Dispersing Billiards with Small Holes

Mark F. Demers

Abstract We study several classes of dispersing billiards with holes, including both finite and infinite horizon Lorentz gases and tables with corner points. We allow holes in the form of arcs in the boundary and open sets in the interior of the table as well as generalized holes in which escape may depend on the angle of collision as well as the position. For a large class of initial distributions (including Lebesgue measure and the smooth invariant (SRB) measure for the billiard before the introduction of the hole), we prove the existence of a common escape rate and a limiting conditionally invariant distribution. The limiting distribution converges to the SRB measure for the billiard as the hole tends to zero. Finally, we are able to characterize the common escape rate via pressure on the survivor set.

1 Introduction

The study of deterministic dynamical systems with holes was introduced by Pianigiani and Yorke [PY] who posed the following intuitive problem. Consider a point particle on a billiard table with chaotic dynamics. If a small hole is made in the table, what are the statistical properties of the trajectories of this system? At what rate does mass escape from the system with respect to a given reference measure? Given an initial distribution μ_0 and letting μ_n denote the normalized distribution at time n (assuming the particle has not escaped by time n), does μ_n converge to a limiting distribution μ? Such a limiting measure may constitute a *conditionally invariant measure* for the open system.

These initial questions in turn motivate many others. For example, does the conditionally invariant measure converge to an invariant measure for the closed system as the size of the hole tends to zero? Such a result views the open system as a pertur-

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bation of the closed system and examines the stability of invariant measures under this type of perturbation. Alternatively, one can consider the *survivor set*, the set of points which never escapes through the hole, and ask if there is a notion of pressure on the survivor set which characterizes the escape rate with respect to a certain class of initial distributions.

Such questions have been studied and affirmative or partial answers obtained for a wide variety of dynamical systems. The first results are for uniformly hyperbolic systems admitting a finite Markov partition: Expanding maps on $\mathbb{R}^n$ [PY, CMS1, CMS2], Smale horseshoes [C1, C2], billiards with convex scatterers satisfying a non-eclipsing condition [LM, R], and Anosov diffeomorphisms [CM1, CM2, CMT1, CMT2]. Subsequent studies include piecewise expanding maps of the interval [BK, CV, LiM, D1] and certain unimodal maps [HY, D2, BDM]. Recently, such questions were answered in the context of the finite horizon periodic Lorentz gas [DWY1, DWY2].

The purpose of this paper is to answer the questions posed above in the context of two-dimensional dispersing billiards. Our results apply to both finite and infinite horizon periodic Lorentz gases as well as billiard tables with corner points. In order to answer these questions, we will study the transfer operator associated with the billiard systems in question and use the recent Banach spaces constructed in [DZ1, DZ2] on which these operators are known to admit a spectral gap. We show that the spectral gap persists for maps with small holes satisfying mild conditions. This unified approach allows us to study a wide variety of holes, including escape through partially absorbing boundaries, i.e. boundaries which may allow particles to escape from any position, but depending on the angle of collision. In addition to the new results in the case of the infinite horizon Lorentz gas and billiards with corner points, the current technique also strengthens results for the finite horizon case obtained in [DWY1]. For example, left open in [DWY1] is whether the escape rates with respect to Lebesgue measure and the smooth invariant measure (the SRB measure) for the closed system are equal. Additionally, does the push-forward of Lebesgue measure (renormalized to condition on non-escape) in the open system converge to a conditionally invariant measure? We answer both questions in the affirmative here.

In order to study the notion of pressure on the survivor set, we invoke another useful tool in the study of dynamical systems known as the Young tower, which is a type of Markov extension for the system. Young towers were introduced in [Y] and constructed for piecewise hyperbolic systems (including billiards with corner points) under general conditions in [Ch1]. Young towers have also been used to study open systems [D1, BDM, DWY1, DWY2], although in such cases, the towers must be constructed after the introduction of the hole due to the extra cutting created by the boundary of the hole. Young towers were used to prove a variational principle for the finite horizon Lorentz gas in [DWY2]. We build on this work here to extend such results to more general dispersing billiards with holes and to complete the answers to the questions posed above.

1.1 Preliminaries

In this section, we recall some basic definitions that we shall use throughout this paper. Given a map $T : M \circlearrowleft$ and a hole $H \subset M$, we define $\mathring{M} = M \setminus H$ and $\mathring{M}^n = \cap_{i=0}^n T^{-i}(M \setminus H)$ to be the set of points in M that have not escaped by time n, $n \geq 1$.

We let $\mathring{T}^n = T^n|\mathring{M}^n$, for $n \geq 1$, denote the map with holes which loses track of points once they enter H. In this sense, we will refer to $\mathring{T}^n$ as the iterates of $\mathring{T}$ despite the fact that the domain of $\mathring{T}$ is not invariant.

Rates of escape. Let μ be a measure on M (not necessarily invariant with respect to T). We define the *exponential rate of escape* with respect to μ to be $-\rho(\mu)$ where

$$\rho(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \mu(\mathring{M}^n), \tag{1}$$

when the limit exists.

Conditionally invariant measures. Given a Borel probability measure μ on M, define $\mathring{T}_*\mu(A) = \mu(T^{-1}A \cap \mathring{M}^1)$ for any Borel $A \subset M$. We say μ is conditionally invariant for $\mathring{T}$ if

$$\frac{\mathring{T}_*\mu(A)}{\mathring{T}_*\mu(M)} = \mu(A) \quad \text{for all Borel } A \subset M. \tag{2}$$

The normalizing constant $\lambda = \mu(\mathring{M}^1)$ is often referred to as the eigenvalue of μ since iterating the above equation yields $\mathring{T}_*^n\mu(A) = \lambda^n \mu(A)$ for $n \in \mathbb{N}$. In particular, $\mu(\mathring{M}^n) = \mathring{T}_*^n \mu(M) = \lambda^n$ so that $-\log\lambda$ is the escape rate with respect to μ according to (1).

We remark that infinitely many conditionally invariant measures have been shown to exist under quite general conditions for any $0 \leq \lambda < 1$ [DY] so that we are not interested in existence results for such measures. Rather, we will be interested in conditionally invariant measures with physical properties: Measures that can be realized as the (renormalized) limit of Lebesgue measure or other physically relevant initial distributions. Such conditionally invariant measures will also describe a common rate of escape with respect to a large class of reference measures.

Pressure on the survivor set. The survivor set[1] $\mathring{M}^\infty := \cap_{i=-\infty}^{\infty} T^i(M \setminus H)$ is a $\mathring{T}$-invariant (and also T-invariant) set which supports all the invariant measures that persist after the introduction of the hole. We define the *pressure* on $\mathring{M}^\infty$ with respect to a class of invariant measures $\mathscr{C}$ to be

$$\mathscr{P}_{\mathscr{C}} = \sup_{\nu\in\mathscr{C}} P_\nu \quad \text{where} \quad P_\nu = h_\nu(T) - \int \chi^+(T)\, d\nu.$$

Here $h_\nu(T)$ denotes the Kolmogorov-Sinai entropy of T with respect to ν and $\chi^+(T)$ represents the sum of positive Lyapunov exponents, counted with multiplicity.

[1] We give the definition for invertible T. When T is not invertible, define $\mathring{M}^\infty = \cap_{i=0}^{\infty} T^{-i}(M \setminus H)$.

We say the open system satisfies a *variational principal* if $\rho(\mu) = \mathscr{P}_{\mathscr{C}}$ for some physically relevant reference measure μ and a class of invariant measures $\mathscr{C}$. If there is an invariant measure $\nu \in \mathscr{C}$ such that $\rho(\mu) = P_\nu$, we say that ν satisfies an *escape rate formula*.

Escape rate formulas have been proved for many of the uniformly hyperbolic systems described in the introduction; for such systems, variational principles are often formulated in terms of the associated symbolic dynamics. The recent reference [DWY2] contains variational principles and inequalities for nonuniformly hyperbolic systems without appealing to symbolic dynamics.

2 Setting and Results

In this section, we describe the classes of billiards that we study in this paper and formulate precise conditions on the holes we introduce. We include a variety of examples of holes that meet these conditions.

2.1 Classes of Dispersing Billiards

We identify a domain $Q \subset \mathbb{R}^2$ or $\mathbb{T}^2$ (the 2-torus with Euclidean metric) as the billiard table and assume that ∂Q has d connected components, $\Gamma_1, \dots, \Gamma_d$, each of which comprises a finite number of $\mathscr{C}^3$ smooth, compact arcs. The dynamics of the billiard flow on Q is induced by a particle traveling at unit speed and undergoing elastic collisions at the boundary. The phase space for the billiard flow is $\mathscr{M} = Q \times \mathbb{S}^1/\sim$ with the conventional identifications at collisions (see for example [CM3, Sect. 2.5]).

Define $M = \cup_{i=1}^d (\Gamma_i \times [-\pi/2, \pi/2])$ to be a union of cylinders. The billiard map $T : M \to M$ is the Poincaré map corresponding to collisions with the scatterers. We will denote coordinates on M by (r, φ), where $r \in \Gamma_i$ is parametrized by arclength (oriented positively in the usual sense) and φ is the angle that the (post-collision) velocity vector at r makes with the normal pointing into the domain Q. We shall denote normalized Lebesgue measure on M by m throughout. It is a standard fact that T preserves the smooth measure μ_{SRB} defined by $d\mu_{\text{SRB}} = \frac{\pi}{2} \cos\varphi \, dm$ [CM3, Sect. 2.12].

We include two types of dispersing billiard tables Q in the results of this paper.

Periodic Lorentz Gas. Let $\{B_i\}_{i=1}^d$ define a finite number of open convex regions in $\mathbb{T}^2$ such that each ∂B_i is $\mathscr{C}^3$ with strictly positive curvature. Then in this case, $Q = \mathbb{T}^2 \setminus (\cup_i B_i)$ and $\Gamma_i = \partial B_i$ for $i = 1, \dots, d$. We refer to the billiard flow on such a domain as a periodic Lorentz gas.

For each $x = (r, \varphi) \in M$, define $\tau(x)$ to be the time of the first (nontangential) collision of the trajectory starting at x under the billiard flow. The billiard table Q is said to have finite horizon if τ is uniformly bounded above on M. Otherwise, Q is said to have infinite horizon. In this paper, we allow our Lorentz gas to have either finite or infinite horizon.

Billiards with Corner Points. Let $Q' \subset \mathbb{R}^2$ be a compact region whose boundary consists of finitely many $\mathscr{C}^3$ curves that are positioned convex inward to Q' with strictly positive curvature. In the interior of Q', we may also define a finite number of open convex regions $\{B_i\}_{i=1}^{d'}$ such that ∂B_i is also $\mathscr{C}^3$ with strictly positive curvature. These obstacles $\{B_i\}_{i=1}^{d'}$ are either pairwise connected or disjoint. We define the billiard table Q to be the compact region $Q := Q' \setminus (\cup_i B_i)$. We assume Q has a connected interior. Since the obstacles B_i may or may not overlap, the boundary of Q comprises a finite number $d \le d' + 1$ of connected components, Γ_i. Each Γ_i consists of a finite number of smooth curves as described above; the endpoints of these smooth curves are called *corner points*. For such billiard tables, the horizon is always finite.

We make two additional assumptions in the case of billiards with corner points:

(C1) The intersections of the smooth curves comprising ∂Q are transverse, i.e. the angle at each corner point is positive.[2]

(C2) The hyperbolicity of the billiard map dominates the complexity in the sense of equation (10) of Sect. 3.4.

Assumption (C2) says that the number of singularity curves for T^{-n} which intersect at a single point cannot grow too quickly compared to the expansion of stable curves. Both assumptions are standard for billiards with corner points (see [BSC1, BSC2, Ch1, DZ2]).

2.1.1 Hyperbolicity and Singularities of Dispersing Billiards

Set $\mathscr{S}_0 = \{(r,\varphi) \in M : \varphi = \pm\pi/2\}$. Let $r_1, \dots r_k$ denote the arclength coordinates of the corner points in ∂Q and set $P_0 = \{(r,\varphi) \in M : r = r_i, i = 1, \dots k\}$ (for the periodic Lorentz gas, $P_0 = \emptyset$). The sets $\mathscr{S}_{\pm n} = \cup_{i=0}^n T^{\mp i}(\mathscr{S}_0 \cup P_0)$ are the singularity sets for $T^{\pm n}$, $n \ge 1$. The sets $\mathscr{S}_n$ comprise finitely many smooth compact curves in the finite horizon case and countably many in the infinite horizon case. Moreover, the curves $\mathscr{S}_n \setminus \mathscr{S}_0$ are decreasing for $n > 0$ and increasing for $n < 0$.

