}_{j-1}} x) dm_W$$
$$+\overline{\psi}_i \int_{W^n_i} h(J_\mu T_{\overline{\omega}_n})^{-1} J_{W^n_i} T_{\overline{\omega}_n} \prod_{j=1}^n g(\omega_j, T_{\overline{\omega}_{j-1}} x) dm_W \Bigg\} d\nu^n(\overline{\omega}_n),$$

where $\overline{\psi}_i = |W^n_i|^{-1} \int_{W^n_i} \psi \circ T_{\overline{\omega}_n}\, dm_W$. Since for each $\overline{\omega}_n$, $T_{\overline{\omega}_n}$ satisfies properties **(H1)-(H5)** with uniform constants, we may use the estimates of Section 4. Accordingly, $|\psi \circ T_{\overline{\omega}_n} - \overline{\psi}_i|_{\mathcal{C}^q(W^n_i)} \leq C\Lambda^{-qn}|W|^{-\alpha}$ using (4.6). Define $G_{\overline{\omega}_n}(x) = \prod_{j=1}^n g(\omega_j, T_{\overline{\omega}_{j-1}} x)$. We estimate the first term of (5.24) using (4.10)

$$\begin{aligned}
&\sum_i \int_{W^n_i} h(\psi \circ T_{\overline{\omega}_n} - \overline{\psi}_i)\, (J_\mu T_{\overline{\omega}_n})^{-1} J_{W^n_i} T_{\overline{\omega}_n} G_{\overline{\omega}_n}\, dm_W \\
&\leq \sum_i C\|h\|_s |W_i|^\alpha |(J_\mu T_{\overline{\omega}_n})^{-1} J_{W^n_i} T_{\overline{\omega}_n}|_{\mathcal{C}^q(W^n_i)} |\psi \circ T_{\overline{\omega}_n} - \overline{\psi}_i|_{\mathcal{C}^q(W^n_i)} |G_{\overline{\omega}_n}|_{\mathcal{C}^q(W^n_i)} \\
&\leq C\|h\|_s \Lambda^{-qn}\eta^n \sum_i \frac{|W^n_i|^\alpha}{|W|^\alpha} |J_{W^n_i} T_{\overline{\omega}_n}|_{\mathcal{C}^0(W^n_i)} |G_{\overline{\omega}_n}|_{\mathcal{C}^q(W^n_i)}.
\end{aligned} \tag{5.25}$$

The only new term here is $|G_{\overline{\omega}_n}|_{\mathcal{C}^q(W^n_i)}$ which is addressed by the following lemma.

Sublemma 5.7. *There exists $C > 0$, independent of W and $\overline{\omega}_n$, such that if $W^n_i \in \mathcal{G}^{\overline{\omega}_n}_n(W)$, then*

$$|G_{\overline{\omega}_n}|_{\mathcal{C}^1(W^n_i)} \leq C G_{\overline{\omega}_n}(x) \ \text{ for any } x \in W^n_i.$$

Proof of Sublemma. For $x, y \in W^n_i$,

$$\begin{aligned}
\log \frac{\prod_{j=1}^n g(\omega_j, T_{\overline{\omega}_{j-1}} x)}{\prod_{j=1}^n g(\omega_j, T_{\overline{\omega}_{j-1}} y)} &\leq \sum_{j=1}^n a^{-1} |g(\omega_j, \cdot)|_{\mathcal{C}^1(M)} d(T_{\overline{\omega}_{j-1}} x, T_{\overline{\omega}_{j-1}} y) \\
&\leq \sum_{j=1}^\infty a^{-1} A C_e \Lambda^{-n} d(x, y) =: c_0 d(x, y),
\end{aligned}$$

using properties (i) and (iii) of g. The distortion bound yields the lemma with $C = c_0 e^{c_0}$. □

We estimate (5.25) using the sublemma and Lemma 5.5(c),

$$\sum_i \int_{W^n_i} h(\psi \circ T_{\overline{\omega}_n} - \overline{\psi}_i)\, (J_\mu T_{\overline{\omega}_n})^{-1} J_{W^n_i} T_{\overline{\omega}_n} G_{\overline{\omega}_n}\, dm_W \leq C\|h\|_s \eta^n \Lambda^{-qn} G_{\overline{\omega}_n}(x_0), \tag{5.26}$$

where x_0 is some point in $T^{-1}_{\overline{\omega}_n} W$.

Similarly, we estimate the second term in (5.24) using (4.11). In each term, $G_{\overline{\omega}_n}$ plays the role of a test function and we replace the occurrences of $|G_{\overline{\omega}_n}|_{\mathcal{C}^p(W_i^n)}$ and $|G_{\overline{\omega}_n}|_{\mathcal{C}^q(W_i^n)}$ as appropriate according to Sublemma 5.7. Thus following (4.11), we write,

$$\sum_i \overline{\psi}_i \int_{W_i^n} h(J_\mu T_{\overline{\omega}_n})^{-1} J_{W_i^n} T_{\overline{\omega}_n} G_{\overline{\omega}_n} dm_W \le C(\delta_0^{-\alpha}\eta^n|h|_w + \theta_*^{(1-\alpha)n}\eta^n\|h\|_s)G_{\overline{\omega}_n}(x_0),$$

choosing the same x_0 as in (5.26). Now combining this expression with (5.26) and (5.24), we obtain

$$\int_W \mathcal{L}_{T_{\overline{\omega}_n}} h\psi\, dm_W \le C\eta^n(\|h\|_s(\Lambda^{-qn} + \theta_*^{(1-\alpha)n}) + \delta_0^{-\alpha}|h|_w)\prod_{j=1}^n g(\omega_j, T_{\overline{\omega}_{j-1}}x_0).$$

We integrate this expression one ω_j at a time, starting with ω_n. Notice that $\int_\Omega g(\omega_n, T_{\overline{\omega}_{n-1}}x_0)d\nu(\omega_n) = 1$ by property (ii) of g since $T_{\overline{\omega}_{n-1}}$ is independent of ω_n. Similarly, each factor in $G_{\overline{\omega}_n}(x_0)$ integrates to 1 so that

$$\|\mathcal{L}^n_{(\nu,g)}h\|_s \le C\|h\|_s\eta^n(\Lambda^{-qn} + \theta_*^{(1-\alpha)n}) + C\delta_0^{-\alpha}\eta^n|h|_w$$

which is the required inequality for the strong stable norm. The inequalities for the weak norm and the strong unstable norm follow similarly, always using Sublemma 5.7. □

6. Proofs of Applications: Movements and Deformations of Scatterers

In this section we prove Theorems 2.7 and 2.8 and leave Theorems 2.10 and 2.11 regarding external forces and kicks to Section 7 since they require more background material.

6.1. **Proof of Theorem 2.7.** We fix constants $\tau_*, \mathcal{K}_* > 0$ and $E_* < \infty$ and denote $\mathcal{F}_1(\tau_*, \mathcal{K}_*, E_*)$ as simply $\mathcal{F}_1$ for brevity. Note that every $T \in \mathcal{F}_1$ is a billiard map corresponding to a standard Lorentz gas with convex scatterers so that we may recall known facts about such maps to establish **(H1)-(H5)** with constants depending only on the three quantities τ_*, $\mathcal{K}_*$ and E_*.

(H1). For $x \in M$, define

$$C^s(x) = \{(dr, d\varphi) \in \mathcal{T}_x M : -\mathcal{K}_*^{-1} - \tau_*^{-1} \le d\varphi/dr \le -\mathcal{K}_*\}$$
$$\text{and } \ C^u(x) = \{(dr, d\varphi) \in \mathcal{T}_x M : \mathcal{K}_* \le d\varphi/dr \le \mathcal{K}_*^{-1} + \tau_*^{-1}\}.$$

Then for any $T \in \mathcal{F}_1$, $DT_xC^u(x) \subset C^u(Tx)$ and $DT_x^{-1}C^s(u) \subset C^s(T^{-1}x)$ whenever DT_x and DT_x^{-1} are defined. Moreover, (3.2) is satisfied with $\Lambda = 1 + 2\mathcal{K}_*\tau_*$ and

$$C_e = \frac{2\tau_*\mathcal{K}_*}{\Lambda}\frac{\sqrt{1+\mathcal{K}_*^2}}{\sqrt{1+(\mathcal{K}_*^{-1}+\tau_*^{-1})^2}},$$

(see [CM, Section 4.4]). Notice that C^s and C^u are uniformly transverse to each other and to the vertical and horizontal directions in M as required.

The bounds on the first and second derivatives of T required by (3.4) and (3.5) are standard for such maps ([CM, Section 4.4]). Here, the index n corresponds to the free flight time $\tau(T^{-1}x)$. For finite horizon, this has a uniform upper bound, while for infinite horizon, the relation between k and n is satisfied with $v_0 = 1/4$ ([CM, Section 5.10]).

(H2). We say a $\mathcal{C}^2$ curve W in M is stable if its tangent vectors $\mathcal{T}_xW$ lie in $C^s(x)$ as defined above for each $x \in W$. We call a stable curve homogeneous if it is contained in a single homogeneity strip $\mathbb{H}_k$. Since each stable curve W has slope bounded away from infinity, we may identify W with the graph of a function of r, which we denote by $\varphi_W(r)$.

By [CM, Proposition 4.29], we may choose B depending only on τ_*, $\mathcal{K}_*$ and E_* such that if $\frac{d^2\varphi_W}{dr^2} \le B$, then each smooth component W' of $T^{-1}W$ satisfies $\frac{d^2\varphi_{W'}}{dr^2} \le B$.

We define $\widehat{\mathcal{W}}^s$ to be the set of all stable homogeneous curves W such that $\frac{d^2\varphi_W}{dr^2} \le B$. The invariance of the family $\mathcal{C}^s(x)$ as well as the choice of B guarantee that $\widehat{\mathcal{W}}^s$ is invariant as required. The set of unstable curves $\widehat{\mathcal{W}}^u$ is defined similarly.

(H3). Following [CM, Section 5.10], we define the adapted norm in the tangent space at $x \in M$ by

$$\|v\|_* = \frac{\mathcal{K}(x) + |\mathcal{V}|}{\sqrt{1+\mathcal{V}^2}}\|v\|, \quad \forall v \in C^s(x) \cup C^u(x)$$

where, $v = (dr, d\varphi)$ is a tangent vector, $\mathcal{V} = d\varphi/dr$ and $\mathcal{K}(x)$ is the curvature of the scatterer at x. Since the slopes of vectors in $C^s(x)$ and $C^u(x)$ are bounded away from $\pm\infty$, we may extend $\|\cdot\|_*$ to all of $\mathbb{R}^2$ in such a way that $\|\cdot\|_*$ is uniformly equivalent to $\|\cdot\|$. It is straightforward to check that for $v \in C^u(x)$,

$$\frac{\|DT(x)v\|_*}{\|v\|_*} \ge 1 + \mathcal{K}_*\tau_* = \Lambda.$$

Uniform expansion in $C^s(x)$ under $DT^{-1}(x)$ follows similarly. Now (3.6) follows from [CM, Lemma 5.56] and (3.7) follows from [DZ, Sublemma 3.5] with $\varsigma_0 = 1/6$.