To control distortion near $\mathscr{S}_0$, we define the usual homogeneity strips following [BSC1, BSC2],

$$\mathbb{H}_k = \{(r,\varphi) \in M : \pi/2 - 1/(k+1) \le \varphi \le \pi/2 - 1/k^2\},$$

for $k \ge k_0$ to be determined later. We define $\mathbb{H}_{-k}$ near $\varphi = -\pi/2$ similarly. By $\mathbb{H}_0$ we denote the complementary set $M \setminus (\cup_{|k| \ge k_0} \mathbb{H}_k)$.

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[2] In the presence of cusps (corner points whose angle is zero), it was proved in [CM07, CZ08] that the billiard map has polynomial decay of correlations and so in general will have polynomial rates of escape. Thus the present methods will not apply.

The assumption of strict convexity of ∂Q for both classes of dispersing billiards above guarantees the hyperbolicity of T in the following sense: There exist invariant families of cones $C^s(x)$ (stable) and $C^u(x)$ (unstable), continuous on $M \setminus (\mathscr{S}_0 \cup V_0)$ and satisfying $DT^{-1}(x)C^s(x) \subset C^s(T^{-1}x)$ and $DT(x)C^u(x) \subset C^u(Tx)$ wherever T and T^{-1} are defined. Indeed, for all classes of billiards we consider here, the angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero on M.

In terms of the global (r, φ) coordinates, the slopes of vectors in $C^s(x)$ are always negative while those in $C^u(x)$ are always positive. Let $\mathscr{K}_{\min} > 0$ denote the minimum curvature of boundary curves in ∂Q. Although vectors in either cone can become arbitrarily close to vertical near corner points (but never at the same time), in all cases $d\varphi/dr \geq \mathscr{K}_{\min}$ for any $(dr, d\varphi) \in C^u(x)$ and $d\varphi/dr \leq -\mathscr{K}_{\min}$ for all $(dr, d\varphi) \in C^s(x)$ so that both cones are uniformly bounded away from the horizontal. We refer the interested reader to [CM3] for the Lorentz gas and to [Ch1] for the case of billiards with corner points. See also [DZ2, Sect. 6.2] for an explicit calculation of the upper boundaries of the cones during a corner series.

We say a curve $W \subset M$ is a *homogeneous stable curve* if W is contained in one homogeneity strip and the tangent vector to W at x belongs to $C^s(x)$ at every point x in W. Homogeneous unstable curves are defined similarly. It is a well-established fact that for all classes of billiards we consider, there exist families of stable and unstable curves $\mathscr{W}^s$ and $\mathscr{W}^u$ which are invariant in the following sense: For any $W \in \mathscr{W}^s$, $T^{-1}W$ consists of at most countably many connected components, each of which belong to $\mathscr{W}^s$. Indeed, we may choose $\mathscr{W}^s$ to consist of all homogeneous stable curves whose curvature is bounded above by a uniform constant $B_c > 0$ and whose length is no greater than some $\delta_0 > 0$ whose value we fix in Sect. 3.5, eq. (11). $\mathscr{W}^u$ is similarly invariant under T after choosing B_c sufficiently large ([CM3, Ch1]).

2.2 Admissible Holes

We denote by $N_\varepsilon(A)$ the ε-neighborhood of a set A in M. In formulating our conditions below, we consider the maximum length scale δ_0 for stable curves, defined by (11), to be fixed before the introduction of the hole.

A hole $H \subset M$ is an open set with finitely many connected components whose boundary consists of finitely many compact smooth arcs. In addition, we require that:

(H1) *(Complexity)* There exists $B_0 > 0$ such that any stable curve of length less than δ_0 can be cut into at most B_0 pieces by ∂H;

(H2) *(Weak transversality)* There exists $C_0 > 0$ such that for any stable curve W, $m_W(N_\varepsilon(\partial H) \cap W) \leq C_0 \varepsilon^{1/2}$ for all $\varepsilon > 0$ sufficiently small[3], where m_W denotes arclength measure on W.

[3] According to [DZ2, Sect. 2.1], we could take $m_W(N_\varepsilon(\partial H) \cap W) \leq C_0 \varepsilon^{t_0}$ for any exponent $t_0 > 0$. We choose $\varepsilon_0 = 1/2$ here to simplify the exposition and choice of constants in the norms and because this already contains an interesting class of examples (see also (A3)(2) of Sect. 3.3).

Note that if ∂H is uniformly transverse to the stable cone, then (H2) is trivially satisfied. The weaker form of transversality defined above admits the "square-root type" tangencies that appear between singularity curves and stable and unstable curves in billiards with corner points.

Although formally we consider holes in M satisfying the above assumptions, we are especially interested in holes which are actually made in the configuration space Q and in turn induce holes in M that satisfy (H1) and (H2). Such holes are more physically relevant from the point of view of the billiard flow. Below we list several examples of holes in Q which induce holes in M satisfying (H1) and (H2). We follow [DWY1] in our exposition of the first two types of holes.

I. Openings in the boundary of Q. Let ω denote an open arc in one of the Γ_i comprising ∂Q. If we consider ω as an absorbing boundary, then ω induces a hole H_ω in M which is simply a vertical rectangle $H_\omega = (a,b) \times [-\pi/2, \pi/2]$, where (a,b) is the arclength interval corresponding to ω. Such holes are called holes of Type I in [DWY1].

It is clear that (H1) is satisfied with $B_0 = 3$. If ω is a positive distance from a corner point, (H2) is also satisfied since stable cones have strictly negative slopes bounded away from the vertical direction outside any neighborhood of the corner points so that ∂H is uniformly transverse to $C^s(x)$. If ω contains a corner point, (H2) is also satisfied due to the square-root type tangencies that appear between stable curves and the vertical lines in P_0 at corner points [Ch1, Section 9].

II. Holes in the interior of Q. Let ω be an open connected set in the interior of Q. ω does not immediately correspond to a subset of M since it is a positive distance from ∂Q. Thus we have a choice in declaring what we consider the induced hole H_ω to be: Either the backward shadow of the hole, i.e. the set of all points in $(r, \varphi) \in M$ whose forward trajectories under the flow enter ω before they reach ∂Q again; or the forward shadow of the hole, i.e. the set of all points $(r, \varphi) \in M$ whose backward trajectories under the flow enter ω before reaching ∂Q. We choose to define H_ω as the latter, the forward shadow of the hole. Thus those trajectories that are about to enter the hole in forward time are still considered in the system while those which would have passed through the hole on their way to their current collision are considered out. Such holes are called holes of Type II in [DWY1].

We remark that with this definition it does not matter whether ω is convex or not. The induced hole H_ω is always the forward shadow of the convex hull of ω. We thus adopt the convention that ω of this type are always convex. Also, a single ω of this type induces $H_\omega \subset M$ with multiple connected components since each Γ_i with a line of sight to ω contains a component of H_ω.

One can easily obtain geometric properties of ∂H_ω by considering ω as a convex scatterer in Q. Then ∂H_ω will correspond to the forward images of the trajectories that meet $\partial \omega$ tangentially, i.e. the forward image of the set $\partial \omega \times [-\pi/2, \pi/2]$ if ω were considered to be a scatterer. Other components of ∂H_ω may comprise curves in $\mathscr{S}_0$ and $T(\mathscr{S}_0)$. Curves in $T(\mathscr{S}_0)$ and in the forward image of $\partial \omega \times [-\pi/2, \pi/2]$ have positive slopes while curves in $\mathscr{S}_0$ are horizontal. All such curves are uniformly transverse to the stable cone which has vectors with strictly negative slopes bounded

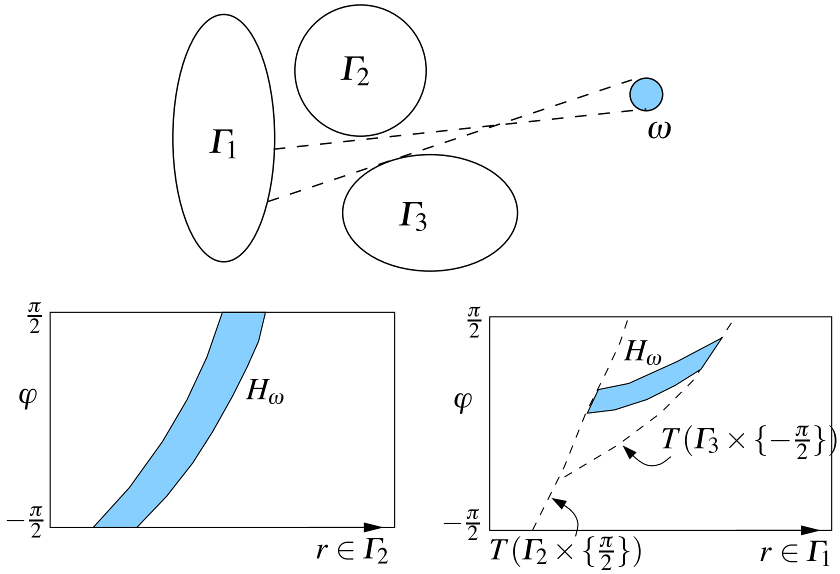

Fig. 1 *Top:* A hole ω of Type II on the billiard table is shown with a line of sight between two scatterers. *Bottom left:* The induced hole H_ω in the component of the phase space M corresponding to Γ_2. *Bottom right:* The induced hole H_ω in the component of the phase space M corresponding to Γ_1. Notice that ∂H_ω is comprised of three types of curves: curves corresponding to $\partial\omega \times \{\pm\frac{\pi}{2}\}$ if ω is considered as a scatterer, curves in $\mathscr{S}_0$ and singularity curves in $T(\mathscr{S}_0)$. In all cases, ∂H_ω is uniformly transverse to the stable cone.

away from 0; thus (H2) is satisfied. See Figure 1 for an example of the geometry of H_ω induced by a hole of Type II. If ω does not lie in an infinite horizon corridor, due to the uniform transversality and the fact that the connected components of H_ω are a positive distance apart, (H1) is also satisfied and one has $B_0 = 3$ if δ_0 is sufficiently small. In any case, $B_0 \leq 2k+1$ where k is the maximal number of connected components of H_ω on a single scatterer.

III. Corner and side pockets. We can combine holes of the first two types described above by placing an open hole ω which intersects ∂Q. Indeed, we even allow ω to contain a corner point. Now $\omega \cap \partial Q$ corresponds to a vertical rectangle in M as described for Type I holes above, while taking the forward image of $\partial\omega \setminus \partial Q$ induces boundaries which are curves with positive slopes in M. These two sets overlap, forming H_ω. See Figure 2 for an example of a hole in the form of a corner pocket.

(H1) is clearly satisfied with $B_0 = 3$ again. Near corner points, the stable cones are not necessarily bounded away from the vertical and so are not uniformly transverse to ∂H_ω. However, they do satisfy the square root type tangency condition required by (H2) (see [Ch1, Sect. 9]).

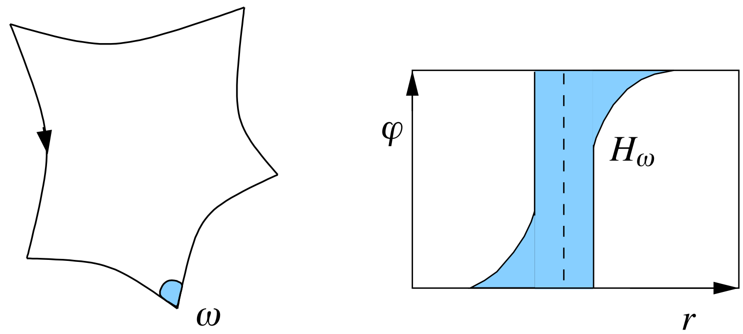

Fig. 2 *Left:* A hole ω positioned as a corner pocket in a dispersing billiard table. *Right:* The induced hole H_ω in the phase space M. The dotted line corresponds to the component of P_0 created by the corner point. The portion of ∂H_ω formed by the forward image of $\partial\omega \times \{\pm\frac{\pi}{2}\}$ has positive slopes, similar to a hole of Type II. Such boundary curves as well as components of ∂H_ω formed by $\mathscr{S}_0$ are uniformly transverse to the stable cone. The vertical segments formed by the rectangle above $\omega \cap \partial Q$ are not uniformly transverse to the stable cone as the diameter of ω shrinks; however, stable curves do satisfy a weaker "square-root type" tangency with components of P_0 so that (H2) is satisfied.

IV. Generalized holes and partially absorbing walls. A generalized hole is a subset ω of configuration space through which trajectories escape if some additional condition is met. As a simple example, we consider ω to be an arc along the boundary and allow trajectories to be absorbed depending on the angle of collision. In other words, if (a,b) denotes the arclength interval corresponding to ω, we choose $\phi_1, \phi_2 \in [-\pi/2, \pi/2]$ and declare $H_\omega = (a,b) \times (\phi_1, \phi_2)$. We may even allow ω to be the entire boundary ∂Q. For large ω, the smallness of the hole, in the sense we shall make precise later, can be guaranteed by choosing $|\phi_1 - \phi_2|$ small.

In any case, since ∂H_ω consists of vertical and horizontal line segments, it is easy to see that (H1) and (H2) are both satisfied. See Figure 3 for an example of partially absorbing walls.

There are many more interesting examples of generalized holes one can construct in a similar manner. For example, one can allow the restriction on the angle to vary depending on the position as long as the resulting holes in M satisfy (H1) and (H2). We leave the generation of such additional examples to the interested reader.

2.3 Transfer Operator

If ψ is a smooth test function, then $\psi \circ T$ is only piecewise smooth due to the singularities of T. For this reason, we introduce scales of spaces, defined using the invariant family of homogeneous stable curves $\mathscr{W}^s$ introduced in Sect. 2.1.1 (and also described in property (A4) in Sect. 3.3) on which to describe the action of the *transfer operator* $\mathcal{L} = \mathcal{L}_T$ associated with T.

Define $T^{-n}\mathcal{W}^s$ to be the set of homogeneous stable curves W such that T^n is smooth on W and $T^iW \in \mathcal{W}^s$ for $0 \le i \le n$. Then $T^{-n}\mathcal{W}^s \subset \mathcal{W}^s$ and it follows from the definition of $\mathcal{W}^s$ that the connected components of $T^{-n}W$ belong to $\mathcal{W}^s$ whenever $W \in \mathcal{W}^s$ (up to subdividing long pieces).

For $W \in T^{-n}\mathcal{W}^s$, a complex-valued test function $\psi : M \to \mathbb{C}$ and $0 < p \le 1$, define $H^p_W(\psi)$ to be the Hölder constant of ψ on W with exponent p measured in the metric d_W. Define $H^p_n(\psi) = \sup_{W \in T^{-n}\mathcal{W}^s} H^p_W(\psi)$ and let $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s) = \{\psi : M \to \mathbb{C} \mid |\psi|_\infty + H^p_n(\psi) < \infty\}$, denote the set of bounded complex-valued functions which are Hölder continuous on elements of $T^{-n}\mathcal{W}^s$. The set $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$ equipped with the norm $|\psi|_{\mathcal{C}^p(T^{-n}\mathcal{W}^s)} = |\psi|_\infty + H^p_n(\psi)$ is a Banach space. We define $\mathcal{C}^p(T^{-n}\mathcal{W}^s)$ to be the closure of $\tilde{\mathcal{C}}^1(T^{-n}\mathcal{W}^s)$ in $\tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s, \mathbb{C})$.[4]

It follows from the hyperbolicity of T (see (A2) from Sect. 3.3) that

$$H^p_{W_i}(\psi \circ T) \le C H^p_W(\psi),$$

for each connected component $W_i \subset T^{-1}W$ and a uniform constant C depending only on T. Thus if $\psi \in \tilde{\mathcal{C}}^p(T^{-(n-1)}\mathcal{W}^s)$, then $\psi \circ T \in \tilde{\mathcal{C}}^p(T^{-n}\mathcal{W}^s)$. Similarly, if $\zeta \in \tilde{\mathcal{C}}^1(T^{-(n-1)}\mathcal{W}^s)$, then $\zeta \circ T \in \tilde{\mathcal{C}}^1(T^{-n}\mathcal{W}^s)$. These two facts together imply that for $p < 1$, if $\psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s)$, then $\psi \circ T \in \mathcal{C}^p(T^{-n}\mathcal{W}^s)$.

If $f \in (\mathcal{C}^p(T^{-n}\mathcal{W}^s))'$, is an element of the dual of $\mathcal{C}^p(T^{-n}\mathcal{W}^s)$, then $\mathcal{L} : (\mathcal{C}^p(T^{-n}\mathcal{W}^s))' \to (\mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s))'$ acts on f by

$$\mathcal{L}f(\psi) = f(\psi \circ T) \quad \forall \psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}^s).$$

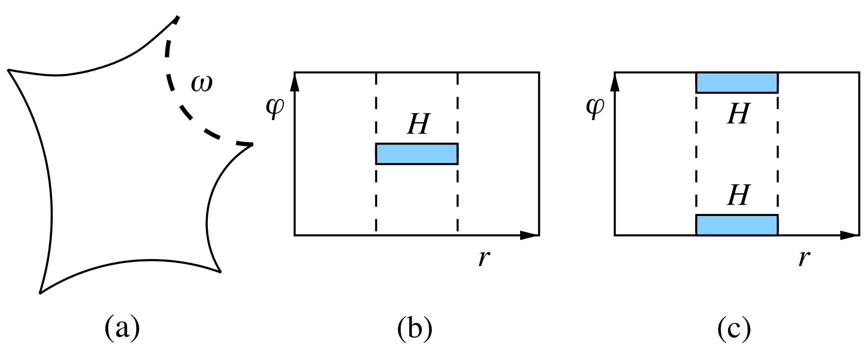

Fig. 3 (a) A partially absorbing hole ω (dashed curve) in a dispersing billiard table. (b) The induced hole H in M if reflections close to normal are absorbed. (c) The two components of the induced hole H in M if reflections close to tangential are absorbed. The dotted vertical lines in (b) and (c) refer to singularity lines corresponding to corner points in the table.

[4] Here by $\tilde{\mathcal{C}}^1(\mathcal{W}^s)$ we mean to indicate $\tilde{\mathcal{C}}^p(\mathcal{W}^s)$ with $p = 1$, i.e. functions which are Lipschitz on elements of $\mathcal{W}^s$.

Recall that m denotes (normalized) Lebesgue measure on M. If $f \in L^1(m)$, then f is canonically identified with a signed measure absolutely continuous with respect to Lebesgue, which we shall also call f, i.e.,

$$f(\psi) = \int_M \psi f \, dm.$$

With the above identification, we write $L^1(M,m) \subset (\mathscr{C}^p(T^{-n}\mathscr{W}^s))'$ for each $n \in \mathbb{N}$. Then restricted to $L^1(M,m)$, $\mathscr{L}$ acts according to the familiar expression

$$\mathscr{L}^n f = f \circ T^{-n} \, |\det DT^n(T^{-n})|^{-1} \quad \text{for any } n \geq 0 \text{ and any } f \in L^1(M,m).$$

When we wish to be explicit about the dependence of $\mathscr{L}$ on a map T, we will use the notation $\mathscr{L}_T$.

When we introduce a hole $H \subset M$, the transfer operator corresponding to $\mathring{T}$, which we shall denote by $\mathring{\mathscr{L}}$, is defined by the equivalent expressions,

$$\mathring{\mathscr{L}} = \mathbf{1}_{\mathring{M}} \mathscr{L} \mathbf{1}_{\mathring{M}} = \mathscr{L} \mathbf{1}_{\mathring{M}^1}, \tag{3}$$

where $\mathbf{1}_A$ indicates the indicator function for the set A. Thus for any test function $\psi \in \mathscr{C}^p(\mathscr{W}^s)$ and $f \in (\mathscr{C}^p(T^{-1}\mathscr{W}^s))'$, we have

$$\mathring{\mathscr{L}} f(\psi) = \mathbf{1}_{\mathring{M}} \mathscr{L}(\mathbf{1}_{\mathring{M}} f)(\psi) = \mathscr{L}(\mathbf{1}_{\mathring{M}} f)(\psi \mathbf{1}_{\mathring{M}}) = f(\psi \circ T \cdot \mathbf{1}_{\mathring{M}^1}),$$

since $\mathbf{1}_{\mathring{M}} \cdot \mathbf{1}_{\mathring{M}} \circ T = \mathbf{1}_{\mathring{M}^1}$. Iterating this expression we obtain,

$$\mathring{\mathscr{L}}^n f(\psi) = f(\psi \circ T^n \cdot \mathbf{1}_{\mathring{M}^n}), \quad \text{for each } n \in \mathbb{N}.$$

When we wish to be explicit about the dependence of $\mathring{\mathscr{L}}$ on a hole H, we will use the notation $\mathring{\mathscr{L}}_H$.

2.4 Main Results

In Section 3.2 we shall define the Banach spaces $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ and $(\mathscr{B}_w, |\cdot|_w)$ used in [DZ2]. It was proved there that $\mathscr{L}_T$ is quasi-compact on $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ under certain abstract assumptions on the map T. We postpone the definition of these norms and the abstract assumptions on the map (which will be satisfied by all classes of billiards we consider here) and first state our results.

In order to obtain information about the spectrum of $\mathring{\mathscr{L}}$ from the spectrum of $\mathscr{L}$, we will use the perturbative framework of Keller and Liverani [KL]. This framework requires two ingredients: (1) uniform Lasota-Yorke inequalities along a sequence of holes; (2) smallness of the perturbation in the following norm:

$$|||\mathscr{L}||| := \{|\mathscr{L} f|_w : \|f\|_{\mathscr{B}} \leq 1\}. \tag{4}$$

The first two propositions establish these ingredients.

Let $\mathscr{H}(B_0,C_0)$ denote a family of holes in M satisfying (H1) and (H2) with uniform constants B_0 and C_0. Throughout this section, T is assumed to be a billiard map corresponding to either a periodic Lorentz gas or a bounded domain with corner points as described in Sect. 2.1.

Proposition 1. *Fix $B_0, C_0 > 0$ and let $\mathscr{H}(B_0,C_0)$ denote the corresponding family of holes satisfying (H1) and (H2). Then there exist constants $C > 0$, $\sigma < 1$ depending only on T, B_0 and C_0 such that for all $H \in \mathscr{H}(B_0,C_0)$ and $n \in \mathbb{N}$,*

$$\|\mathring{\mathscr{L}}_H^n f\|_{\mathscr{B}} \le C\sigma^n \|f\|_{\mathscr{B}} + C|f|_w \quad \text{for all } f \in \mathscr{B}; \tag{5}$$

$$\|\mathscr{L}^n f\|_{\mathscr{B}} \le C\sigma^n \|f\|_{\mathscr{B}} + C|f|_w \quad \text{for all } f \in \mathscr{B}; \tag{6}$$

$$|\mathring{\mathscr{L}}_H^n f|_w \le C|f|_w \quad \text{and} \quad |\mathscr{L}^n f|_w \le C|f|_w \quad \text{for all } f \in \mathscr{B}_w. \tag{7}$$

For a hole $H \subset M$, we define $\mathrm{diam}_s(H) = \sup_{W \in \mathscr{W}^s} |W \cap H|$ and refer to this quantity as the *stable diameter* of H. We define the unstable diameter $\mathrm{diam}_u(H)$ similarly.

Proposition 2. *Suppose H is a hole satisfying (H1) and (H2) and let $h = \mathrm{diam}_s(H)$. Then there exists $C > 0$, depending only on C_0, B_0 and T, such that*

$$|||\mathscr{L} - \mathring{\mathscr{L}}_H||| \le Ch^{\alpha-\gamma},$$

where $0 < \gamma < \alpha$ are from the norms, Sect. 3.2.

Theorem 1. *Fix $B_0, C_0 > 0$. Then for all $H \in \mathscr{H}(B_0,C_0)$ with $\mathrm{diam}_s(H)$ sufficiently small, $\mathring{\mathscr{L}}_H$ has a spectral gap. Its eigenvalue of maximum modulus λ_H is real and its associated eigenvector μ_H is a conditionally invariant measure for $\mathring{T}$ which is singular with respect to Lebesgue measure.*

Moreover, for any probability measure $\mu \in \mathscr{B}$ such that $\lim_{n\to\infty} \lambda_H^{-n} \mathring{\mathscr{L}}_H^n \mu \neq 0$, we have,

1. $\rho(\mu) = \log \lambda_H$;
2. $\left\| \dfrac{\mathring{\mathscr{L}}_H^n \mu}{|\mathring{\mathscr{L}}_H^n \mu|} - \mu_H \right\|_{\mathscr{B}} \le C\sigma_1^n$, *for some $C > 0$, $\sigma_1 < 1$.*

In particular, both Lebesgue measure and the smooth invariant measure μ_{SRB} for T have the same escape rate and converge to μ_H under the normalized action of $\mathring{T}$.

Theorem 2. *Let H_ε be a sequence of holes in $\mathscr{H}(B_0,C_0)$ such that $\mathrm{diam}_s(H_\varepsilon) \le \varepsilon$. Let μ_ε denote the conditionally invariant measures corresponding to λ_ε from Theorem 1. Then*

$$\lim_{\varepsilon\to 0} |\mu_\varepsilon - \mu_{SRB}|_w = 0,$$

and $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0$.

We remark that convergence in the weak norm $|\cdot|_w$ implies the weak convergence of measures.