The reason that the constant δ_0 from (3.10) can be chosen uniformly is that all infinite horizon points are uniformly bounded away from one another for all maps in the family $\mathcal{F}$. Once we specify a minimum curvature $\mathcal{K}_*$ and the arclengths given by $|I_i|$, $i = 1, \dots, d$, then every scatterer corresponding to an admissible configuration for $\mathcal{F}$ must have a minimum diameter uniformly bounded away from 0. Thus two infinite horizon points cannot converge as we move and deform scatterers in this fixed family $\mathcal{F}$, and indeed they must maintain a minimum distance from one another.

From this point forward, we consider k_0 to be fixed.

(H4). The bounded distortion constant C_d in (3.8) and (3.9) depends only on the choice of k_0 from **(H3)** and the uniform hyperbolicity constants C_e and Λ ([CM, Lemma 5.27]).

(H5). For maps in $\mathcal{F}_1$, $DT(x) \equiv 1$ so we may take $\eta = 1$.

6.2. **Proof of Theorem 2.8.** Fix constants $\tau_*, \mathcal{K}_* > 0$ and $E_* < \infty$ and consider a configuration $Q_0 \in \mathcal{Q}_1(\tau_*, \mathcal{K}_*, E_*)$ with scatterers $\Gamma_1, \dots, \Gamma_d$. Choose $\gamma \le \frac{1}{2}\min\{\tau_*, \mathcal{K}_*\}$ and let $\tilde{Q} \in \mathcal{F}_B(Q_0, E_*; \gamma)$ with scatterers $\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_d$. Since $\ell(I_i) = |\partial\Gamma_i| = |\partial\tilde{\Gamma}_i|$ we may take the corresponding functions $u_i, \tilde{u}_i$ to be arclength parametrizations of $\partial\Gamma_i$ and $\partial\tilde{\Gamma}_i$ respectively. We denote by u_i' and u_i'' the first and second derivatives of u_i with respect to the arclength parameter r. Then the curvature of $\partial\Gamma_i$ is simply given by $\mathcal{K}(r) = \|u_i''(r)\|$ at each point $u_i(r) \in \partial\Gamma_i$, and similarly for $\partial\tilde{\Gamma}_i$.

Thus on $\partial\tilde{\Gamma}_i$, we have by assumption on $\tilde{Q}$ and γ,

$$\tilde{\mathcal{K}}(r) = \|\tilde{u}_i''\| = \|u_i'' + \tilde{u}_i'' - u_i''\| \ge \mathcal{K}(r) - \gamma \ge \mathcal{K}_*/2.$$

Also, $\tau_{\min}(\tilde{Q}) \ge \tau_{\min}(Q_0) - \gamma \ge \tau_*/2$ since $\|u_i - \tilde{u}_i\| \le \gamma$. Thus $\mathcal{F}_A(Q_0, E_*; \gamma) \subset \mathcal{F}_1(\tau_*/2, \mathcal{K}_*/2, E_*)$.

Next we must show that $\tilde{Q} \in \mathcal{F}_A(Q_0, E_*; \gamma)$ represents a small perturbation in the distance $d_{\mathcal{F}}(\cdot,\cdot)$. We do this by first fixing $\Gamma_2, \dots, \Gamma_d$ and considering a deformation of Γ_1 into $\tilde{\Gamma}_1$ such that $|u_1 - \tilde{u}_1|_{\mathcal{C}^2} \le \gamma$.

Let T_0 be the map corresponding to Q_0 and let T_1 be the map corresponding to $\tilde{Q}$. We fix $x = (r, \varphi) \in I_1 \times [-\pi/2, \pi/2]$ and compare $T_0^{-1}x$ with $T_1^{-1}x$. To do this, we let Φ_t^0 and Φ_t^1 denote the flow on the tables Q_0 and $\tilde{Q}$ respectively. We denote by $\pi_0(x)$ the projection of x onto the flow space $\mathbb{T}^2 \times S^1$ corresponding to Q_0 and by π_0^q and π_0^θ the projections onto the position and angular coordinates respectively. Let $\tau_0(x)$ denote the free flight time of x under Φ_t^0 and let $\mathcal{K}_0(\cdot)$ denote the curvature of the scatterers in Q_0. The analogous objects, $\pi_1, \pi_1^q, \pi_1^\theta, \tau_1(\cdot)$ and $\mathcal{K}_1(\cdot)$ are defined for the table $\tilde{Q}$.

First suppose that $T_0^{-1}x$ and $T_1^{-1}x$ lie on the same scatterer Γ_j. Notice that the trajectories $\Phi_{-t}^0(\pi_0 x)$ and $\Phi_{-t}^1(\pi_1 x)$ begin from two points in $\mathbb{T}^2$ at most γ apart and make an angle of at most

γ with one another. We decompose this motion into the sum of (I) two parallel trajectories starting a distance γ apart and (II) two trajectories starting at the same point and making an angle γ.

I. Parallel trajectories. It is an elementary estimate that two parallel lines a distance γ apart will intersect a convex scatterer at a distance at most

$$d_{\mathbb{T}^2}(\pi_0^q(T_0^{-1}x), \pi_1^q(T_1^{-1}x)) \le \sqrt{3\gamma/\mathcal{K}_{\min}(\Gamma_j)} \le \sqrt{3\gamma/\mathcal{K}_*}, \tag{6.1}$$

where $d_{\mathbb{T}^2}$ denotes distance on $\mathbb{T}^2$.

II. Nonparallel trajectories making an angle $\gamma \neq 0$. After time t under the flow, the two trajectories will be at most $t\gamma$ apart in $\mathbb{T}^2$. Let $\tau(x_{-1}) = \max\{\tau_0(T_0^{-1}x), \tau_1(T_1^{-1}x)\}$. Then in the case of a finite horizon Lorentz gas, by the same estimate as in (6.1),

$$d_{\mathbb{T}^2}(\pi_0^q(T_0^{-1}x), \pi_1^q(T_1^{-1}x)) \le \sqrt{3\gamma\tau(x_{-1})/\mathcal{K}_{\min}(\Gamma_j)} \le \sqrt{3\gamma\tau_{\max}/\mathcal{K}_*}. \tag{6.2}$$

In the infinite horizon case, define $\hat{\tau} = \gamma^{-1/3}$. If $\tau(x_{-1}) \le \hat{\tau}$, then (6.2) implies $d_{\mathbb{T}^2}(\pi_0^q(T_0^{-1}x), \pi_1^q(T_1^{-1}x)) \le \sqrt{3/\mathcal{K}_*}\gamma^{1/3}$. On the other hand, suppose $\tau_0(T_0^{-1}x) > \hat{\tau}$. Then x lies in a cell D_n such that $c^{-1}n \le \tau_0(T_0^{-1}y) \le cn$ for some $c > 0$ and all $y \in D_n$, and the width of D_n in the stable direction is at most C'/n (see [CM, Section 4.10]). Thus

$$d_M(x, \mathcal{S}_{-1}^{T_0}) \le C'n^{-1} \le C'c\tau_0^{-1}(T_0^{-1}x) \le C'c\hat{\tau}^{-1} \le C'c\gamma^{1/3}. \tag{6.3}$$

An identical estimate holds if $\tau_1(T_1^{-1}x) > \hat{\tau}$. Thus either $x \in N_{C\gamma^{1/3}}(\mathcal{S}_{-1}^{T_0} \cup \mathcal{S}_{-1}^{T_1})$ or

$$d_{\mathbb{T}^2}(\pi_0^q(T_0^{-1}x), \pi_1^q(T_1^{-1}x)) \le \sqrt{3/\mathcal{K}_*}\gamma^{1/3}. \tag{6.4}$$

Concatenating these two estimates (I) and (II), we see that in terms of position coordinates, $T_0^{-1}x$ and $T_1^{-1}x$ in I_j are of order $\gamma^{1/2}$ in the finite horizon case and of order $\gamma^{1/3}$ in the infinite horizon case. Since the normal direction of Γ_j varies smoothly with the position, we have $d_M(T_0^{-1}x, T_1^{-1}x)$ of the same order. Similar estimates hold when starting from $x \in \Gamma_j$ and comparing images in Γ_1 and $\tilde{\Gamma}_1$.

In the case when $T_0^{-1}x$ and $T_1^{-1}x$ do not lie on the same scatterer Γ_j, we must have $x \in N_{C\gamma^{1/3}}(\mathcal{S}_{-1}^{T_0} \cup \mathcal{S}_{-1}^{T_1})$ by the preceding arguments where $C = 4\mathcal{K}_*^{-3/2}$ is sufficient. We have thus shown **(C1)** holds with $\varepsilon = C\gamma^{1/3}$. Indeed, **(C1)** holds with $\varepsilon = C\gamma^b$ for any $0 < b \le 1/3$ by the same argument.

We can consider the deformation of d scatterers as the concatenation of errors induced by deforming one scatterer at a time. The preceding analysis holds with C increased by a factor of d.

Condition **(C2)** is trivial to check since $J_\mu T_i \equiv 1$ for $i = 0, 1$.

Next we prove **(C4)**. By [CM, eq. (2.26)], $DT_0^{-1}(x) = \frac{-1}{\cos\varphi(T_0^{-1}x)} A_0(x)$, where

$$A_0(x) = \begin{bmatrix} \tau_0(T_0^{-1}x)\mathcal{K}_0(x) + \cos\varphi(x) & -\tau_0(T_0^{-1}x) \\ -\mathcal{K}_0(T_0^{-1}x)(\tau_0(T_0^{-1}x)\mathcal{K}_0(x) + \cos\varphi(x)) - \mathcal{K}_0(x)\cos\varphi(T_0^{-1}x) & \tau_0(T_0^{-1}x)\mathcal{K}_0(T_0^{-1}x) + \cos\varphi(T_0^{-1}x) \end{bmatrix},$$

and $DT_1^{-1}x = \frac{-1}{\cos\varphi(T_1^{-1}x)} A_1(x)$, with a similar definition for $A_1(x)$. Thus

$$\|DT_0^{-1}(x) - DT_1^{-1}x\| \le \left| \frac{1}{\cos\varphi(T_0^{-1}x)} - \frac{1}{\cos\varphi(T_1^{-1}x)} \right| \|A_0(x)\| + \frac{1}{\cos\varphi(T_1^{-1}x)} \|A_0(x) - A_1(x)\| \tag{6.5}$$

Note that $\|A_i(x)\|$ is bounded by a uniform constant times $\tau_i(T_i^{-1}x)$. Now to estimate $\|A_0(x) - A_1(x)\|$, we focus on the lower left entry of the matrix since that contains all the differences to be estimated in the other entries as well. We estimate one difference at a time. Again letting Γ_j denote the scatterer on which $T_0^{-1}x$ and $T_1^{-1}x$ lie, we have

$$\begin{aligned}|\mathcal{K}_0(T_0^{-1}x) - \mathcal{K}_1(T_1^{-1}x)| &\le |\mathcal{K}_0(T_0^{-1}x) - \mathcal{K}_0(T_1^{-1}x)| + |\mathcal{K}_0(T_1^{-1}x) - \mathcal{K}_1(T_1^{-1}x)| \\ &\le E_* d_M(T_0^{-1}x, T_1^{-1}x) + \|u_i''(T_1^{-1}x) - \tilde{u}_i''(T_1^{-1}x)\| \le E_* d_M(T_0^{-1}x, T_1^{-1}x) + \gamma,\end{aligned}$$

by definition of γ. Next,

$$|K_0(x) - K_1(x)| \le \gamma \qquad \text{and} \qquad |\cos\varphi(T_0^{-1}x) - \cos\varphi(T_1^{-1}x)| \le d_M(T_0^{-1}x, T_1^{-1}x)$$

follow immediately. Finally, since $\tau_i(T_i^{-1}x)$ is the length of the line segment connecting $\pi_i^q(x)$ to $\pi_i^q(T_i^{-1}x)$, $i = 1, 2$, we have

$$|\tau_0(T_0^{-1}x) - \tau_1(T_1^{-1}x)| \le d_{\mathbb{T}^2}(\pi_0^q(T_0^{-1}x), \pi_1^q(T_1^{-1}x)) + d_{\mathbb{T}^2}(\pi_0^q(x), \pi_1^q(x)) \le d_M(T_0^{-1}x, T_1^{-1}x) + 2\gamma.$$

Putting these estimates together, we conclude

$$\|A_0(x) - A_1(x)\| \le K\tau(x_{-1})(d_M(T_0^{-1}x, T_1^{-1}x) + \gamma) \tag{6.6}$$

where K is a uniform constant depending on E_* and $\mathcal{K}_*$.