Next we proceed to study the connection between escape rate and pressure on the survivor set, $\mathring{M}^\infty$. Let $\mathscr{M}_{\mathring{T}}$ denote the set of ergodic, $\mathring{T}$-invariant probability measures supported on $\mathring{M}^\infty$. Following [DWY2], we define a class of invariant measures by

$$\mathscr{G}_H = \{\nu \in \mathscr{M}_{\mathring{T}} : \exists C, r > 0 \text{ such that } \forall \varepsilon > 0, \nu(N_\varepsilon(\mathscr{S}_0 \cup P_0 \cup \partial H)) \le C\varepsilon^r\}. \quad (8)$$

If we omit ∂H, the condition on $N_\varepsilon(\mathscr{S}_0 \cup P_0)$ is the same as that used in [KS] to ensure the existence of Lyapunov exponents and stable and unstable manifolds for ν-a.e. point. Thus this restriction, or something like it, on the class of invariant measures is necessary for maps with singularities.

Theorem 3. *Suppose $H \in \mathscr{H}(B_0, C_0)$ satisfies the assumptions of Theorem 1. If $\mathrm{diam}_u(H)$ is sufficiently small, then*

$$\rho(m) = \log \lambda_H = \sup_{\nu \in \mathscr{G}_H} \{h_\nu(T) - \chi_\nu^+(T)\}.$$

Moreover, we may define a measure ν_H via the limit,

$$\nu_H(\psi) = \lim_{n\to\infty} \lambda_H^{-n} \mu_H(\psi) \quad \text{for all } \psi \in \mathscr{C}^0(M),$$

and ν_H is an invariant probability measure for $\mathring{T}$ belonging to $\mathscr{G}_H$ that achieves the supremum in the variational principle above, i.e. $\rho(m) = h_{\nu_H}(T) - \chi_{\nu_H}^+(T)$.

3 Analytical Framework

In this section we introduce the necessary definitions and abstract assumptions on the class of maps studied in [DZ2]. We will then show in Sect. 4 that an iterate of our map T with the expanded singularity set induced by ∂H satisfies these abstract conditions.

3.1 Representation of Admissible Stable Curves

Recall from Sect. 2.1.1 that the stable cone $C^s(x)$ is bounded away from the horizontal direction in all cases we consider. Thus any curve $W \in \mathscr{W}^s$ can be viewed as the graph of a function $r_W(\varphi)$ of the angular coordinate φ with derivative uniformly bounded above. For each homogeneous stable curve W, let I_W denote the φ-interval on which r_W is defined and define $G_W(\varphi) = (r_W(\varphi), \varphi)$ so that $W = \{G_W(\varphi) : \varphi \in I_W\}$.[5]

[5] Our treatment of stable curves here differs from that in [DZ2]. In that abstract setting, stable curves are defined via graphs in charts of the given manifold. In the present more concrete setting, we dispense with charts and use the global (r, φ) coordinates.

With this view of stable curves, we may redefine $\mathscr{W}^s$ to be the set of homogeneous stable curves satisfying $|W| \le \delta_0$ and $|\frac{d^2 r_W}{d\varphi^2}| \le B_c$ for some $\delta_0, B_c > 0$. $\mathscr{W}^s$ is invariant under T^{-1} in the sense described in Sect. 2.1.1 as long as B_c is chosen sufficiently large [Ch1]. From this point forward, we fix such a choice of B_c once and for all.

The family $\mathscr{W}^u$ of unstable curves has an analogous characterization.

We define a distance in $\mathscr{W}^s$ as follows. Let $W_i = G_{W_i}(I_i)$, $i = 1,2$ be two curves in $\mathscr{W}^s$ with defining functions r_{W_i}. We denote by $\ell(I_1 \triangle I_2)$ the length of the symmetric difference of the φ-intervals on which they are defined. Then the distance between W_1 and W_2 is defined as

$$d_{\mathscr{W}^s}(W_1, W_2) = \eta(W_1, W_2) + \ell(I_1 \triangle I_2) + |r_{W_1} - r_{W_2}|_{\mathscr{C}^1(I_1 \cap I_2)},$$

where $\eta(W_1, W_2) = 0$ if W_1 and W_2 lie in the same homogeneity strip in the same component of M and $\eta = \infty$ otherwise.

For two functions $\psi_i \in \mathscr{C}^p(W_i)$, we define the distance between them to be,

$$d_q(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathscr{C}^q(I_1 \cap I_2)},$$

where $q < 1$ is from the definition of the strong stable norm in Sect. 3.2.

3.2 Norms

Given a curve $W \in \mathscr{W}^s$ and $0 \le p \le 1$, we define $\tilde{\mathscr{C}}^p(W)$ to be the set of complex valued Hölder continuous functions on W with exponent p, with distance measured in the Euclidean metric along W. We set $\mathscr{C}^p(W)$ to be the closure of $\tilde{\mathscr{C}}^1(W)$ in the $\tilde{\mathscr{C}}^p$-norm: $|\psi|_{\mathscr{C}^p(W)} = |\psi|_{\mathscr{C}^0(W)} + H^p_W(\psi)$, where $H^p_W(\psi)$ denotes the Hölder constant of ψ on W as in Sect. 2.3. $\tilde{\mathscr{C}}^p(M)$ and $\mathscr{C}^p(M)$ are defined similarly.

For $\alpha, p \ge 0$, define the following norms for test functions,

$$|\psi|_{W,\alpha,p} := |W|^\alpha \cdot \cos W \cdot |\psi|_{\mathscr{C}^p(W)},$$

where $\cos W$ denotes the average value of $\cos\varphi$ along W, with respect to arclength.

We choose constants to define our norms as follows. Choose $\alpha, \gamma > 0$ such that $\gamma < \alpha < \frac{1}{3}$. Next choose $p, q > 0$ such that $q < p < \gamma$ and note that $p < \frac{1}{3}$ necessarily by the restriction on γ. Finally, choose $\beta > 0$ such that $\beta < \min\{(\alpha - \gamma)/6, p - q, \frac{1}{3} - \alpha\}$.

Given a function $f \in \mathscr{C}^1(M)$, define the *weak norm* of f by

$$|f|_w := \sup_{W \in \mathscr{W}^s} \sup_{\substack{\psi \in \mathscr{C}^p(W) \\ |\psi|_{W,\gamma,p} \le 1}} \int_W f \psi \, dm_W.$$

We define the *strong stable norm* of f by

$$\|f\|_s := \sup_{W \in \mathscr{W}^s} \sup_{\substack{\psi \in \mathscr{C}^q(W) \\ |\psi|_{W,\alpha,q} \le 1}} \int_W f \psi \, dm_W ,$$

and the *strong unstable norm* of f by

$$\|f\|_u := \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathscr{W}^s \\ d_{\mathscr{W}^s}(W_1, W_2) < \varepsilon}} \sup_{\substack{\psi_i \in \mathscr{C}^p(W_i) \\ |\psi_i|_{W_i,\gamma,p} \le 1 \\ d_q(\psi_1, \psi_2) < \varepsilon}} \varepsilon^{-\beta} \left| \int_{W_1} f \psi_1 \, dm_W - \int_{W_2} f \psi_2 \, dm_W \right| ,$$

where $\varepsilon_0 > 0$ is chosen less than δ_0, the maximum length of $W \in \mathscr{W}^s$. We then define the *strong norm* of f by

$$\|f\|_{\mathscr{B}} = \|f\|_s + b\|f\|_u ,$$

where b is a small constant chosen in (12).

We define $\mathscr{B}$ to be the completion of $\mathscr{C}^1(M)$ in the strong norm and $\mathscr{B}_w$ to be the completion of $\mathscr{C}^1(M)$ in the weak norm.

3.3 Uniform Properties of T

In this section we recall the uniform properties (A1)-(A5) for a hyperbolic map T used in [DZ2] to prove the required Lasota-Yorke inequalities for $\mathscr{L}_T$. The following properties refer to the map before the introduction of a hole. Rather than recall the properties in the abstract setting used in [DZ2], we translate them into the concrete setting of dispersing billiards which we have adopted here. This will make the properties easier to verify and will avoid cumbersome additional notation which serves no purpose here. Translated into this setting, most of the abstract assumptions read as well-known facts about dispersing billiards.

(A1) Jacobian. $|DT(x)| := |\det DT(x)| = \cos\varphi(x)/\cos\varphi(Tx)$ wherever $DT(x)$ exists.

(A2) Hyperbolicity. The set $\mathscr{S}_0 \cup P_0$ consists of finitely many curves, although $\mathscr{S}_{\pm n}$, $n \ge 1$, may be finite or countable. There exist families of stable and unstable cones, continuous on the closure of each component of $M \setminus (\mathscr{S}_0 \cup P_0)$, such that the angle between $C^s(x)$ and $C^u(x)$ is uniformly bounded away from 0 on M. Furthermore, there exist constants $C > 0$, $\Lambda > 1$ such that the following hold.

1. $DT(C^u(x)) \subset C^u(Tx)$ and $DT^{-1}(C^s(x)) \subset C^s(T^{-1}x)$ whenever DT and DT^{-1} exist.
2. $\|DT(x)v\|_* \ge \Lambda\|v\|_*, \forall v \in C^u(x)$ and $\|DT^{-1}(x)v\|_* \ge \Lambda\|v\|_*, \forall v \in C^s(x)$, where $\|\cdot\|_*$ is an adapted norm, uniformly equivalent to the Euclidean norm, $\|\cdot\|$.

(A3) Structure of Singularities.

1. There exists $C_1 > 0$ such that for all $x \in M$,

$$C_1 \frac{\tau(T^{-1}x)}{\cos\varphi(T^{-1}x)} \le \frac{\|DT^{-1}(x)v\|}{\|v\|} \le C_1^{-1} \frac{\tau(T^{-1}x)}{\cos\varphi(T^{-1}x)}, \quad \forall v \in C^s(x).$$

Let $\exp_x$ denote the exponential map from the tangent space $\mathscr{T}_x M$ to M. Then,

$$\|D^2 T^{-1}(x)v\| \le C_1^{-1} \tau^2(T^{-1}x)(\cos\varphi(T^{-1}x))^{-3},$$

for all $v \in \mathscr{T}_x M$ such that $T^{-1}(\exp_x(v))$ and $T^{-1}x$ lie in the same homogeneity strip.

In the infinite horizon case, let x_∞ denote one of the finitely many infinite horizon points: points in $\mathscr{S}_0$ which are the accumulation points of a sequence of curves $S_n \subset \mathscr{S}_{-1}$. Let $D_{n,k}$ denote the set of points between S_n and S_{n+1} and whose image under T^{-1} lies in $\mathbb{H}_k$. Then there exists a constant $c_s > 0$ such that $k \ge c_s n^{1/4}$.

2. There exists $C_2 > 0$ such that for any stable curve $W \in \mathscr{W}^s$ and any smooth curve $S \subset \mathscr{S}_{-n}$, we have $m_W(N_\varepsilon(S) \cap W) \le C_2 \varepsilon^{1/2}$ for all $\varepsilon > 0$ sufficiently small.
3. $\partial \mathbb{H}_k$ are uniformly transverse to the stable cones.
4. $\exists C > 0$ such that for all $k > k_0$, if $W \in \mathscr{W}^s$ and $W \subset \mathbb{H}_k$, then $|W| \le Ck^{-3}$.
5. The sum $\sum_{k \ge k_0} \cos(\mathbb{H}_k) < \infty$, where $\cos(\mathbb{H}_k)$ is the average value of $\cos\varphi$ on $\mathbb{H}_k$.

(A4) Invariant families of stable and unstable curves. There are invariant families of curves $\mathscr{W}^s$ and $\mathscr{W}^u$ with the properties described in Sect. 2.1.1 and Sect. 3.1. Moreover, we require the following distortion bounds along stable curves.

There exists $C_d > 0$ such that for any $W \in \mathscr{W}^s$ with $T^i W \in \mathscr{W}^s$ for $i = 0, 1, \dots, n$, and any $x, y \in W$,

$$\left| \frac{J_W T^n(x)}{J_W T^n(y)} - 1 \right| \le C_d d_W(x,y)^{1/3} \ \text{ and } \ \left| \frac{|DT^n(x)|}{|DT^n(y)|} - 1 \right| \le C_d d_W(x,y)^{1/3},$$

where $J_W T(x)$ denotes the Jacobian of T along W.