Notice that if $W \in \mathcal{W}^s$, then $|T_i^{-1}W| \le C|W|^{1/3}$ in the infinite horizon case and $|T_i^{-1}W| \le C|W|^{1/2}$ in the finite horizon case. Thus for $\delta < 1/k_0$, if $T_i^{-1}x \in N_\delta(\mathcal{S}_0)$, then $d_M(x, \mathcal{S}_{-1}^{T_i}) \le C_t\delta^2$ where C_t is a uniform constant depending on the transversality of $C^s(x)$ with the horizontal direction and of $\mathcal{S}_{-1}^{T_i}$ with $C^s(x)$.

Now choose $\varepsilon = \gamma^a$, where $a \le 1/3$ will be determined shortly. Suppose $x \notin N_\varepsilon(\mathcal{S}_{-1}^{T_0} \cup \mathcal{S}_{-1}^{T_1})$. Then by the above observation, $\cos\varphi(T_i^{-1}x) \ge C\varepsilon^{1/2}$, $i = 0, 1$, and also by (6.3), $\tau(x_{-1}) \le C\varepsilon^{-1}$. Thus recalling that $d_M(T_0^{-1}x, T_1^{-1}x) \le C\gamma^{1/3}$, we estimate the first term of (6.5),

(6.7)
$$\begin{aligned}\|A_0(x)\|\left|\frac{1}{\cos\varphi(T_0^{-1}x)} - \frac{1}{\cos\varphi(T_1^{-1}x)}\right| &\le \frac{K\tau(T_0^{-1}x)}{\cos\varphi(T_0^{-1}x)\cos\varphi(T_1^{-1}x)}|\cos\varphi(T_1^{-1}x) - \cos\varphi(T_0^{-1}x)| \\ &\le C\varepsilon^{-2} d_M(T_0^{-1}x, T_1^{-1}x) \le C'\gamma^{1/3-2a}.\end{aligned}$$

To estimate the second term of (6.5), we use (6.6) to estimate,

$$\frac{1}{\cos\varphi(T_1^{-1}x)}\|A_0(x) - A_1(x)\| \le C\varepsilon^{-3/2}\gamma^{1/3} = C\gamma^{1/3-3a/2}.$$

Putting these estimates together, we have

$$\|DT_0^{-1}(x) - DT_1^{-1}(x)\| \le C''\gamma^{1/3-2a}.$$

Choosing $a = 2/15$ establishes **(C4)**.

Condition **(C3)** follows similarly using the fact that the stable Jacobian along $W \in \mathcal{W}^s$ is simply the norm of the tangent vector to W times $DT_i(x)$, $i = 0, 1$. The improved estimate in **(C3)** comes from the fact that instead of estimating (6.7) as above, we must estimate instead

$$\tau(x_{-1})\left|\frac{\cos\varphi(T_0^{-1}x)}{\cos\varphi(T_1^{-1}x)} - 1\right| \le C\varepsilon^{-3/2} d_M(T_0^{-1}x, T_1^{-1}x) \le C'\gamma^{1/3-3a/2} = C'\gamma^{2/15} = C'\varepsilon$$

with our choice of $a = 2/15$.

If we restrict perturbations to the finite horizon case with horizons uniformly bounded by some $\tau_{\max} < \infty$, then our estimates above improve by omitting a factor of ε^{-1} and $d(T_0^{-1}x, T_1^{-1}x) \le C\gamma^{1/2}$ by (6.2). In this case, the optimal choice of $a = 1/3$.

7. Proofs of Applications: External Forces with Kicks and Slips

In this section we prove Theorem 2.10 and 2.11 for the perturbed dispersing billiards under external forces with kicks and slips. To simplify the analysis, for any fixed force $\mathbf{F}$, we will consider our system, denoted as $T_{\mathbf{F},\mathbf{G}}$, as a perturbation of the map $T_{\mathbf{F},\mathbf{0}}$. We say a constant C is uniform if $C = C(\varepsilon_1, \tau_*, \mathcal{K}_*, E_*)$, where $\varepsilon_1, \tau_*, \mathcal{K}_*$ and E_* are from **(A2)** and **(A3)**.

We begin by reviewing some properties of $T_{\mathbf{F}} = T_{\mathbf{F},\mathbf{0}}$ proved in [Ch2] and proving some additional ones that we shall need.

7.1. **Properties of $T_{\mathbf{F}}$.** We assume the setup described in Section 2.4.B, which is the billiard flow given by (2.4) and (2.5) with $\mathbf{G} = \mathbf{0}$.

Let $\mathbf{x} = (\mathbf{q}, \theta) \in \mathcal{M}$ be any phase point with position $\mathbf{q}$, and $V \in \mathcal{T}_{\mathbf{x}}\mathcal{M}$ a tangent vector at $\mathbf{x}$. Pick a small number $\delta_0 > 0$ and a C^3 curve $c_s(0) = (\mathbf{q}_s, \theta_s) \subset \mathcal{M}$ tangent to the vector V, such that $c_0 = \mathbf{x}$ and $\frac{dc_s}{ds}|_{s=0} = V$, and $s \in [0, \delta_0]$. Now we define $c_s(t) = \Phi^t c_s(0)$, for any $t \geq 0$. Since τ is the free path function, we have $d\tau = p dt$. In the calculation below, we denote differentiation with respect to s by primes and that with respect to τ by dots. In particular, $\dot{c}_s(t) = (\dot{\mathbf{q}}, \dot{\theta}) = (\mathbf{v}, h)$, where $\mathbf{v} = \mathbf{p}/p = (\cos\theta, \sin\theta)$ and $h = h(\mathbf{q}, \theta)$ is the geometric curvature of the billiard trajectory with initial condition $(\mathbf{q}, \theta)$ on the table.

If we assume t_s to be the time that the trajectory of $c_s(0)$ hits the wall of the billiard table, then $\{c_s(t) \,|\, t \in [0, t_s], s \in [0, \delta_0]\}$ is a C^3 smooth 2-d manifold in $\mathcal{M}$. We introduce two quantities $u = \mathbf{q}' \cdot \mathbf{v}$, and $w = \mathbf{q}' \cdot \mathbf{v}^\perp$, where $\mathbf{v}^\perp = (-\sin\theta, \cos\theta)$. Clearly $\mathbf{q}' = u\mathbf{v} + w\mathbf{v}^\perp$. Now let $\kappa = (\theta' - uh)/w$. We consider two vectors of the surface $U = (\mathbf{v}, h)$ and $R = (\mathbf{v}^\perp, \kappa)$. Clearly $\dot{c}_s = U$ and $c_s' = uU + wR$. Define $p_U = \mathrm{grad}(p) \cdot U$, $p_R = \mathrm{grad}(p) \cdot R$, and $h_U = \mathrm{grad}(h) \cdot U$, $h_R = \mathrm{grad}(h) \cdot R$, respectively. Then it is straight forward to check that

$$p' = \mathrm{grad}(p) \cdot c_s' = p_U u + p_R w \qquad h' = h_U u + h_R w \qquad \text{and} \quad \theta' = \kappa w + hu. \tag{7.1}$$

In addition $\dot{p} = p_U$ and $\dot{h} = h_U$. The derivation of these formulas can be found in [Ch2]. The following lemma was proved in [Ch2, Lemmas 3.1, 3.2].

Lemma 7.1 ([Ch2])**.** *The evolution of the quantities κ and w between collisions is given by the equations*

$$\dot{\kappa} = -\kappa^2 + a + b\kappa \quad \text{and} \quad \dot{w} = \kappa w, \tag{7.2}$$

where $a = a(h)$, $b = b(h)$ are smooth functions whose $\mathcal{C}^0$ norms are bounded by $c_0\varepsilon_1$ for some uniform $c_0 > 0$. Furthermore, at the moment of collision,

$$u^+ = u^-, \;\; w^+ = -w^- \quad \text{and} \quad \kappa^+ = \kappa^- + \frac{2\mathcal{K}(r) + (h^+ + h^-)\sin\varphi}{\cos\varphi} \tag{7.3}$$

In addition the derivative of r and φ satisfies

$$dr/ds = \mp w^\pm / \cos\varphi \quad \text{and} \quad d\varphi/dr = \mp\mathcal{K}(r) + \kappa^\pm \cos\varphi \mp h^\pm \sin\varphi. \tag{7.4}$$

We will calculate the differential of the map $T_{\mathbf{F}}$ (which is not contained in [Ch2]). It follows from (7.2) that

$$\frac{d\dot{w}}{d\tau} = \frac{d}{d\tau}(\kappa w) = \dot{\kappa} w + \kappa \dot{w} = \kappa \dot{w} - \kappa^2 w + (a + b\kappa) w = aw + b\dot{w}. \tag{7.5}$$

This implies that

$$\begin{cases} \dot{w}(\tau) = \dot{w}(0) + \int_0^\tau aw + b\dot{w}\, d\gamma \\ w(\tau) = w(0) + \dot{w}(0)\tau + \int_0^\tau \int_0^\xi aw + b\dot{w}\, d\gamma\, d\xi \end{cases} \tag{7.6}$$

At the moment of collision, (7.3) implies that

$$\begin{cases} w^+ = -w^- \\ \dot{w}^+ = -\dot{w}^- - \frac{2\mathcal{K} + (h^+ + h^-)\sin\varphi}{\cos\varphi} w^- \end{cases} \tag{7.7}$$

In addition (7.4) implies that

$$\frac{d\varphi}{ds} = \frac{\mathcal{K}(r) + h^{\pm}\sin\varphi}{\cos\varphi} w^{\pm} \mp \dot{w}^{\pm} \tag{7.8}$$

Lemma 7.2. *For $x = (r, \varphi)$, let $\tau_1(x)$ denote the distance to the next collision under the flow. There exist constants $\hat{C}_1, \hat{C}_2 > 0$ independent of x, such that $|w(\tau)|$ and $|\dot{w}(\tau)|$ are uniformly bounded from above by $\hat{C}_1|w^+(0)| + \hat{C}_2|\dot{w}^+(0)|$ for $\tau \in [0, \tau_1(x)]$.*

Proof. We fix x and abbreviate $\tau_1(x)$ as τ_1. We begin by adapting [Ch2, Lemma 3.4], to show that if for some $\tau_0 \in [0, \tau_1)$, $\kappa(\tau_0)$ is bounded away from zero, then κ is bounded away from zero and infinity on $[\tau_0, \tau_1]$. More precisely, (7.2) implies that if $\kappa > 0$, then

$$-(\kappa + \varepsilon_2)^2 \le \dot{\kappa} = -\kappa^2 + b\kappa + a = -\Big(\kappa - \frac{b}{2}\Big)^2 + \frac{b^2}{4} + a \le -(\kappa - c_0\varepsilon_1)^2 + \varepsilon_2^2$$

where $\varepsilon_2^2 = 2c_0\varepsilon_1$.