We also require an analogous distortion bound along unstable curves. If $T^i W \in \mathscr{W}^u$ for $0 \le i \le n$, then for any $x, y \in W$,

$$\left| \frac{|DT^n(x)|}{|DT^n(y)|} - 1 \right| \le C_d d_W(T^n x, T^n y)^{1/3}.$$

(A5) One-step expansion. Let $W \in \mathscr{W}^s$ and partition the connected components of $T^{-1}W$ into maximal pieces V_i such that each V_i is a homogeneous stable curve (not necessarily of length at most δ_0). Let $|J_{V_i} T|_*$ denote the minimum contraction on V_i under T in the metric induced by the adapted norm from (A2)(2). There exists a choice of k_0 for the homogeneity strips such that

$$\lim_{\delta \to 0} \sup_{\substack{W \in \mathscr{W}^s \\ |W| < \delta}} \sum_i |J_{V_i} T|_* < 1. \tag{9}$$

3.4 Verifying (A1)-(A5) for our classes of maps

For both the finite and infinite horizon periodic Lorentz gas, (A1)-(A5) are well-established properties of the billiard map T (see [CM3]). In this concrete setting, (A3)(3) is due to the fact that stable cones are bounded away from the horizontal, (A3)(4) follows directly from (A3)(3), and the series in (A3)(5) is majorized by the series C/k^2 for some $C > 0$.

For billiards with corner points, the uniform transversality between P_0 and the stable cones fails. However, the tangencies between stable curves and vertical lines in P_0 are at worst of a square-root type so that (A3)(2) holds [Ch1, Section 9].

In addition, the uniform expansion (A2)(2) and the one-step expansion (A5) fail. In order to regain uniform hyperbolicity, the usual solution is to consider a higher iterate of T, $T_1 = T^{n_1}$, and prove the above properties hold for T_1. This is done in the present specific setting in [DZ2, Sect. 6] based on the facts established in [Ch1, Sect. 9]. Indeed, in this case one may take the adapted norm $\|\cdot\|_*$ to be simply the Euclidean norm. Below, we formulate precisely the complexity condition referred to in condition (C2) in Sect. 2.1 for billiards with corner points which allows us to regain uniform hyperbolicity for this higher iterate of T.

A consequence of (C1) is that there is a uniform upper bound on the number of consecutive collisions near a corner point. More precisely, let $\phi_0 > 0$ be the minimum angle of intersection of ∂Q at the corner points and define $m_0 = [\pi/\phi_0] + 1$. Then there exists a constant $\tau_c > 0$ such that for each $x \in M$, there is an $i \in \{0, \dots, m_0 - 1\}$ such that $\tau(T^i x) \geq \tau_c$ [BSC1, Ch1]. Define

$$\Lambda_0 := (1 + \tau_c \mathscr{K}_{\min})^{1/m_0} > 1 \quad \text{and} \quad n_0 := \left[\frac{\ln(1 + \mathscr{K}_{\min}^{-2})}{2 \ln \Lambda_0} \right].$$

Let K_n denote the maximum number of smooth curves in $\cup_{i=0}^n T^i(\mathscr{S}_0 \cup V_0)$ that intersect or terminate at any one point in M. We assume there exists $n_2 \geq 1$ such that

$$K_{n_2} < \Lambda_0^{n_2}. \tag{10}$$

This inequality can be iterated since $K_{\ell n_2} \leq (K_{n_2})^\ell$ for each $\ell \in \mathbb{N}$. Let $s = K_{n_2} \Lambda_0^{-n_2} < 1$. Choose ℓ_1 so that $n_2 \ell_1 > m_0 + n_0$ and $\max\{s^{\ell_1}, \Lambda_0^{-\ell_1 n_2}\} < \frac{1}{2} \Lambda_0^{-m_0 - n_0}$. We define $n_1 = n_2 \ell_1$ and estimate,

$$K_{n_1} \leq s^{\ell_1} \Lambda_0^{n_1} < \Lambda_0^{n_1} (\Lambda_0^{-m_0 - n_0} - \Lambda_0^{-n_1}) = \Lambda_0^{n_1 - m_0 - n_0} - 1.$$

Now defining $T_1 = T^{n_1}$, it is shown explicitly in [DZ2, Sect. 6] that T_1 satisfies (A2)(2) with respect to the Euclidean metric with $\Lambda := \Lambda_0^{n_1 - m_0 - n_0} > 1$. Also, since $T_1^{-1} W$ can be cut into at most $K_{n_1} + 1 < \Lambda$ pieces by $\mathscr{S}_{-n_1}$, it follows that the one step expansion (9) of (A5) holds by choosing k_0 large enough.

3.5 Properties of the Banach Spaces

Properties (A1)-(A5) are sufficient to prove the following set of Lasota-Yorke inequalities, with constants depending only on the quantities appearing in (A1)-(A5). We have one more constant to fix.

In light of (A5), we fix $\delta_0 > 0$ in the definition of $\mathscr{W}^s$ sufficiently small so that,

$$\sup_{W \in \mathscr{W}^s} \sum_i |\lambda_{V_i} T|_* =: \theta_* < 1. \tag{11}$$

The following proposition is [DZ2, Proposition 2.2].

Proposition 3. *Let T satisfy properties (A1)-(A5). There exists $C > 0$, depending only on the quantities in (A1)-(A5) such that for all $n \in \mathbb{N}$ and all $f \in \mathscr{B}$,*

$$\begin{aligned}
|\mathscr{L}^n f|_w &\le C|h|_w \\
\|\mathscr{L}^n f\|_s &\le C(\theta_*^{n(1-\alpha)} + \Lambda^{-qn})\|f\|_s + C\delta_0^{\gamma-\alpha}|f|_w \\
\|\mathscr{L}^n f|_u &\le C\Lambda^{-\beta n}\|h\|_u + Cn\|h\|_s,
\end{aligned}$$

where θ_* is from (11).

The proposition is enough to conclude the required Lasota-Yorke inequality in (6) since if we choose $1 > \sigma > \max\{\Lambda^{-\beta}, \theta_*^{1-\alpha}, \Lambda^{-q}\}$, then there exists $N \ge 0$ such that

$$\begin{aligned}
\|\mathscr{L}^N h\|_{\mathscr{B}} = \|\mathscr{L}^N h\|_s + b\|\mathscr{L}^N h\|_u &\le \frac{\sigma^N}{2}\|h\|_s + C\delta_0^{\gamma-\alpha}|h|_w + b\sigma^N\|h\|_u + bCN\|h\|_s \\
&\le \sigma^N\|h\|_{\mathscr{B}} + C\delta_0^{\gamma-\alpha}|h|_w,
\end{aligned} \tag{12}$$

provided b is chosen small enough with respect to N.

It is proved in [DZ2, Lemma 3.9] that the unit ball of $\mathscr{B}$ is compactly embedded in $\mathscr{B}_w$. Thus it follows from standard arguments (see [B, HH]) that the essential spectral radius of $\mathscr{L}$ on $\mathscr{B}$ is bounded by σ, while the spectral radius of $\mathscr{L}$ is at most 1. The fact that the spectral radius actually equals 1 is proved in [DZ2, Lemma 5.1].

Next we state two lemmas proved in [DZ2] that we shall need.

Lemma 1. *([DZ2, Lemma 3.4]) Let $\mathscr{P}$ be a (mod 0) countable partition of M into open, simply connected sets such that (1) for each $k \in \mathbb{N}$, there is an $N_k < \infty$ such that at most N_k elements $P \in \mathscr{P}$ intersect $\mathbb{H}_k$; (2) there are constants $K, C_3 > 0$ such that for each $P \in \mathscr{P}$ and $W \in \mathscr{W}^s$, $P \cap W$ comprises at most K connected components and for any $\varepsilon > 0$, $m_W(N_\varepsilon(\partial P) \cap W) \le C_3\varepsilon^{1/2}$.*

Let $s > \beta/(1-\beta)$. If $f \in \mathscr{C}^s(P)$ for each $P \in \mathscr{P}$ and $\sup_{P \in \mathscr{P}} |f|_{\mathscr{C}^s(P)} < \infty$, then $f \in \mathscr{B}$. In particular, $\mathscr{C}^s(M) \subset \mathscr{B}$ and both Lebesgue measure and the smooth SRB measure for T are in $\mathscr{B}$.

Lemma 2. *([DZ2, Lemma 5.3]) If $f \in \mathscr{B}$ and $\psi \in \mathscr{C}^s(M)$, $s > \max\{\beta/(1-\beta), p\}$, then $\psi f \in \mathscr{B}$. Moreover, $\|\psi f\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}|\psi|_{\mathscr{C}^p(M)}$ for some $C > 0$ independent of ψ and f.*

Other properties of $\mathscr{B}$ and $\mathscr{B}_w$ proved in [DZ2] include:

- $\mathscr{L}$ is well-defined as a continuous linear operator on both $\mathscr{B}$ and $\mathscr{B}_w$. Moreover, there is a sequence of continuous and injective embeddings $\mathscr{C}^s(M) \hookrightarrow \mathscr{B} \hookrightarrow \mathscr{B}_w \hookrightarrow (\mathscr{C}^p(M))'$ for all $s > \beta/(1-\beta)$.
- The elements of the peripheral spectrum of $\mathscr{L}$ on $\mathscr{B}$ are measures and all physical measures belong to $\mathscr{B}$.[6]
- If T is mixing, then $\mathscr{L}$ has a spectral gap, i.e. 1 is a simple eigenvalue and all other eigenvalues have modulus strictly less than 1.

3.5.1 Working with Higher Iterates of T

As mentioned in Sect. 3.4, (A2)(2) and (A5) fail for billiards with corner points so that we must work with a higher iterate of T, $T_1 = T^{n_1}$. In this context, the spectral gap is originally proved for $\mathscr{L}_1 = \mathscr{L}^{n_1}$. In order to conclude that $\mathscr{L}$ also has a spectral gap, one must show that $\|\mathscr{L}\|_{\mathscr{B}}$ is finite. This is done for billiards with corner points in [DZ2, Sect. 6.5].

The proof there relies on the fact that although (A4) for T fails in the sense that the sum in (9) cannot be made less than 1 (so θ_* would be greater than 1 in Prop. 3), the sum in (9) does remain finite. Also, in (A2)(2), Λ is not strictly greater than 1. Yet the rest of the Lasota-Yorke inequalities go through without any problem for T so that one obtains the inequalities of the same form as Prop. 3, but without contraction. This suffices to conclude that $\mathscr{L}$ is bounded on $\mathscr{B}$.

We will make a similar argument for our map with holes since the added discontinuities coming from ∂H will cause (9) to fail, but due to the finiteness assumption (H1), the sum will still converge, keeping $\mathring{\mathscr{L}}_H$ bounded. In order to regain contraction for the open system, we will work with a higher iterate of T even in the case of the Lorentz gas.

4 Extension to Open Systems

In this section we will prove Propositions 1 and 2 and Theorems 1 and 2. Proposition 3 yields the required inequalities for Proposition 1 for $\mathscr{L}$. We begin by explaining how to extend those inequalities to $\mathring{\mathscr{L}}_H$.

[6] Recall that a physical measure for T is an ergodic, invariant probability measure μ for which there exists a positive Lebesgue measure set B_μ, with $\mu(B_\mu) = 1$, such that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \psi(T^i x) = \mu(\psi)$ for all $x \in B_\mu$ and all continuous functions ψ.

In this section, we *assume* that we have a map T (without holes) satisfying (A1)-(A5) (in the case of corner points this is already an iterate $T_1 = T^{n_1}$) and then show how to choose a higher iterate of T once we introduce the additional cuts made by ∂H so that (A1)-(A5) are still satisfied.

We fix $B_0, C_0 > 0$ and the set of holes $\mathscr{H}(B_0, C_0)$ satisfying (H1) and (H2). We choose $H \in \mathscr{H}(B_0, C_0)$ and define $\mathring{T}$ as in Sect. 1.1.

We want to think of ∂H as an extended singularity set for T. To this end, we define a map $\hat{T}$ which is equal to T everywhere, except $\hat{T}$ has the expanded singularity set induced by $\mathscr{S}_0 \cup P_0 \cup \partial H$. Thus when iterating $\hat{T}$, we introduce artificial cuts according to ∂H. When we want to consider the map with a hole, we simply drop the pieces that would have entered H.

Note that by (H2) and (A3)(2), ∂H has the same properties as $\mathscr{S}_0 \cup P_0$. Since $\hat{T}$ and T are the same map everywhere, properties (A1)-(A4) hold for $\hat{T}$ with essentially the same constants as for T (we may have to replace C_2 by C_0 in (A3)(2), but taking C_2' to be the larger of these two numbers, we note that both maps satisfy (A3)(2) with respect to C_2').

Thus the only thing which we need to address is (A5) and in particular (9) which may fail for $\hat{T}$ due to the additional cuts. Note that since ∂H increases the sums in (A5) by at most a factor of B_0, both sums are still finite. This is sufficient to ruin contraction in the Lasota-Yorke inequalities, but still yields a finite bound on $\|\mathscr{L}_{\hat{T}}\|$ according to the discussion in Sect. 3.5.1, where $\mathscr{L}_{\hat{T}}$ denotes the transfer operator corresponding to $\hat{T}$.

4.1 Complexity Bound and Proof of Proposition 1

In this section, we prove the following lemmas, which will allow us to regain (A5) for a higher iterate of $\hat{T}$. This in turn will allow us to prove Proposition 1.

Lemma 3. *There exists a sequence $\delta_n \downarrow 0$ such that*

$$\sup_{\substack{W \in \mathscr{W}^s \\ |W| \le \delta_n}} \sum_i |J_{V_i^n} T^n|_* \le \theta_*^n, \tag{13}$$

where V_i^n denote the maximal homogeneous stable curves in $T^{-n}W$ on which T^n is smooth.

Proof. We prove the lemma by induction on n. The case $n = 1$ follows from (A5) and (11) by taking $\delta_1 = \delta_0$.

Now assume (13) holds for all $0 \le k \le n$. In order to extend this inequality to $n+1$, we claim that $\delta_{n+1} \le \delta_n$ can be chosen so small that $|V_i^n| \le \delta_0$ whenever $|W| \le \delta_{n+1}$. In this way, V_i^n will belong to $\mathscr{W}^s$ and we may apply (A5) to each such curve without additional artificial subdivisions. Let $A(V_i^n)$ comprise those indices j such that $TV_j^{n+1} \subset V_i^n$. Then grouping V_j^{n+1} according to the sets $A(V_i^n)$, we have

$$\sum_j |J_{V_j^{n+1}} T^{n+1}|_* \le \sum_i \sum_{j \in A(V_i^n)} |J_{V_j^{n+1}} T|_* |J_{V_i^n} T^n|_* \le \sum_i |J_{V_i^n} T^n|_* \theta_* \le \theta_*^{n+1},$$

as required. It remains to prove the claim.

The claim follows from the fact that if T^{-1} is smooth on a stable curve W, then there exists a uniform constant C, depending only on T, such that $|T^{-1}W| \le C|W|^{1/2}$ in the finite horizon case and $|T^{-1}W| \le C|W|^{1/3}$ in the infinite horizon case [CM3, Sect. 4.9].[7] Thus the lemma follows if we inductively choose $\delta_{n+1} = \delta_n^2$ in the finite horizon case and $\delta_{n+1} = \delta_n^3$ in the infinite horizon case. □

For $W \in \mathscr{W}^s$, Let $\hat{V}_i^n$ denote the maximal homogeneous stable curves in $\hat{T}^{-n}W$ on which $\hat{T}^n$ is smooth.

Lemma 4. *For $n \in \mathbb{N}$, let δ_n be from Lemma 3. Then*

$$\sup_{\substack{W \in \mathscr{W}^s \\ |W| \le \delta_n}} \sum_i |J_{\hat{V}_i^n} \hat{T}^n|_* \le (1 + n(B_0 - 1))\theta_*^n.$$

Proof. Fix $W \in \mathscr{W}^s$ with $|W| \le \delta_n$. Each V_i^n comprises one or more $\hat{V}_j^n$ due to the expanded singularity set for $\hat{T}$. For a fixed V_i^n, we must estimate the cardinality of the curves $\hat{V}_j^n \subset V_i^n$.

Let $U_i^n = T^n V_i^n$ and $\hat{U}_j^n = T^n \hat{V}_j^n$ for each i and j. Note that if $\hat{V}_{j_1}^n$ and $\hat{V}_{j_2}^n$ belong to the same curve V_i^n, then in fact $T^{-k}\hat{U}_{j_1}^n$ and $T^{-k}\hat{U}_{j_2}^n$ belong to the same smooth curve $T^{-k}U_i^n$ for each $0 \le k \le n$ since the additional cuts due to $\hat{T}$ are artificial and do not change the orbits of points. Also, $|T^{-k}U_i^n| \le \delta_0$ for each $k \le n-1$ by choice of δ_n from the proof of Lemma 3.

Applying (H2) to $T^{-k+1}U_i^n$, the total number of new cuts in $T^{-k}U_i^n$ compared to $T^{-k+1}U_i^n$ can be no more than $B_0 - 1$. Inductively, the total number of cuts introduced into V_i^n by time n can be no more than $n(B_0 - 1)$. Thus the cardinality of the set of j such that $\hat{V}_j^n \subset V_i^n$ is at most $1 + n(B_0 - 1)$. This, plus the fact that $|J_{\hat{V}_j^n} \hat{T}^n|_* \le |J_{V_i^n} T^n|_*$ whenever $\hat{V}_j^n \subset V_i^n$ completes the proof of the lemma. □

Proof (Proof of Proposition 1). Now we choose n_0 such that $(1 + n_0(B_0 - 1))\theta_*^{n_0} = \theta_0 < 1$. Then setting $\hat{T}_0 = \hat{T}^{n_0}$, and choosing δ_{n_0} from Lemma 3 to be the maximum length scale of curves in $\mathscr{W}^s$, we have (A1)-(A5) satisfied for $\hat{T}_0$. Thus the results of [DZ2] imply the uniform Lasota-Yorke inequalities for $\mathscr{L}_{\hat{T}_0}$ given by Proposition 3 with δ_{n_0} in place of δ_0 and θ_0 in place of θ_* with the same choices of constants in the norms.

Notice that we need not change the definition of the Banach spaces $\mathscr{B}$ and $\mathscr{B}_w$. This is because once we have the uniform Lasota-Yorke inequalities measured on $|W| \le \delta_{n_0}$, we can quickly extend them to $|W| \le \delta_0$ by subdividing such curves into at most $[\delta_0/\delta_{n_0}] + 1$ pieces of length at most δ_{n_0} and then applying the estimates on

[7] Indeed, [CM3, Sect. 4.9] does not address the infinite horizon case explicitly, but a quick calculation shows that an exponent of $1/3$ is in fact needed.

the shorter pieces. This has the effect of multiplying all the inequalities in Proposition 3 by the factor $[\delta_0/\delta_{n_0}]+1$ which affects neither the essential spectral radius nor the spectral radius of $\mathcal{L}_{\hat{T}_0}$.

Since $\mathcal{L}_{\hat{T}}$ is bounded as an operator on $\mathcal{B}$ as mentioned previously, this implies that $\mathcal{L}_{\hat{T}}$ also satisfies a uniform set of Lasota-Yorke inequalities with the contraction factor in (12) weakened to σ^{1/n_0}.

Now the transfer operator $\mathring{\mathcal{L}}_{\hat{T}}$ corresponding to the map with a hole satisfies the same Lasota-Yorke inequalities as $\mathcal{L}_{\hat{T}}$ with the same constants since the pieces we must sum over are fewer (we drop the pieces that pass through H), but the estimates on each surviving piece remain the same (the maps T, $\mathring{T}$ and $\hat{T}$ are all the same on such pieces). The equalities of Proposition 1 now follow with constants depending only on (A1)-(A5), B_0 and C_0, as required. □

4.2 Proof of Proposition 2

Fix $H \in \mathcal{H}(B_0, C_0)$ with $\mathrm{diam}_s(H) \le h$. We must estimate $|||\mathcal{L} - \mathring{\mathcal{L}}|||$. We do this estimate for T directly rather than some power of T that we have been working with in previous sections. This is because we do not need contraction for the present estimate, but only use the smallness of the hole. We assume only that T satisfies (A1)-(A5) with $\Lambda = 1$ and the sum in (A5) finite, but not necessarily contracting. As noted previously, these weaker conditions are satisfied for all our classes of billiard maps (see Sects. 3.4 and 3.5.1).

We choose $f \in \mathcal{B}$ and recalling (3), we estimate,

$$\begin{aligned} |\mathcal{L}f - \mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_{\mathring{M}}f)|_w &\le |\mathcal{L}f - \mathbf{1}_{\mathring{M}}\mathcal{L}f|_w + |\mathbf{1}_{\mathring{M}}\mathcal{L}f - \mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_{\mathring{M}}f)|_w \\ &\le |\mathbf{1}_H\mathcal{L}f|_w + |\mathbf{1}_{\mathring{M}}\mathcal{L}(\mathbf{1}_H f)|_w, \end{aligned} \tag{14}$$

by linearity since $H = M \setminus \mathring{M}$.

In order to estimate the two terms in (14), we will need the following two lemmas. The first lemma shows that the indicator functions $\mathbf{1}_H$ and $\mathbf{1}_{\mathring{M}}$ are bounded multipliers in both spaces $\mathcal{B}$ and $\mathcal{B}_w$. The second lemma shows that the hole is a small perturbation in the sense of $|||\cdot|||$.

Lemma 5. *Suppose $f \in \mathcal{B}$ and $H \in \mathcal{H}(B_0, C_0)$. There exists $C > 0$, depending only on B_0 and C_0 and (A1)-(A5), such that $\|\mathbf{1}_H f\|_{\mathcal{B}} \le C\|f\|_{\mathcal{B}}$ and $|\mathbf{1}_H f|_w \le C|f|_w$. Similar bounds hold for $\mathbf{1}_{\mathring{M}} f$.*

Lemma 6. *If $f \in \mathcal{B}$ and $H \in \mathcal{H}(B_0, C_0)$, then $|\mathbf{1}_H f|_w \le Ch^{\alpha-\gamma}\|f\|_s$, where $h = \mathrm{diam}_s(H)$ and $C > 0$ depends only on B_0.*

Postponing the proofs of these two lemmas, we first show how they complete the proof of Proposition 2. Let $C_w = \sup\{|\mathcal{L}f|_w : f \in \mathcal{B}_w, |f|_w \le 1\}$ and $C_B = \sup\{\|\mathcal{L}f\|_{\mathcal{B}} : f \in \mathcal{B}, \|f\|_{\mathcal{B}} \le 1\}$ denote the norm of $\mathcal{L}$ in the two spaces $\mathcal{B}_w$ and $\mathcal{B}$, respectively. Then using the two lemmas together with (14), we have,

$$
\begin{aligned}
|\mathscr{L}f - \mathbf{1}_{\mathring{M}}\mathscr{L}(\mathbf{1}_{\mathring{M}}f)|_w &\le |\mathbf{1}_H\mathscr{L}f|_w + |\mathbf{1}_{\mathring{M}}\mathscr{L}(\mathbf{1}_H f)|_w \\
&\le Ch^{\alpha-\gamma}\|\mathscr{L}f\|_{\mathscr{B}} + CC_w|1_H f|_w \\
&\le CC_B h^{\alpha-\gamma}\|f\|_{\mathscr{B}} + CC_w h^{\alpha-\gamma}\|f\|_{\mathscr{B}}.
\end{aligned}
$$

Now taking the supremum over $f \in \mathscr{B}$, $\|f\|_{\mathscr{B}} \le 1$, completes the proof of the proposition.

4.2.1 Proofs of Lemmas 5 and 6

We will use below that there exists a constant $C_c > 0$ such that

$$
C_c^{-1} \le \frac{\cos\varphi(x)}{\cos\varphi(y)} \le C_c \tag{15}
$$

whenever x and y lie in the same homogeneity strip.

Proof (Proof of Lemma 5). We begin by checking that the partition $\mathscr{P}$ formed by the open, connected components of H and $M \setminus H$ satisfies assumptions (1) and (2) of Lemma 1. Assumption (1) holds with N_k uniformly bounded in k due to the fact that H has finitely many connected components. Also, assumption (2) of Lemma 1 is satisfied due to (H2) and (A3)(2) with $C_3 = \max\{C_0, C_2\}$ and $K = B_0$.

By density, it suffices to prove the lemma for $f \in \mathscr{C}^1(M)$. We fix $f \in \mathscr{C}^1(M)$ and note that $\mathbf{1}_H f$ and $\mathbf{1}_{\mathring{M}} f$ have the type of singularity admitted in Lemma 1 so that $\mathbf{1}_H f, \mathbf{1}_{\mathring{M}} f \in \mathscr{B}$.

We now proceed to estimate $\|\mathbf{1}_H f\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}$. The estimate for the weak norm is similar to that for the strong stable norm and is omitted. From these, the estimates on $\mathbf{1}_{\mathring{M}} f$ follow by linearity since $\mathbf{1}_{\mathring{M}} = \mathbf{1}_M - \mathbf{1}_H$.