So if we assume that for some $\tau_0 \in [0, \tau_1)$, $\kappa^+(\tau_0) > c_1$ for a fixed $c_1 > 5\sqrt{\varepsilon_0}$, then we may integrate these inequalities to obtain

$$\frac{1}{(\kappa^+(\tau_0) + \varepsilon_2)^{-1} + (\tau - \tau_0)} - \varepsilon_2 \le \kappa(\tau) \le \varepsilon_2 \frac{Ae^{2\varepsilon_2(\tau - \tau_0)} + 1}{Ae^{2\varepsilon_2(\tau - \tau_0)} - 1} + c_0\varepsilon_1,$$

where $A = (\kappa^+(\tau_0) - c_0\varepsilon_1 + \varepsilon_2)/(\kappa^+(\tau_0) - c_0\varepsilon_1 - \varepsilon_2)$. Then since ε_0 is small compared to $\kappa^+(\tau_0)$, this reduces to

$$\frac{1}{(\kappa^+(\tau_0))^{-1} + (\tau - \tau_0)} - \varepsilon_3 \le \kappa(\tau) \le \frac{1}{(\kappa^+(\tau_0))^{-1} + (\tau - \tau_0)} + \varepsilon_3 \tag{7.9}$$

where $\varepsilon_3 = 2\varepsilon_2 + 2c_0\varepsilon_1$.

Now (7.2) implies that for any $0 \le \tau' < \tau \le \tau_1$,

$$w(\tau) = w(\tau') \exp\left(\int_{\tau'}^{\tau} \kappa d\gamma\right). \tag{7.10}$$

Also, (7.5) implies that $\frac{\dot{w}}{w} d\ln\dot{w} = (a + b\kappa)\,d\tau$ and since $\dot{w} = \kappa w$, we integrate this to obtain,

$$\dot{w}(\tau) = \dot{w}(0) \exp\left(\int_0^{\tau} (\frac{a}{\kappa} + b)\,d\gamma\right) \quad \text{for any } \tau \in [0, \tau_1]. \tag{7.11}$$

Integrating again, it follows that

$$w(\tau) = w(0) + \dot{w}(0) \int_0^{\tau} \exp\left(\int_0^{\xi} (\frac{a}{\kappa} + b)\,d\gamma\right) d\xi. \tag{7.12}$$

This implies that both $w(\tau), \dot{w}(\tau)$ are functions of $(w^+(0), \dot{w}^+(0))$.

To show that $|w|$ and $|\dot{w}|$ are uniformly bounded, we consider three cases.

Case I: κ is finite on $[0, \tau_1)$ and $\kappa(\tau) < 1/\tau_{\min}$ for all $\tau \in [0, \tau_1)$ (κ can be positive or negative). Then by (7.10), $|w(\tau)| \le |w(0)|e^{\tau/\tau_{\min}} \le |w(0)|e^{\tau_{\max}/\tau_{\min}}$ for all $\tau \in [0, \tau_1]$.

Once we know $|w|$ is bounded on $[0, \tau_1]$, we may use it to bound $|\dot{w}|$ as follows. We integrate (7.5) using the integrating factor $\exp(-\int_0^{\tau} b\,d\gamma)$ to obtain,

$$\dot{w}(\tau) = \dot{w}(0)e^{\int_0^{\tau} b\,d\gamma} + e^{\int_0^{\tau} b\,d\gamma} \int_0^{\tau} aw(\xi)e^{-\int_0^{\xi} b\,d\gamma}\,d\xi. \tag{7.13}$$

Thus

$$|\dot{w}(\tau)| \le |\dot{w}^+(0)|e^{c_0\varepsilon_1\tau_{\max}} + |w^+(0)|e^{(2c_0\varepsilon_1 + 1/\tau_{\min})\tau_{\max}} c_0\varepsilon_1\tau_{\max} =: C_1|\dot{w}^+(0)| + C_2|w^+(0)|. \tag{7.14}$$

Case II: κ is finite on $[0,\tau_1)$, $\kappa(\tau_0) \geq 1/\tau_{\min}$ for some $\tau_0 \in [0,\tau_1]$ and τ_0 is the least τ in the interval with this property. Then by (7.9), $\kappa(\tau) \geq (\tau_{\min} + 2\tau_{\max})^{-1}$ for all $\tau \in [\tau_0, \tau_1]$. As a consequence, by (7.12),

$$|w(\tau)| \leq |w(\tau_0)| + |\dot{w}(\tau_0)|\tau_{\max} e^{c_0\varepsilon_1(\tau_{\min}+2\tau_{\max}+1)\tau_{\max}} \tag{7.15}$$

for each $\tau \in [\tau_0, \tau_1]$. On the other hand, for $\tau \in [0, \tau_0]$, we have $\kappa(\tau) \leq 1/\tau_{\min}$, so that both $|w(\tau)|$ and $|\dot{w}(\tau)|$ are uniformly bounded on this interval by Case I. This together with (7.15) proves Case II for $|w|$. The estimate for $|\dot{w}|$ follows again from (7.13) and (7.14).

Case III: $\kappa(\tau_0) = \pm\infty$ for some $\tau_0 \in (0, \tau_1)$. According to (7.2) and (7.9), the only way this case can occur is if κ reaches $-\infty$ in finite time and changes from $-\infty$ to ∞ at τ_0. (7.10) implies in particular that $w(\tau_0) = 0$.

On the interval $[0, \tau_0]$, κ clearly satisfies the assumption of Case I so that both $|w|$ and $|\dot{w}|$ are uniformly bounded as in the statement of the lemma on this interval. Indeed, this is true on any interval in which κ remains negative. Thus the only case left to consider is when $\kappa(\tau) > 0$ for $\tau \in (\tau_0, \tau_1]$.

In this case, (7.2) guarantees that κ initially decreases and (7.9) guarantees that $\kappa(\tau) \geq \tau_{\min}^{-1}$ on this interval. Thus by (7.12), we estimate as in (7.15) to bound $|w|$ by a linear combination of $|w(\tau_0)|$ and $|\dot{w}(\tau_0)|$. But since these two quantities are in turn bounded by $|w^+(0)|$ and $|\dot{w}^+(0)|$ by the previous paragraph, the proof of Case III is complete for $|w|$. The estimate on $|\dot{w}|$ now follows again from (7.13) and (7.14). □

Combining the above facts, we can show the following.

Lemma 7.3. *If we denote $x_1 = (r_1, \varphi_1) = T_{\mathbf{F}}x$, then there exits $C = C(\mathcal{K}_*, \tau_*) > 0$ such that for any unit vector $(dr/ds, d\varphi/ds)$,*

$$\begin{cases} -\cos\varphi_1 \frac{dr_1}{ds} = (\cos\varphi + \tau\mathcal{K} + a_1)\,\frac{dr}{ds} + (\tau + a_2)\frac{d\varphi}{ds} \\ -\cos\varphi_1 \frac{d\varphi_1}{ds} = (\tau\mathcal{K}_1\mathcal{K} + \mathcal{K}_1\cos\varphi + \mathcal{K}\cos\varphi_1 + b_1)\,\frac{dr}{ds} + (\mathcal{K}_1\tau + \cos\varphi_1 + b_2)\,\frac{d\varphi}{ds} \end{cases} \tag{7.16}$$

where $a_i \leq C\varepsilon_1$ and $b_i \leq C\varepsilon_1$, for $i = 1, 2$. In addition

$$(1 - C\varepsilon_1)\frac{\cos\varphi}{\cos\varphi_1} \leq |\det D_x T_{\mathbf{F}}| \leq (1 + C\varepsilon_1)\frac{\cos\varphi}{\cos\varphi_1} \tag{7.17}$$

Proof. Let $x_1 = T_{\mathbf{F}}x$, and $\tau_1(x)$ be the length of the free path of x. By (7.11) and (7.12), there exists a linear transformation D_x such that

$$D_x(w^+, \dot{w}^+)^T = (w_1^-, \dot{w}_1^-)^T \tag{7.18}$$

where $w_1^- = w^-(\tau_1)$ and $\dot{w}_1^- = \dot{w}^-(\tau_1)$. Indeed, by Lemma 7.2, there exist smooth functions c_i, $i = 1, \ldots 4$ with $|c_i| \leq C\varepsilon_1$ for some $C = C(\mathcal{K}_*, \tau_*) > 0$ such that

(7.19)

$$I := \int_0^\tau aw + b\dot{w}\, d\gamma = c_1 w^+(0) + c_2\dot{w}^+(0), \quad II := \int_0^\tau \int_0^\xi aw + b\dot{w}\, d\gamma d\xi = c_3 w^+(0) + c_4\dot{w}^+(0),$$

so that using (7.6), we may write D_x as

$$D_x = \begin{pmatrix} 1 + c_3 & \tau + c_4 \\ c_1 & 1 + c_2 \end{pmatrix}. \tag{7.20}$$

Using (7.7) and (7.8), the differential of $DT_{\mathbf{F}}$ satisfies

$$DT_{\mathbf{F}} = N_{x_1}^{-1} L_{x_1} D_x N_x \tag{7.21}$$

where

$$N_x = -\begin{pmatrix} \cos\varphi & 0 \\ \mathcal{K} + h^+\sin\varphi & 1 \end{pmatrix}$$

is the coordinate transformation matrix on $\mathcal{T}_x M$, such that $(w^+(0), \dot{w}^+(0))^T = N_x(dr/ds, d\varphi/ds)^T$, and

$$L_{x_1} = \begin{pmatrix} -1 & 0 \\ -\frac{2\mathcal{K}_1 + (h_1^+ + h_1^-)\sin\varphi_1}{\cos\varphi_1} & -1 \end{pmatrix} \quad \text{and} \quad N_{x_1}^{-1} = \begin{pmatrix} -\frac{1}{\cos\varphi_1} & 0 \\ \frac{\mathcal{K}_1 + h_1^+ \sin\varphi_1}{\cos\varphi_1} & -1 \end{pmatrix}. \tag{7.22}$$

Now combining (7.6) with (7.19) and (7.21), we get

$$\begin{aligned} -\cos\varphi_1 \frac{dr_1}{ds} &= \left(\cos\varphi + \tau\mathcal{K} + \tau h^+ \sin\varphi\right)\frac{dr}{ds} + \tau\frac{d\varphi}{ds} - II \\ &= \left(\cos\varphi + \tau\mathcal{K} + \tau h^+ \sin\varphi\right)\frac{dr}{ds} + \tau\frac{d\varphi}{ds} - c_3 w^+ - c_4 \dot{w}^+ \\ &= (\cos\varphi + \tau\mathcal{K} + a_1)\frac{dr}{ds} + (\tau + a_2)\frac{d\varphi}{ds} \end{aligned}$$

where $a_1 = c_3\cos\varphi + c_4(\mathcal{K} + h^+ \sin\varphi) + \tau h^+ \sin\varphi$ and $a_2 = c_4$. Similarly we obtain

$$\begin{aligned} -\cos\varphi_1 \frac{d\varphi_1}{ds} &= -(\mathcal{K}_1 + h_1^- \sin\varphi_1)w_1^- - \cos\varphi_1 \dot{w}_1^- \\ &= -(\mathcal{K}_1 + h_1^- \sin\varphi_1)(w^+ + \dot{w}^+\tau + II) - \cos\varphi_1(\dot{w}^+ + I) \\ &= \left[(\mathcal{K}_1 + h_1^- \sin\varphi_1)\cos\varphi + (\tau(\mathcal{K}_1 + h_1^- \sin\varphi_1) + \cos\varphi_1)(\mathcal{K} + h^+ \sin\varphi)\right]\frac{dr}{ds} \\ &\quad + \left(\tau\mathcal{K}_1 + \tau h_1^- \sin\varphi_1 + \cos\varphi_1\right)\frac{d\varphi}{ds} - II(\mathcal{K}_1 + h_1^- \sin\varphi_1) - I\cos\varphi_1 \\ &= (\tau\mathcal{K}_1\mathcal{K} + \mathcal{K}_1\cos\varphi + \mathcal{K}\cos\varphi_1 + b_1)\frac{dr}{ds} + (\mathcal{K}_1\tau + \cos\varphi_1 + b_2)\frac{d\varphi}{ds} \end{aligned}$$

where

$$\begin{aligned} b_1 &= (\cos\varphi + \tau\mathcal{K})h_1^- \sin\varphi_1 + \cos\varphi_1\left(c_1\cos\varphi + c_2\mathcal{K} + (1 + c_2)h^+ \sin\varphi\right) \\ &\quad + \left(c_3\cos\varphi + \tau h^+ \sin\varphi + c_4(\mathcal{K} + h^+ \sin\varphi)\right)(\mathcal{K}_1 + h_1^- \sin\varphi_1) \end{aligned}$$

and $b_2 = (\tau + c_4)h_1^- \sin\varphi_1 + c_4\mathcal{K}_1 + c_2\cos\varphi_1$.

Now we use the assumption that the quantities $\mathcal{K}, \tau$ are uniformly bounded from above, and $|h^\pm| = \mathcal{O}(\varepsilon_1)$, to obtain that for any unit vector $(dr/ds, d\varphi/ds)$, the quantities $|a_i| \le C\varepsilon_1$ and $|b_i| \le C\varepsilon_1$, $i = 1, 2$, for some uniform $C > 0$.

Finally we use (7.21) to calculate the determinant of the differential $D_x T_{\mathbf{F}}$,

$$\begin{aligned} \det D_x T_{\mathbf{F}} &= \det N_{x_1}^{-1} \cdot \det L_{x_1} \cdot \det D_x \cdot \det N_x = \frac{\cos\varphi}{\cos\varphi_1}\det D_x \\ &= \frac{\cos\varphi}{\cos\varphi_1}\left((1 + c_2)(1 + c_3) - c_1(\tau + c_4)\right) \end{aligned} \tag{7.23}$$

which implies the last inequality (7.17). □

It follows from the above lemma that the differential $D_x T_{\mathbf{F}} : \mathcal{T}_x M \to \mathcal{T}_{x_1} M$ at any point $x = (r, \varphi) \in M$ is the 2×2 matrix:

$$DT_{\mathbf{F}}(x) = -\frac{1}{\cos\varphi_1}\begin{pmatrix} \tau\mathcal{K} + \cos\varphi + a_1 & \tau + a_2 \\ \mathcal{K}(r_1)(\tau\mathcal{K} + \cos\varphi) + \mathcal{K}\cos\varphi_1 + b_1 & \tau\mathcal{K}(r_1) + \cos\varphi_1 + b_2 \end{pmatrix} \tag{7.24}$$

where $x_1 = T_{\mathbf{F}}(x) = (r_1, \varphi_1)$.

Furthermore it was shown in [Ch01] that the map $T_{\mathbf{F}}$ has two families of cones $\bar{\mathcal{C}}^u(x)$ (unstable) and $\bar{\mathcal{C}}^s(x)$ (stable) in the tangent spaces $\mathcal{T}_x M$, for all $x \in M$. More precisely, the unstable cone $\bar{\mathcal{C}}^u(x)$ contains all tangent vectors based at x whose images generate dispersing wave fronts:

$$\bar{\mathcal{C}}^u(x) = \{(dr, d\varphi) \in \mathcal{T}_x M : B_0^{-1} \le d\varphi/dr \le B_0\}. \tag{7.25}$$

The unstable cone $\bar{\mathcal{C}}^u(x)$ is strictly invariant under $DT_{\mathbf{F}}$. Similarly the stable cone

$$\bar{\mathcal{C}}^s(x) = \{(dr, d\varphi) \in \mathcal{T}_x M : -B_0^{-1} \geq d\varphi/dr \geq -B_0\}$$

is strictly invariant under $DT_{\mathbf{F}}^{-1}$. Here $B_0 = B_0(\varepsilon_1, \tau_*, \mathcal{K}_*) > 1$ is a uniform constant. Indeed, there exists a uniform constant $C > 0$ such that we can choose $B_0 = \mathcal{K}_*^{-1} + 2\tau_*^{-1} + C\varepsilon_1$ for all ε_1 sufficiently small.

Let $dx = (dr, d\varphi) \in \mathcal{T}_x M$. Following [CM, Section 5.10], we define the adapted norm $\|\cdot\|_*$ by

$$\|dx\|_* = \frac{\mathcal{K}(x) + |\mathcal{V}|}{\sqrt{1 + \mathcal{V}^2}} \|dx\|, \quad \forall dx \in C^s(x) \cup C^u(x), \tag{7.26}$$

where $\|dx\| = \sqrt{dr^2 + d\varphi^2}$ is the Euclidean norm. Since the slopes of vectors in $C^s(x)$ and $C^u(x)$ are bounded away from $\pm\infty$, we may extend $\|\cdot\|_*$ to all of $\mathbb{R}^2$ in such a way that $\|\cdot\|_*$ is uniformly equivalent to $\|\cdot\|$. It is straightforward to check that for $dx \in C^u(x)$,

$$\frac{\|dx_1\|_*}{\|dx\|_*} \geq \hat{\Lambda} := 1 + \mathcal{K}_{\min}\tau_{\min}/2. \tag{7.27}$$

Finally, a simple calculation using (7.24) shows that there exists a constant $B_1 = B_1(\mathcal{K}_*, \tau_{\min}, \tau_{\max}) > 0$ such that

$$\frac{B_1^{-1}}{\cos\varphi(x_1)} \leq \frac{\|dx_1\|}{\|dx\|} \leq \frac{B_1}{\cos\varphi(x_1)}, \quad \text{for all } dx \in C^u(x). \tag{7.28}$$

Uniform expansion in $C^s(x)$ under $DT^{-1}(x)$ follows similarly. (See also [Ch2, Sect. 3].)

7.2. **Hyperbolicity of the perturbed map $T_{\mathbf{F},\mathbf{G}}$.** We are now ready to verify conditions **(H1)-(H5)** for the map $T_{\mathbf{F},\mathbf{G}}$. We do this fixing $\mathbf{F}$, $\mathbf{G}$ satisfying assumptions **(A1)-(A4)** with $|\mathbf{F}|_{\mathcal{C}^1}, |\mathbf{G}|_{\mathcal{C}^1} \leq \varepsilon$ for some $\varepsilon \leq \varepsilon_1$. We then compare $T = T_{\mathbf{F},\mathbf{G}}$ with the related map $T_{\mathbf{F}} = T_{\mathbf{F},\mathbf{0}}$.

Since $\mathbf{G}$ preserves tangential collisions, the discontinuity set of T is the same as that of $T_{\mathbf{F}}$, which comprises the preimage of $\mathcal{S}_0 := \{\varphi = \pm\pi/2\}$. Similarly, the singularity sets of T^{-1} and $T_{\mathbf{F}}^{-1}$ are the same due to **(A4)**. But the singular sets for higher iterates are not the same. Let $\mathcal{S}_{\pm n}^T = \cup_{i=0}^n T^{\mp i}\mathcal{S}_{0,H}$ with $n \in \mathbb{N}$. Then $T^{\pm n}$ is smooth on $M \setminus \mathcal{S}_{\pm n}^T$.

For any phase point $x = (r, \varphi) \in M$, let $Tx = (\bar{r}_1, \bar{\varphi}_1)$ and $T_{\mathbf{F}}x = (r_1, \varphi_1)$. According to **(A3)** and **(A4)** and since we are on a fixed integral surface, we may express $\mathbf{G}$ in local coordinates via two smooth functions g^1 and g^2 such that $g^i(r, \pm\pi/2) = 0$, $i = 1, 2$, and

$$\bar{r}_1 = r_1 + g^1(r_1, \varphi_1) \quad \text{and} \quad \bar{\varphi}_1 = \varphi_1 + g^2(r_1, \varphi_1) \tag{7.29}$$

where g^i is a C^2 function with C^1 norm uniformly bounded from above by $c_g\varepsilon$, for some uniform constant $c_g > 0$.

According to (7.29), the differential of T satisfies

$$d\bar{r}_1 = \left(1 + g_1^1(r_1, \varphi_1)\right) dr_1 + g_2^1(r_1, \varphi_1) d\varphi_1 \quad \text{and} \quad d\bar{\varphi}_1 = g_1^2(r_1, \varphi_1) dr_1 + \left(1 + g_2^2(r_1, \varphi_1)\right) d\varphi_1 \tag{7.30}$$

where $g_1^i(r_1, \varphi_1) = \partial g^i/\partial r_1$ and $g_2^i(r_1, \varphi_1) = \partial g^i/\partial \varphi_1$. This implies

$$DT(x) = \begin{pmatrix} 1 + g_1^1(r_1, \varphi_1) & g_2^1(r_1, \varphi_1) \\ g_1^2(r_1, \varphi_1) & 1 + g_2^2(r_1, \varphi_1) \end{pmatrix} DT_{\mathbf{F}}(x) \tag{7.31}$$

Note that T is not a $\mathcal{C}^1$ perturbation of $T_{\mathbf{F}}$ around the boundary of M. Furthermore, T no longer preserves $\mu_{\mathbf{F}}$, the SRB measure for $T_{\mathbf{F}}$. However, it follows from (7.17) and (7.31) that

$$|\det DT(x)| \leq \frac{\cos\varphi(x)}{\cos\bar{\varphi}_1(x)} \frac{\cos\bar{\varphi}_1(x)}{\cos\varphi_1(x)} (1 + C\varepsilon) \leq \frac{\cos\varphi(x)}{\cos\bar{\varphi}_1(x)} (1 + C_1\varepsilon) \tag{7.32}$$

since by (7.29),

$$\frac{\cos\bar\varphi_1(x)}{\cos\varphi_1(x)} = \frac{\cos(\varphi_1(x)+g^2(x_1))}{\cos\varphi_1(x)} \le (1+C'\varepsilon) \tag{7.33}$$

since $g^2(r,\pm\pi/2)=0$ and $|\nabla g^2|\le C\varepsilon$. Clearly this implies condition **(H5)**.