To estimate the strong stable norm of $\mathbf{1}_H f$, let $W \in \mathscr{W}^s$ and $\psi \in \mathscr{C}^q(W)$ with $|\psi|_{W,\alpha,q} \le 1$. Note that $|\psi|_{\mathscr{C}^q(W)} \le (\cos W)^{-1}|W|^{-\alpha}$. Then since $W \cap H$ comprises at most B_0 curves $W_i \in \mathscr{W}^s$, we have

$$
\begin{aligned}
\int_W \mathbf{1}_H f\,\psi\,dm_W &= \sum_i \int_{W_i} f\,\psi\,dm_W \le \|f\|_s \sum_i |W_i|^\alpha \cos W_i\,|\psi|_{\mathscr{C}^q(W)} \\
&\le \|f\|_s \sum_i \frac{|W_i|^\alpha}{|W|^\alpha}\frac{\cos W_i}{\cos W} \le \|f\|_s C_c B_0,
\end{aligned}
$$

since $\cos W_i / \cos W$ is uniformly bounded by (15) and $|W_i| \le |W|$. Taking the supremum over $W \in \mathscr{W}^s$ and $\psi \in \mathscr{C}^q(W)$ yields $\|\mathbf{1}_H f\|_s \le C_c B_0 \|f\|_s$.

Next we estimate the strong unstable norm of $\mathbf{1}_H f$. Let $\varepsilon \le \varepsilon_0$ and choose $W^1, W^2 \in \mathscr{W}^s$ with $d_{\mathscr{W}^s}(W^1, W^2) \le \varepsilon$. For $\ell = 1,2$, let $\psi_\ell \in W^\ell$ with $|\psi_\ell|_{W^\ell,\gamma,p} \le 1$ and $d_q(\psi_1, \psi_2) \le \varepsilon$. We must estimate

$$
\int_{W^1} \mathbf{1}_H f\,\psi_1\,dm_W - \int_{W^2} \mathbf{1}_H f\,\psi_2\,dm_W .
$$

Recalling the notation of Section 3.1, we consider W^ℓ as graphs of functions of their angular coordinates, $r_{W^\ell}(\varphi)$, and write $W^\ell = G_{W^\ell}(I_{W^\ell})$, $\ell = 1,2$. We subdivide $W^1 \cap H$ and $W^2 \cap H$ into matched pieces U_j^1 and U_j^2 and unmatched pieces V_k^1 and V_k^2 respectively using a foliation of horizontal line segments in M. Thus U_j^1 and U_j^2 are matched if both are defined over the same φ-interval I_j.

Due to (H1), there are at most B_0 matched pieces and $2B_0+2$ unmatched pieces V_k^ℓ created by ∂H and near the endpoints of W^ℓ. Note that due to (H2) and (A3)(2) we have $|V_k^\ell| \le C_2' \varepsilon^{1/2}$, where $C_2' = \max\{C_0, C_2\}$, since $d_{\mathscr{W}^s}(W_1, W_2) \le \varepsilon$. We split the estimate into matched and unmatched pieces,

$$\int_{W^1} \mathbf{1}_H f\,\psi_1\,dm_W - \int_{W^2} \mathbf{1}_H f\,\psi_2\,dm_W = \sum_j \int_{U_j^1} f\,\psi_1\,dm_W - \int_{U_j^2} f\,\psi_2\,dm_W \\ + \sum_{\ell,k} \int_{V_k^\ell} f\,\psi_\ell\,dm_W.$$

We estimate the integrals on unmatched pieces first,

$$\begin{aligned}\int_{V_k^\ell} f\,\psi_\ell\,dm_W &\le \|f\|_s |V_k^\ell|^\alpha \cos V_k^\ell |\psi_\ell|_{\mathscr{C}^q(W_\ell)} \le \|f\|_s \frac{|V_k^\ell|^\alpha \cos V_k^\ell}{|W^\ell|^\gamma \cos W^\ell} \\ &\le C_c C_2' \|f\|_s \varepsilon^{(\alpha-\gamma)/2},\end{aligned} \tag{16}$$

where we have used (15) and $|V_k^\ell| \le |W^\ell|$ in the last step.

To estimate the integrals on matched pieces, note that

$$d_{\mathscr{W}^s}(U_j^1, U_j^2) \le d_{\mathscr{W}^s}(W^1, W^2) \le \varepsilon.$$

Also,

$$|\psi_1 \circ G_{U_j^1} - \psi_2 \circ G_{U_j^2}|_{\mathscr{C}^q(I_j)} \le |\psi_1 \circ G_{W^1} - \psi_2 \circ G_{W^2}|_{\mathscr{C}^q(I_{W^1} \cap I_{W^2})} \le \varepsilon,$$

since $r_{U_j^1}$ and $r_{U_j^2}$ are simply the restrictions of r_{W^1} and r_{W^2} to I_j respectively. Thus,

$$\left| \int_{U_j^1} f\,\psi_1\,dm_W - \int_{U_j^2} f\,\psi_2\,dm_W \right| \le \|h\|_u \varepsilon^\beta.$$

Putting this estimate together with (16) and using the fact that the number of matched and unmatched pieces are finite as mentioned earlier, we obtain,

$$\left| \int_{W^1} \mathbf{1}_H f\,\psi_1\,dm_W - \int_{W^2} \mathbf{1}_H f\,\psi_2\,dm_W \right| \le C_c C_2'(2B_0+2)\|h\|_s \varepsilon^{(\alpha-\gamma)/2} + B_0 \|h\|_u \varepsilon^\beta,$$

which, since $\beta \le (\alpha-\gamma)/2$, means we may divide through by ε^β to complete the estimate on the strong unstable norm and the proof of the lemma. $\square$

Proof (Proof of Lemma 6). Again, by density, it suffices to do this estimate for $f \in \mathscr{C}^1(M)$.

Let $f \in \mathscr{C}^1(M)$ and $W \in \mathscr{W}^s$. Take $\psi \in \mathscr{C}^p(W)$ with $|\psi|_{W,\gamma,p} \le 1$. Let W_i denote the at most B_0 connected components of $W \cap H$. Then each W_i belongs to $\mathscr{W}^s$ and $|W_i| \le h$ by definition of the stable diameter. We thus estimate,

$$\begin{aligned}\int_W \mathbf{1}_H f\,\psi\,dm_W &= \sum_i \int_{W_i} f\psi\,dm_W \le \sum_i \|f\|_s |W_i|^\alpha \cos W_i |\psi|_{\mathscr{C}^q(W_i^1)} \\ &\le \sum_i \|f\|_s h^{\alpha-\gamma} \frac{|W_i|^\gamma \cos W_i}{|W|^\gamma \cos W} \le B_0 C_c \|f\|_s h^{\alpha-\gamma}\end{aligned}$$

where we have used (15) in the last line.

Taking the supremum over $W \in \mathscr{W}^s$ and $\psi \in \mathscr{C}^p(W)$, the lemma is proved. □

4.3 Proof of Theorems 1 and 2

Proof (Proof of Theorem 1). Propositions 1 and 2 allow us to apply the perturbative framework of Keller and Liverani [KL] in the following way. We fix B_0, C_0 and consider the family of holes $\mathscr{H}(B_0,C_0)$. Proposition 1 guarantees uniform Lasota-Yorke inequalities for $\mathring{\mathscr{L}}_H$ for all $H \in \mathscr{H}(B_0,C_0)$. Then for $H \in \mathscr{H}(B_0,C_0)$ with $\mathrm{diam}_s(H)$ sufficiently small, Proposition 2 and [KL, Corollary 1] imply that the spectra outside the disk of radius $\sigma < 1$ and the corresponding spectral projectors of $\mathring{\mathscr{L}}_H$ move Hölder continuously for $H \in \mathscr{H}(B_0,C_0)$. Thus for $\mathrm{diam}_s(H)$ sufficiently small, $\mathring{\mathscr{L}}_H$ has a spectral gap. We prove the remainder of the theorem assuming that $\mathring{\mathscr{L}}_H$ has a spectral gap in this context.

Since $\mathring{\mathscr{L}}_H$ is real, its eigenvalue of maximum modulus, λ_H, must persist in being real and positive. We claim that its corresponding eigenvector $\mu_H \in \mathscr{B}$ is a measure. It follows from the spectral decomposition of $\mathring{\mathscr{L}}_H$ that for each $f \in \mathscr{B}$, there exists a constant c_f such that

$$\lim_{n\to\infty} \lambda_H^{-n} \mathring{\mathscr{L}}_H^n f(\psi) = c_f \mu_H(\psi), \quad \forall \psi \in \mathscr{C}^p(M). \tag{17}$$

Indeed, the above limit defines the spectral projector Π_{λ_H} onto the eigenspace corresponding to λ_H for $\mathring{\mathscr{L}}_H$. Letting Π_1 denote the eigenprojector onto the eigenspace corresponding to eigenvalue 1 for $\mathscr{L}$, we know that these projectors vary Hölder continuously in the $|||\cdot|||$-norm from (4) according to [KL, Corollary 2]. Recall that μ_{SRB} denotes the smooth invariant measure for T before the introduction of the hole (see Sect. 2.1). Then since $\Pi_1 m(1) = \mu_{\text{SRB}}(1) = 1$, where m denotes Lebesgue measure, we must have $\Pi_{\lambda_H} m(1) = c_m \mu_H(1) > 0$. Indeed, the positivity of $\mathring{\mathscr{L}}$ requires both $c_m > 0$ and $\mu_H(1) > 0$.

Now (17) with 1 (the density of m) in place of f implies $|\mu_H(\psi)| \le |\psi|_\infty |\mu_H(1)|$, which implies that μ_H is a measure. Since $\mu_H(1) > 0$ by the positivity of $\mathring{\mathscr{L}}_H$ we may normalize μ_H to be a probability measure, $\mu_H(1) = 1$. It is now clear that μ_H is a conditionally invariant measure for $\mathring{T}$.

From (2) and the injectivity of T, it follows that μ_H cannot be supported on the set $\cup_{i\geq 0} T^i(H)$. Since this set has full Lebesgue measure by the ergodicity of μ_{SRB}, μ_H must be singular with respect to Lebesgue.

Now suppose that $\mu \in \mathscr{B}$ is a probability measure such that $c_\mu > 0$. Then by (17),

$$c_\mu \mu_H(1) = \lim_{n\to\infty} \lambda_H^{-n} \mathring{\mathscr{L}}_H^n \mu(1) = \lim_{n\to\infty} \lambda_H^{-n} \mu(\mathring{M}^n),$$

so that the escape rates with respect to μ_H and μ are equal, i.e., $\rho(\mu) = \log \lambda_H$. Moreover,

$$\frac{\mathring{\mathscr{L}}_H^n \mu}{|\mathring{\mathscr{L}}_H^n \mu|} = \frac{\mathring{\mathscr{L}}_H^n \mu}{\lambda_H^n} \frac{\lambda_H^n}{\mathring{\mathscr{L}}_H^n \mu(1)} = c_\mu \mu_H \cdot \frac{1}{c_\mu \mu_H(1)} = \mu_H,$$

and the convergence is at an exponential rate in $\mathscr{B}$ due to the spectral decomposition of $\mathring{\mathscr{L}}_H$.

We complete the proof by remarking that $c_m, c_{\mu_{\text{SRB}}} > 0$ by continuity of the spectral projectors so that Lebesgue and the smooth SRB measure for T are both included in this class of measures in $\mathscr{B}$. □

Proof (Proof of Theorem 2). With Propositions 1 and 2 established, the convergence of μ_H to μ_{SRB} and λ_H to 1 follows immediately from the continuity of the spectral projectors corresponding to $\mathring{\mathscr{L}}_H$ given by [KL, Corollary 2] as long as we take a sequence of holes in $\mathscr{H}(B_0, C_0)$ with B_0 and C_0 fixed. □

5 Variational Principle

For all the maps and holes that we consider, we have $\rho(m) \geq \mathscr{P}_{\mathscr{G}_H}$, where $\mathscr{G}_H$ is from (8) by [DWY2, Theorem C]. (Indeed, the setting in terms of both the maps and the permissible holes considered in [DWY2] is much more general than the classes of billiard maps we consider here.) Thus, in order to show that a variational principle holds, we need to find $\nu_H \in \mathscr{G}_H$ such that $\rho(m) = P_{\nu_H}$. We proceed to construct such a measure.

5.1 Definition of ν_H

Let $s > \max\{\beta/(1-\beta), p\}$. We define a linear functional on $\mathscr{C}^s(M)$ by

$$\nu_H(\psi) = \lim_{n\to\infty} \lambda_H^{-n} \mu_H(\mathbf{1}_{\mathring{M}^n} \psi), \qquad \forall \psi \in \mathscr{C}^s(M). \tag{18}$$

The limit is well-defined by (17) since $\mu_H(\mathbf{1}_{\mathring{M}^n} \psi) = \mathring{\mathscr{L}}^n(\psi \mu_H)(1)$ and $\psi \mu_H \in \mathscr{B}$ by Lemma 2. Indeed, $\nu_H(\psi) = c_{\psi \mu_H}$ in the notation of (17).