The next proposition shows that although the perturbed maps do not have the same families of stable/unstable manifolds, they do share common families of stable and unstable cones.

Proposition 7.4. *There exist two families of cones $C^u(x)$ (unstable) and $C^s(x)$ (stable) in the tangent spaces $\mathcal{T}_xM$ and $\Lambda>1$, such that for all $x\in M$:*

1. *$DT(C^u(x))\subset C^u(Tx)$ and $DT(C^s(x))\supset C^s(Tx)$ whenever DT exists.*
2. *These families of cones are continuous on M and the angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero.*
3. *$\|D_xT(v)\|_*\ge\Lambda\|v\|_*, \forall v\in C^u(x)$ and $\|D_xT^{-1}(v)\|_*\ge\Lambda\|v\|_*, \forall v\in C^s(x)$.*

Proof. For $x\in M$ and any unit vector $dx\in\mathcal{T}_xM$, let $dx_1=D_xT_{\mathbf{F}}dx$. Then by (7.30) the slope $\bar{\mathcal{V}}_1$ of the vector $d\bar x_1$ at $\bar x_1:=Tx=(\bar r_1,\bar\varphi_1)$ satisfies

$$\bar{\mathcal{V}}_1=\frac{g_1^2+(1+g_2^2)\mathcal{V}_1}{1+g_1^1+g_2^1\mathcal{V}_1}=\mathcal{V}_1+\mathcal{O}(\varepsilon) \tag{7.34}$$

So the cone $\bar C^u(x)$ from (7.25) may not be invariant under $DT(x)$. Accordingly, we define a slightly bigger cone,

$$C^u(x)=\{(dr,d\varphi)\in\mathcal{T}_xM:\ B_0^{-1}(1-c_1\varepsilon_1))\le d\varphi/dr\le B_0(1+c_2\varepsilon_1)$$

for some constants $c_1,c_2>0$, and we use assumption (**A2**) to ensure that $c_i\varepsilon_1<1/2$, $i=1,2$. By (7.24), $DT_{\mathbf{F}}$ maps the first and third quadrants strictly inside themselves and shrinks any cones larger than the unstable cones. More precisely, let V be a unit vector on the upper boundary of $C^u(x)$, with slope $\mathcal{V}=B_0(1+c_2\varepsilon_1)$. Then by (7.24) the slope of $DT_{\mathbf{F}}V$ satisfies $\mathcal{V}_1=\frac{C+D\mathcal{V}}{A+B\mathcal{V}}$, where we denote

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}=\begin{pmatrix}\tau\mathcal{K}+\cos\varphi+a_1 & \tau+a_2\\ \mathcal{K}(r_1)(\tau\mathcal{K}+\cos\varphi)+\mathcal{K}\cos\varphi_1+b_1 & \tau\mathcal{K}(r_1)+\cos\varphi_1+b_2\end{pmatrix}.$$

It follows from the invariance of $\bar{\mathcal{C}}^u$ that $\frac{C+DB_0}{A+BB_0}<B_0$. One can easily check that

$$\mathcal{V}_1=\frac{C+DB_0(1+c_2\varepsilon_1)}{A+BB_0(1+c_2\varepsilon_1)}<\mathcal{V}=B_0(1+c_2\varepsilon_1)$$

Similarly we can check the lower boundary of the cone is also mapped inside the cone C^u. Thus C^u is invariant under DT.

Similarly we define the stable cone $C^s(x)$ as

$$C^s(x)=\{(dr,d\varphi)\in\mathcal{T}_xM:\ -B_0^{-1}(1-c_1\varepsilon_1))\ge d\varphi/dr\ge -B_0(1+c_2\varepsilon_1)\}.$$

Then one can check that the stable cone $\mathcal{C}^s$ is strictly invariant under DT^{-1} whenever DT^{-1} exists for any $T\in\mathcal{F}$. From the definitions of $C^s(x)$ and $C^u(x)$, it is clear that the angle between them is bounded away from 0 on M. Thus items (1) and (2) of the lemma are proved.

To prove (3), note that (7.26) implies,

$$\frac{\|d\bar x_1\|_*}{\|dx\|_*}=\frac{\|d\bar x_1\|_*}{\|dx_1\|_*}\frac{\|dx_1\|_*}{\|dx\|_*}=\frac{\|dx_1\|_*}{\|dx\|_*}\frac{\mathcal{K}(\bar r_1)+|d\bar\varphi_1|}{\mathcal{K}(r_1)+|d\varphi_1|}.$$

Using (7.29), (7.30), (7.27) and the fact that $\mathcal{K}(\cdot)$ is a $\mathcal{C}^1$ function on M, we conclude that for $\epsilon_0=1$ small enough,

$$\frac{\|d\bar x_1\|_*}{\|dx\|_*}\ge\Lambda:=1+\mathcal{K}_{\min}\tau_{\min}/3. \tag{7.35}$$

Similarly, one can show property (3) for stable cones, which we will not repeat here. □

Near grazing collisions, we have also using (7.28) and (7.33) along with (7.29) and (7.30),

$$\frac{B_1^{-1}(1-C\varepsilon_1)}{\cos\bar\varphi_1} \le \frac{\|d\bar x_1\|}{\|dx\|} = \frac{\|d\bar x_1\|}{\|dx_1\|}\frac{\|dx_1\|}{\|dx\|} \le \frac{B_1(1+C\varepsilon_1)}{\cos\bar\varphi_1}, \tag{7.36}$$

which establishes (3.4) in **(H1)** since in the finite horizon case, there are only finitely many singularity curves so we may take n in that formula to be 1.

The last formula (3.5) in **(H1)** (again with $n=1$) follows directly from differentiating (7.31) and using (7.24) to recover this standard estimate for the unperturbed billiard (see [KS] or [Ch2, Sect. 9.9] for the classical result). This finishes the verification of **(H1)**.

7.3. **Regularity of stable and unstable curves.** It follows from Proposition 7.4 that we may define common families of stable and unstable cones for all perturbations $T\in\mathcal{F}_B(Q_0,\tau_*,\varepsilon_1)$. Recall the homogeneity strips $\mathbb{H}_k$ defined in Section 3.1 and that a homogeneous curve in M is a curve that lies in a single homogeneity strip. In this subsection we will show that there is a class of C^2 smooth unstable homogeneous curves $\widehat{\mathcal{W}}^u$ in M which is invariant under any $T\in\mathcal{F}$. Furthermore these curves are regular in the sense that they have uniformly bounded curvature and distortion bounds. Similarly, there is an invariant class of homogeneous stable curves, $\widehat{\mathcal{W}}^s$.

7.3.1. *Curvature bounds.* The next lemma, proved in $T_{\mathbf{F}}$ in [Ch2], states that the images of an unstable curve are essentially flattened under the map $T_{\mathbf{F}}$.

Lemma 7.5. *Let $W\subset M$ be a C^2-smooth unstable curve with equation $\varphi_0=\varphi_0(r_0)$ such that $T_{\mathbf{F}}^iW$ is a homogeneous unstable curve for each $0\le i\le n$. Then $T_{\mathbf{F}}^nW$ has equation $\varphi_n=\varphi_n(r_n)$ which satisfies:*

$$\left|\frac{d^2\varphi_n}{dr_n^2}\right| \le C_1 + \theta^{3n}\left|\frac{d^2\varphi_0}{dr_0^2}\right| \le C_2 \tag{7.37}$$

where $C_i=C_i(Q)$, $i=1,2$ is a constant and $\theta\in(0,1)$. Furthermore, for any regular unstable curve W, there exists $n_W\ge 1$, such that for any $n>n_W$, every smooth curve of T^nW has uniformly bounded curvature.

One can obtain a similar bounded curvature property for the perturbed map T.

Proposition 7.6. *(Curvature bounds) Let W be any C^2 smooth unstable curve. Then there exists $n_W\ge 1$ and $C_b>0$ such that every smooth curve $W'\subset T^nW$ with equation $\bar\varphi_n=\bar\varphi_n(\bar r_n)$ satisfies*

$$|d^2\bar\varphi_n/d\bar r_n^2| \le C_b, \quad \textit{for } n>n_W. \tag{7.38}$$

Proof. We fix any phase point $\bar x_0 := x\in W$, denote $x_n=(r_n,\varphi_n)=T_{\mathbf{F}}^nx$ and $\bar x_n=(\bar r_n,\bar\varphi_n)=T^nx$. According to (7.30), the slope of the vector $DT\,d\bar x$ satisfies

$$\frac{d\bar\varphi_1}{d\bar r_1} = \frac{g_1^2+(1+g_2^2)\mathcal{V}_1}{1+g_1^1+g_2^1\mathcal{V}_1} = \mathcal{V}_1 + \frac{g_1^2+g_2^2\mathcal{V}_1-g_1^1\mathcal{V}_1-g_2^1\mathcal{V}_1}{1+g_1^1+g_2^1\mathcal{V}_1}, \tag{7.39}$$

where $\mathcal{V}_1=d\varphi_1/dr_1$, $\bar{\mathcal{V}}_1=d\bar\varphi_1/d\bar r_1$. We differentiate the above equality with respect to r_1, using the fact that by (7.30), $\dfrac{d\bar r_1}{dr_1}=1+g_1^1+g_2^1\mathcal{V}_1$. Now use the same notation as in Lemma 7.5 to get for some $C_0>0$ and $C_3>0$

$$\left|\frac{d^2\bar\varphi_1}{d\bar r_1^2}\right| \le C_0 + (1+C_3\varepsilon_1)\theta^3\left|\frac{d^2\bar\varphi_0}{d\bar r_0^2}\right|, \tag{7.40}$$

since $d^2\bar{\varphi}_0/d\bar{r}_0^2 = d^2\varphi_0/dr_0^2$. By choosing ε_1 small one can make $(1+\varepsilon_1 C_3)\theta^2 < 1$. Then we have for any $n \geq 1$,

$$\left|\frac{d^2\bar{\varphi}_n}{d\bar{r}_n^2}\right| \leq \frac{C_0}{1-\theta} + \theta^n \left|\frac{d^2\bar{\varphi}_0}{d\bar{r}_0^2}\right|$$

Since W is $\mathcal{C}^2$, there exists $C_1 = C_1(W) > 0$ such that $|\frac{d^2\bar{\varphi}_0}{d\bar{r}_0^2}| < C_1$. We fix a constant $C_b = C_b(Q) > 0$ and define

$$n_W = \left|\frac{\ln(C_b/C_1)}{\ln\theta}\right|.$$

Then for any $n > n_W$, connected components of T^nW have equation $\bar{\varphi}_n = \bar{\varphi}_n(\bar{r}_n)$ with second derivative bounded from above by C_b. □

We now fix the constant $C_b > 0$, then define $\widehat{\mathcal{W}}^u$ be the class of all homogeneous unstable curves W whose curvature is uniformly bounded by C_b. It follows from Propositions 7.4 and 7.5 that the class $\widehat{\mathcal{W}}^u$ is invariant under any $T \in \mathcal{F}$. Any unstable curve $W \in \widehat{\mathcal{W}}^u$ is called a regular unstable curve. Similarly one defines $\widehat{\mathcal{W}}^s$. This verifies condition **(H2)**.

7.3.2. *Distortion bounds.* In this section, we establish the distortion bounds for T required by **(H4)**. For any stable curve $W \in \widehat{\mathcal{W}}^s$ and $x \in W$, denote by $J_W T_{\mathbf{F}}(x)$ (resp. $J_W T(x)$) the Jacobian of $T_{\mathbf{F}}$ (resp. T) along W at $x \in W$. It was shown in [Ch2] that there exists $C_1 > 0$, such that for any regular stable curve W for which $T_{\mathbf{F}}W$ is also a regular stable curve,

$$|\ln J_W T_{\mathbf{F}}(x) - \ln J_W T_{\mathbf{F}}(y)| \leq C_1 d_W(x,y)^{\frac{1}{3}} \tag{7.41}$$

where $d_W(x,y)$ is the arclength between x and y along W. We show that T has the same properties on the set of all regular stable curves $\widehat{\mathcal{W}}^s$.

Lemma 7.7. *(Distortion bounds) Let $T \in \mathcal{F}$ and $W \in \widehat{\mathcal{W}}^s$ be such that T is smooth on W and $TW \in \widehat{\mathcal{W}}^s$. There exists $C_J > 0$ independent of W and T such that*

$$|\ln J_W T(x) - \ln J_W T(y)| \leq C_J d_W(x,y)^{\frac{1}{3}}.$$

Proof. Fix $T \in \mathcal{F}$ and $W \in \mathcal{W}^s$ for which $TW \in \widehat{\mathcal{W}}^s$. This implies in particular that both T and $T_{\mathbf{F}}$ are smooth on W. For any $x = (r,\varphi) \in W$, let $x_1 := T_{\mathbf{F}}x = (r_1,\varphi_1)$ and $\bar{x}_1 = Tx = (\bar{r}_1,\bar{\varphi}_1)$. Similarly, let $dx = (dr, d\varphi) \in \mathcal{T}_x W$ be a unit vector and define $dx_1 = DT_{\mathbf{F}}(x)dx = (dr_1, d\varphi_1)$ and $d\bar{x}_1 = DT(x)dx = (d\bar{r}_1, d\bar{\varphi}_1)$. Then

$$\frac{J_W T(x)}{J_W T_{\mathbf{F}}(x)} = \sqrt{\frac{1+\bar{\mathcal{V}}_1^2}{1+\mathcal{V}_1^2}}\frac{|d\bar{r}_1|}{|dr_1|}$$

where $\mathcal{V}_1 = d\varphi_1/dr_1$ and $\bar{\mathcal{V}}_1 = d\bar{\varphi}_1/d\bar{r}_1$. Then it follows from (7.30) that

$$\ln J_W T(x) = \ln J_W T_{\mathbf{F}}(x) + \frac{1}{2}\ln(1+\bar{\mathcal{V}}_1^2) - \frac{1}{2}\ln(1+\mathcal{V}_1^2) + \ln|1+g_1^1+g_2^1\mathcal{V}_1|. \tag{7.42}$$

By the smoothness of W and the curvature bounds, there exists $C > 0$ such that for any $x, y \in W$,

$$|\ln(1+\mathcal{V}_1^2(x_1)) - \ln(1+\mathcal{V}_1^2(y_1))| \leq |\mathcal{V}_1^2(x_1) - \mathcal{V}_1^2(y_1)| \leq C d_{T_{\mathbf{F}}W}(x_1,y_1) \leq C' d_W(x,y),$$

where $y_1 = T_{\mathbf{F}}y$, and similarly for $\bar{\mathcal{V}}_1$. Since $\mathbf{G}$ is C^2, the terms involving g_1^1 and g_2^1 satisfy a Lipschitz bound as well. Putting this together with (7.41) and (7.42) proves the lemma. □

In general, for $W \in \widehat{\mathcal{W}}^s$ and $n \in \mathbb{N}$, suppose T^n is smooth on W and that $T^k W \in \widehat{\mathcal{W}}^s$, $0 \le k \le n$. Define $T^k W = W_k$ and for $x, y \in W$, let $x_k = T^k x$ and $y_k = T^k y$. Then

$$\begin{aligned} |\ln J_W T^n(x) - \ln J_W T^n(y)| &\le \sum_{k=0}^{n-1} |\ln J_{W_k} T(x_k) - \ln J_{W_k} T(y_k)| \\ &\le C \sum_{k=0}^{n-1} d_{W_k}(x_k, y_k)^{1/3} \le C d_W(x,y)^{1/3} \sum_{k=0}^{\infty} \Lambda^{-k/3}, \end{aligned} \tag{7.43}$$

due to (7.35). This completes the required estimate on $J_W T$.

Finally, we prove the required bounded distortion estimate for $J_\mu T$. By (7.23) and (7.31), we have

$$\det DT(x) = \frac{\cos\varphi}{\cos\varphi_1}\big((1+c_2)(1+c_3) - c_1(\tau + c_4)\big)\big((1+g_1^1)(1+g_2^2) - g_2^1 g_1^2\big) =: \frac{A(x)}{\cos\bar{\varphi}_1}, \tag{7.44}$$

where $c_1, \dots, c_4$ are defined by (7.19) and we have replaced $\cos\varphi_1$ with $\cos\bar{\varphi}_1$ times a smooth function on $M \setminus \mathcal{S}_1^T$ due to (7.33). Note that $A(x)$ is a smooth function of its argument wherever T is smooth and has bounded C^1 norm on $M \setminus \mathcal{S}_1^T$. It follows that $J_\mu T$ is a smooth function on $M \setminus \mathcal{S}_1^T$ whose C^1-norm is bounded between $1 \pm C\varepsilon_1$ for some uniform constant C depending on the table (recall that $d\mu = c \cos\varphi \, dm$ is the smooth invariant measure for the unperturbed billiard $T_{\mathbf{0},\mathbf{0}}$). The required distortion estimates (3.8) and (3.9) for $J_\mu T$ follow using this smoothness and the uniform hyperbolicity of T as in (7.43). Indeed, (7.43) holds with exponent 1 rather than 1/3 for $J_\mu T$. This completes the verification of **(H4)**.

Distortion bounds for $\det DT$ with exponent 1/3 follow from the above considerations in addition to recalling that $1/\cos\varphi$ is of order k^2 in $\mathbb{H}_k$, while the width of such a strip along a stable or unstable curve is k^{-3}. Similarly, one may prove absolute continuity of the holonomy map between unstable leaves as in [Ch2], but we do not do that here since we do not need this fact.

7.4. **One step expansion.** Since we have established the expansion factors given by (7.35) and (7.36), the one-step expansion condition (3.6) follows from an argument similar to the unperturbed case (see [CM, Lemma 5.56]) and fixes the choice of $k_0 \in \mathbb{N}$, the minimum index of the homogeneity strips. We will not reprove that lemma here. Instead, we focus on the second part of **(H3)**, given by (3.7).

Fix $\delta_0 > 0$ and k_0 satisfying (3.10) and define $\mathcal{W}^s$ accordingly. For $W \in \mathcal{W}^s$, let V_i denote the maximal homogeneous connected components of $T^{-1}W$.

Lemma 7.8. *For any $\varsigma > 1/2$, there exists $C = C(\delta_0, \varsigma, \varepsilon_0) > 0$ such that for any $W \in \mathcal{W}^s$, any $T \in \mathcal{F}_B(Q_0, \tau_*, \varepsilon_1)$,*

$$\sum_i \frac{|TV_i|^\varsigma}{|V_i|^\varsigma} < C. \tag{7.45}$$

Proof. According to the structure of singular curves, a stable curve of length $\le \delta_0$ can be cut by at most $N \le \tau_{\max}/\tau_{\min}$ singularity curves in $\mathcal{S}_{-1}^T$ (see [CM, §5.10]). For each $s \in \mathcal{S}_{-1}^T$ intersecting W, W is cut further by images of the boundaries of homogeneity strips S_k^H, $k \ge k_0$. For one such s, we relabel the components V_i of $T^{-1}W$ on which T is smooth by V_k, k corresponding to the homogeneity strip $\mathbb{H}_k$ containing V_k. By (7.36), there exists $c_1 = c_1(\varepsilon_1) > 0$ such that on TV_k, the expansion under T^{-1} is $\ge c_1 k^2$. So for all $\varsigma > 1/2$,

$$\sum_{k \ge k_0} \frac{|TV_k|^\varsigma}{|V_k|^\varsigma} \le c_1 \sum_{k \ge k_0} \frac{1}{k^{2\varsigma}} \le \frac{c_1}{k_0^{2\varsigma - 1}}. \tag{7.46}$$

An upper bound for (7.45) in this case is given by N times the bound in (7.46). $\square$

This completes the verification of **(H1)-(H5)** and completes the proof of Theorem 2.10.

7.5. **Smallness of the perturbation.** In this section, we check that conditions **(C1)-(C4)** are satisfied for ε_1 sufficiently small. We will then be able to apply Theorem 2.11 to any map $T \in \mathcal{F}_B(Q_0, \tau_*, \varepsilon_1)$.

We fix $\varepsilon \in (0, \varepsilon_1)$ and choose any $T := T_{\mathbf{F},\mathbf{G}} \in \mathcal{F}_B(Q_0, \tau_*, \varepsilon_1)$, such that $|\mathbf{F}|_{\mathcal{C}^1}, |\mathbf{G}|_{\mathcal{C}^1} \leq \varepsilon$. By the triangle inequality, it suffices to estimate $d_{\mathcal{F}}(T_0, T)$ where $T_0 = T_{\mathbf{0},\mathbf{0}}$ is the unperturbed billiard map.

Denote by Φ^t the flow corresponding to T and by Φ_0^t the flow corresponding to T_0. Let $x \in M \setminus (S_{-1}^T \cup S_{-1}^{T_0})$. By the facts summarized in Section 7.1, $\Phi^t(x)$ and $\Phi_0^t(x)$ can be no further than a uniform constant times εt on the billiard table. Thus since T has finite horizon bounded by $\tau_{\max}$ and the scatterers have uniformly bounded curvature, $T(x)$ and $T_{\mathbf{F},\mathbf{0}}(x)$ can be no more than a constant times $\sqrt{\varepsilon}$ apart if they lie on the same scatterer. By the smallness of $\mathbf{G}$ and (7.29), we have $d_M(T_{\mathbf{F},\mathbf{0}}(x), T_{\mathbf{F},\mathbf{G}}) < C\varepsilon$ and thus by the triangle inequality, $d_M(T(x), T_0(x)) < C_f\sqrt{\varepsilon}$ for some uniform $C_f > 0$ as long as they lie on the same scatterer. A similar bound holds for $T^{-1}x$ and $T_0^{-1}x$.

Let $\epsilon = C_f \varepsilon^{1/3}$. It then follows that for any $x \notin N_\epsilon(S_{-1}^T \cup S_{-1}^{T_0})$, $d(T^{-1}(x), T_0^{-1}(x)) < \epsilon$. This is **(C1)**.

To establish **(C2)**, we use the fact that $J_\mu T_0 \equiv 1$ while

$$J_\mu T(x) = \big((1 + c_2)(1 + c_3) - c_1(\tau + c_4)\big)\big((1 + g_1^1)(1 + g_2^2) - g_2^1 g_1^2\big)$$

by (7.44). Since the functions here are all bounded by uniform constants times ε and our horizon is bounded by $\tau_{\max}$, **(C2)** is satisfied.

Next, we prove **(C4)**. Inverting (7.31) and (7.24) and using (7.44), we have

$$DT^{-1}(x) = \frac{-1}{A(T^{-1}x)\cos\varphi(T^{-1}x)} \begin{pmatrix} B + b_2 & C - a_2 \\ D - b_1 & E + a_1 \end{pmatrix} \begin{pmatrix} 1 + g_2^2 & -g_2^1 \\ -g_1^2 & 1 + g_1^1 \end{pmatrix},$$

where A is the smooth function from (7.44) and $B = \tau(T^{-1}x)K(x) + \cos\varphi(x)$, $C = -\tau(T^{-1}x)$,

$$D = -\mathcal{K}(T^{-1}x)(\tau(T^{-1}x)\mathcal{K}(x) + \cos\varphi(x)) - \mathcal{K}(x)\cos\varphi(T^{-1}x), \quad \text{and} \quad E = \tau(T^{-1}x)\mathcal{K}(T^{-1}x) + \cos\varphi(T^{-1}x)$$

match the corresponding entries of $DT_0^{-1}x$ with T replaced by T_0.

We split the matrix product as

$$\left(\begin{pmatrix} B & C \\ D & E \end{pmatrix} + \begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix}\right)\left(I + \begin{pmatrix} g_2^2 & -g_2^1 \\ -g_1^2 & g_1^1 \end{pmatrix}\right) =: F + R,$$

where $F = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and R is a matrix whose entries are smooth functions, all bounded by a uniform constant times ε. Now defining F_0 to be the matrix F with T_0 replacing T, we write,

(7.47)

$$\|DT^{-1}(x) - DT_0^{-1}(x)\| = \left\|\frac{F + R}{A(T^{-1}x)\cos\varphi(T^{-1}x)} - \frac{1}{\cos\varphi(T_0^{-1}x)}F_0\right\|$$

$$\leq \frac{\|F - F_0\|}{|A(T^{-1}x)\cos\varphi(T^{-1}x)|} + \|F_0\|\left|\frac{1}{A(T^{-1}x)\cos\varphi(T^{-1}x)} - \frac{1}{\cos\varphi(T_0^{-1}x)}\right| + \frac{\|R\|}{|A(T^{-1}x)\cos\varphi(T^{-1}x)|}.$$

Notice that if $x \notin N_\epsilon(\mathcal{S}_{-1}^T \cup \mathcal{S}_{-1}^{T_0})$, then due to the uniform expansion given by (7.36) and the uniform transversality of the stable cone with $\mathcal{S}_0$, we have $d_M(T^{-1}x, \mathcal{S}_0) \geq C\sqrt{\epsilon}$, for some uniform constant C. Thus $\cos\varphi(T^{-1}x) \geq C'\sqrt{\epsilon}$ for some uniform constant $C' > 0$. The same fact is true for $T_0^{-1}x$.

Using this, plus the fact that the entries of F and F_0 are smooth functions of their arguments with uniformly bounded $\mathcal{C}^1$ norms, we estimate the first term of (7.47) by

$$\frac{\|F - F_0\|}{|A(T^{-1}x)\cos\varphi(T^{-1}x)|} \le C\epsilon^{-1/2} d_M(T^{-1}x, T_0^{-1}x) \le C'\epsilon^{-1/2}\varepsilon^{1/2} = C'C_f\epsilon$$

since the C^1 norm of A is bounded above and below by $1 \pm C\varepsilon$ by (7.44). Similarly, the third term of (7.47) is bounded by $C\epsilon$.

Since $\|F_0\|$ is uniformly bounded, we split the middle term of (7.47) into the sum of two terms,

$$\left|\frac{1}{A(T^{-1}x)\cos\varphi(T^{-1}x)} - \frac{1}{\cos\varphi(T_0^{-1}x)}\right| \le \frac{1}{\cos\varphi(T^{-1}x)}\left|\frac{1}{A(T^{-1}x)} - 1\right| + \left|\frac{1}{\cos\varphi(T^{-1}x)} - \frac{1}{\cos\varphi(T_0^{-1}x)}\right|.$$

As noted earlier, the $\mathcal{C}^1$ norm of A is bounded above and below by $1 \pm C\varepsilon$ so that the first difference above is bounded by $C\epsilon^{-1/2}\varepsilon \le CC_f\epsilon$. The second difference is bounded by $C\epsilon^{-1} d_M(T^{-1}x, T_0^{-1}x) \le C'\epsilon^{-1}\varepsilon^{1/2} = C'C_f\epsilon^{1/2}$, similar to the estimate (6.7).

Putting these estimates together in (7.47) proves **(C4)** with respect to ϵ. Condition **(C3)** follows similarly using the fact that $J_W T(x) = \|DT(x)v\|$ where $v \in \mathcal{T}_x W$ is a unit vector. The exponent of ϵ in **(C3)** is better than in **(C4)** by a factor of $\epsilon^{1/2}$ since we must estimate $\left|\frac{\cos\varphi(T^{-1}x)}{\cos\varphi(T_0^{-1}x)} - 1\right|$ in place of $\left|\frac{1}{\cos\varphi(T^{-1}x)} - \frac{1}{\cos\varphi(T_0^{-1}x)}\right|$.

References

[B] V. Baladi, *Positive transfer operators and decay of correlations*, Advanced Series in Nonlinear Dynamics, **16**, World Scientific (2000).

[BM] P. Bálint and I. Melbourne, *Decay of correlations for flows with unbounded roof function, including the infinite horizon planar periodic Lorentz gas*, Preprint.

[Ch1] Chernov, N., *Decay of correlations in dispersing billiards*, J. Statist. Phys. **94** (1999), 513–556.

[Ch2] N. Chernov, *Sinai billiards under small external forces*, Ann. Henri Poincare, 197–236 **2** (2001).

[Ch4] N. Chernov, *Sinai billiards under small external forces II* Annales Henri Poincare, **9** (2008), 91-107.

[CD1] N. Chernov and D. Dolgopyat, *Brownian Brownian Motion – I*, Memoirs of American Mathematical Society, **198**: 927 (2009).

[CD2] N. Chernov and D. Dolgopyat, *Anomalous current in periodic Lorentz gases with infinite horizon*, Russian Mathematical Surveys, **64** (2009), 651-699.

[CELS1] N. Chernov, G. Eyink, J. Lebowitz and Ya. Sinai, *Derivation of Ohms law in a deterministic mechanical model,* Phys. Rev. Lett. **70** (1993), 22092212.

[CELS2] N. Chernov, G. Eyink, J. Lebowitz and Ya. Sinai, *Steady-state electrical conduction in the periodic Lorentz gas*, Comm. Math. Phys. **154** (1993), 569–601.

[CH] Chernov N. and Haskell C. *Nonuniformly hyperbolic K-systems are Bernoulli*, Ergod. Th. Dynam. Sys. **16** (1996), 19–44.

[CM] N. Chernov and R. Markarian, *Chaotic Billiards*, Mathematical Surveys and Monographs, **127**, AMS, Providence, RI, 2006.

[DL] M.F. Demers and C. Liverani, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps*, Trans. Amer. Math. Soc. **360**:9 (2008), 4777-4814.

[DZ] M.F. Demers and H.-K. Zhang, *Spectral analysis of the transfer operator for the Lorentz gas*, Journal of Modern Dynamics **5**:4 (2011), 665-709.

[DSV] D. Dolgopyat, D. Szász and T. Varjú, *Limit Theorems for Locally Perturbed Lorentz processes*, Duke Math. J. **148** (2009) 459-499.

[GO] G. Gallavotti G. and D. Ornstein, *Billiards and Bernoulli schemes*, Commun. Math. Phys. **38** (1974), 83–101.

[GL] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Th. Dynam. Sys. **26**:1 (2006), 189-217.

[G] S. Gouëzel, *Almost sure invariance principle for dynamical systems by spectral methods*, Ann. Prob. **38**:4 (2010), 1639-1671.

[KS] A. Katok and J.-M. Strelcyn, with the collaboration of F. Ledrappier and F. Przytycki, *Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities*, Lecture Notes in Math. **1222**, Springer-Verlag: Berlin, 283 pages, 1986.

[KL] G. Keller, C. Liverani, *Stability of the spectrum for transfer operators*, Annali della Scuola Normale Superiore di Pisa, Scienze Fisiche e Matematiche, (4) **XXVIII** (1999), 141-152.

[L] C. Liverani, *Invariant measures and their properties. A functional analytic point of view*, Dynamical Systems. Part II: Topological Geometrical and Ergodic Properties of Dynamics. Pubblicazioni della Classe di Scienze, Scuola Normale Superiore, Pisa. Centro di Ricerca Matematica "Ennio De Giorgi": Proceedings. Published by the Scuola Normale Superiore in Pisa (2004).

[MPS] R. Markarian, E.J. Pujals, M. Sambarino, *Pinball billiards with dominated splitting*, Ergodic Th. Dynam. Sys. **30** (2010), 1757-1786.

[MN1] I. Melbourne and M. Nicol, *Almost sure invariance principle for nonuniformly hyperbolic systems*, Commun. Math. Phys. **260** (2005), 393-401.

[MN2] I. Melbourne and M. Nicol, *Large deviations for nonuniformly hyperbolic systems*, Trans. Amer. Math. Soc. **360** (2008), 6661-6676.

[RY] L. Rey-Bellet and L.-S. Young, *Large deviations in nonuniformly hyperbolic dynamical systems*, Ergod. Th. and Dynam. Sys. **28** (2008), 587-612.

[S] Ya. Sinai, *Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards*, Russ. Math. Surv. **25** (1970), 137–189.

[SC] Sinai Ya. G. and Chernov N., *Ergodic properties of some systems of two-dimensional discs and three-dimensional spheres*, Russian Math. Surveys **42** (1987), 181–207.

[Y] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. Math. **147** (1998), 585–650.

[Z] H.-K. Zhang, *Current in periodic Lorentz gases with twists*, Comm. Math. Phys. **306**:3 (2011), 747-776.

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