Since $|\nu_H(\psi)| \le |\psi|_\infty$, ν_H extends as a bounded linear functional to $\mathscr{C}^0(M)$ so that by the Riesz representation theorem, ν_H is a measure. Also, the fact that $\nu_H(1) = 1$ and the positivity of the limit in the definition of ν_H implies that indeed, ν_H is a probability measure supported on the survivor set, $\mathring{M}^\infty$.

Now $\mathbf{1}_{\mathring{M}^{n-1}} \circ \mathring{T} = \mathbf{1}_{\mathring{M}^n}$ due to the nested nature of the sequence $\mathring{M}^n$. We use this to write,

$$\begin{aligned}\nu_H(\psi \circ \mathring{T}) &= \lim_{n\to\infty} \lambda_H^{-n} \mu_H(\mathbf{1}_{\mathring{M}^n} \psi \circ \mathring{T}) = \lim_{n\to\infty} \lambda_H^{-n} \mathring{\mathscr{L}}_H \mu_H(\mathbf{1}_{\mathring{M}^{n-1}} \psi) \\ &= \lim_{n\to\infty} \lambda_H^{-(n-1)} \mu_H(\mathbf{1}_{\mathring{M}^{n-1}} \psi) = \nu_H(\psi),\end{aligned}$$

so that ν_H is an invariant measure for $\mathring{T}$, and also for T since $\mathring{T} = T$ on $\mathring{M}^\infty$.

In the subsequent sections, we will show that ν_H defined in this way belongs to $\mathscr{G}_H$ and that $\rho(m) = P_{\nu_H}$. To do this, we will use a Young tower constructed for the open system. This approach combines the work of [Y, Ch1] to define and construct the tower under general assumptions on the map and singularities (all of which are satisfied in the present setting), and then applies the results of [DWY2] to conclude the variational principle for the open system.

5.2 Review: Young Towers with Holes

In this section, we recall some of the important definitions regarding Young towers for piecewise hyperbolic maps. We refer the reader to [Y] for details.

Generalized Horseshoes. Given a piecewise $C^{1+\varepsilon}$ diffeomorphism $T : M \circlearrowleft$ of a Riemannian manifold M, the tower is built on a compact set X with a hyperbolic product structure: $X = (\cup\Gamma^u) \cap (\cup\Gamma^s)$ where Γ^u and Γ^s are continuous families of local unstable and stable manifolds, respectively, with $m_\omega(\omega \cap \Delta_0) > 0$ for every $\omega \in \Gamma^u$, where m_ω is the Riemannian volume on ω. We define an s-subset of X to be a set $X^s = (\cup\Gamma^u) \cap (\cup\tilde{\Gamma}^s)$ for some $\tilde{\Gamma}^s \subset \Gamma^s$; u-subsets are defined similarly.

Modulo a set of m_ω-measure zero, X is a countable disjoint union of closed s-subsets X_i with the property that for each i, there exists $R_i \in \mathbb{Z}^+$ such that $T^{R_i}(X_i)$ is a u-subset of X. The function $R : X \to \mathbb{Z}^+$ is called the return time function to X. We say the inducing scheme (T, X, R) has *exponential tail* if there exist $C > 0$, $\vartheta < 1$ such that $m_\omega(R > n) \le C\vartheta^n$ for all $\omega \in \Gamma^u$.

In [Y], the return map $T^R : X \circlearrowleft$ satisfies certain uniform hyperbolic properties listed as (P1)-(P5) in that paper. We omit the formal statements of those properties and instead focus on the properties we shall need in what follows. Essentially, the structure of (T^R, X) is that of a generalized horseshoe with countably many branches and variable return times.

For $x \in X$, let $\omega^s(x)$ denote the element of Γ^s containing x. An important quantity in the construction of the horseshoe is the *separation time* $s : X \times X \to \mathbb{Z}^+$ with the following properties: (i) $s(x,y) = s(x',y')$ for $x' \in \omega^s(x)$, $y' \in \omega^s(y)$; (ii) for $x, y \in X_i$, $s(x,y) \ge R_i$; (iii) for $x \in X_i$, $y \in X_j$, $i \ne j$, we have $s(x,y) \le \min(R_i, R_j)$. The separation time is used in the construction of the horseshoe to determine when the derivative along orbits starting in X cease to be comparable, either due to discontinuities or subdivisions to control distortion.

Induced Markov Extensions. To the inducing scheme (T,X,R) described above, it is shown in [Y] that one can associate a Markov extension $F : \Delta \circlearrowleft$ which inherits the uniform hyperbolicity of T^R. The Young tower $\Delta := \cup_{\ell \geq 0} \Delta_\ell$ is the disjoint union of sets $\Delta_\ell := \{(x,\ell) : x \in X, R(x) > \ell\}$. The tower map F is defined by $F(x,\ell) = (x,\ell+1)$ for $\ell < R(x)-1$ and $F(x,R(x)-1) = (T^R x, 0)$. Thus, F maps x up the levels of the tower until the return time is reached. Identifying Δ_0 with X, we have a uniquely defined projection $\pi : \Delta \to M$ such that $T \circ \pi = \pi \circ F$.

The separation time $s(\cdot,\cdot)$ defines a natural countable Markov partition $\{\Delta_{\ell,j}\}$ on Δ: for $x,y \in \Delta_0$, $s(x,y) = \inf\{n > 0 : F^n x, F^n y$ lie in different $\Delta_{\ell,j}\}$. We call the Young tower *mixing* if g.c.d.$\{R\} = 1$.

Towers with Holes Given a hole $H \subset M$, we say that a constructed tower (F,Δ) *respects the hole* if the following two conditions are met.

(R1) $\pi^{-1}H$ is the union of countably many elements of $\{\Delta_{\ell,j}\}$.
(R2) $\pi(\Delta_0) \subset M \setminus H$ and there exist $\delta > 0$, $\xi_1 > 1$ such that all $x \in \Delta_0$ satisfy $d(T^n x, \mathscr{S} \cup \partial H) \geq \delta \xi_1^{-n}$ for all $n \geq 0$, where $\mathscr{S}$ is the singularity set for T.

The first condition above guarantees that the tower map with a hole $\mathring{F}$ is still a Markov map on Δ. The second is a controlled approach condition to ∂H similar to that required for the singularity set for the map. Both conditions have appeared in several previous works, [DWY1, DWY2, DW].

In this setting, the following theorem is proved in [DWY2]. In the setting of [DWY2], T is a $C^{1+\varepsilon}$ piecewise smooth diffeomorphism of a manifold M satisfying the Katok-Strelcyn conditions [KS], which are more general than our assumptions (A1)-(A5) (and indeed include a wide variety of billiard maps) so that we may apply the results of [DWY2] in the present setting.

Theorem 4. *([DWY2, Theorem D]) Suppose T satisfies (A1)-(A4) and that $(T,M;H)$ admits a mixing Young tower $(F,\Delta;\tilde{H})$ with an exponential tail such that (F,Δ) respects the hole. Let μ_{SRB} denote the unique invariant SRB measure of T supported on $\pi(\Delta)$. If the transfer operator on the tower with a hole has a spectral gap with leading eigenvalue λ_Δ, then*

(a) $\rho(\mu_{SRB})$ is well-defined and equals $\log \lambda_\Delta$;
(b) $\mathring{T}^n_* \mu_{SRB} / |\mathring{T}^n_* \mu_{SRB}|$ converges weakly to a conditionally invariant measure $\tilde{\mu}_H$ with eigenvalue λ_Δ;
(c) there exists $\tilde{\nu}_H \in \mathscr{G}_H$ such that

$$\rho(\mu_{SRB}) = P_{\tilde{\nu}_H} := h_{\tilde{\nu}_H}(T) - \chi^+_{\tilde{\nu}_H}(T) \, ;$$

(d) $\tilde{\nu}_H$ is defined by

$$\tilde{\nu}_H(\psi) = \lim_{n\to\infty} \lambda_\Delta^{-n} \int_{\mathring{M}^n} \psi \, d\tilde{\mu}_H \qquad \text{for all } \psi \in \mathscr{C}^0(M).$$

In addition, $\tilde{\nu}_H$ enjoys exponential decay of correlations on Hölder observables.

5.3 Proof of Theorem 3 Assuming a Young Tower Respecting H

We proceed to prove Theorem 3 under the assumption that the tower construction necessary to invoke Theorem 4 holds for some iterate $T_0 = T^{n_0}$, where T is one of the billiard maps considered here. More precisely, we assume that T_0 satisfies (A1)-(A5) and that T_0 admits a mixing Young tower with an exponential tail which respects the hole H. Moreover, we assume the hole is sufficiently small that the transfer operator $\mathring{\mathscr{L}}_T$ has a spectral gap by Theorem 1 and also the transfer operator on the tower for $(\mathring{F}, \Delta; \tilde{H})$ has a spectral gap.

In this context, Theorem 4 gives information about objects with respect to T_0. So for example, we distinguish between the escape rates with respect to T and T_0 via the notation, $\rho(\mu_{\text{SRB}}; T)$ and $\rho(\mu_{\text{SRB}}; T_0)$. Obviously, $\rho(\mu_{\text{SRB}}; T_0) = n_0 \rho(\mu_{\text{SRB}}; T)$.

Since $\mu_{\text{SRB}} = \frac{\pi}{2} \cos\varphi\, m \in \mathscr{B}$ by Lemma 1, we have $\mathring{\mathscr{L}}_T^n \mu_{\text{SRB}} / |\mathring{\mathscr{L}}_T^n \mu_{\text{SRB}}| \to \mu_H$ and $\mu_H \in \mathscr{B}$ is a conditionally invariant measure for $\mathring{T}$ with eigenvalue λ_H by Theorem 1. Thus, applying Theorem 4(b) to T_0, we must have $\tilde{\mu}_H = \mu_H$ and $\lambda_\Delta = \lambda_H^{n_0}$, since λ_Δ is the eigenvalue of μ_H with respect to T_0.

Since $\tilde{\mu}_H = \mu_H$, comparing the definitions of ν_H in (18) and $\tilde{\nu}_H$ in Theorem 4(d), we must have $\tilde{\nu}_H = \nu_H$ as well. Thus by Theorem 4(c), we have $\rho(\mu_{\text{SRB}}; T_0) = h_{\nu_H}(T_0) - \chi^+_{\nu_H}(T_0)$. But since $h_{\nu_H}(T_0) = n_0 h_{\nu_H}(T)$ and $\chi^+_{\nu_H}(T_0) = n_0 \chi^+_{\nu_H}(T)$, this implies

$$\rho(\mu_{\text{SRB}}; T) = h_{\nu_H}(T) - \chi^+_{\nu_H}(T).$$

Now since $\rho(m; T) = \rho(\mu_{\text{SRB}}; T) = \log \lambda_H$ by Theorem 1, this completes the proof of Theorem 3.

5.4 Existence of a Young Tower Respecting the Hole

In this section we prove that a Young tower satisfying the assumptions of Theorem 4 can be constructed for some iterate of our map with holes. Recall the definition of $\hat{T}_0 = \hat{T}^{n_0}$ from Section 4.1, where $\hat{T}$ is the same as T, but with expanded singularity set due to ∂H.

We showed in Section 4.1 that $\hat{T}_0$ satisfies (A1)-(A5) so that the one-step expansion condition (9) is recovered via Lemma 4 even in the presence of the additional cuts due to ∂H.

In this setting, $\hat{T}_0$ satisfies the abstract conditions used in [Ch1] to construct Young towers for a general class of hyperbolic maps with singularities. Below, we recall the simplified two-dimensional version of these properties used in [CZ, Section 4] (see also [Ch1, Section 2] for the more general version).

1. *Smoothness.* Assume that M is a smooth, two-dimensional compact Riemannian manifold and that $\mathscr{F}$ is a C^2 diffeomorphism of $M \setminus D$ onto $\mathscr{F}(M \setminus D)$, where D is a closed set of Lebesgue measure zero that is referred to as the singularity set of $\mathscr{F}$.
2. *Hyperbolicity.* There exist two families of cones $C^s(x)$ and $C^u(x)$ satisfying our assumptions (A2)(1) and (2). The singularity sets D are transverse to stable and unstable cones as in our assumption (A2)(4) and (5). Defining $D_{\pm n} = \cup_{i=1}^n \mathscr{F}^{\mp i}$, the tangent vectors to $D_n \setminus D$ are in stable cones for $n > 0$ and in unstable cones for $n < 0$.
3. *SRB measure.* $\mathscr{F}$ preserves a mixing SRB measure.
4. *Distortion bounds.* Our condition (A3), but for the corresponding unstable Jacobian of $\mathscr{F}$. To obtain this, D necessarily includes the boundaries of homogeneity strips.
5. *Bounded curvature.* The same as what we require in (A3), but for unstable manifolds.
6. *Absolute continuity.* The holonomy map between unstable manifolds is assumed to be absolutely continuous wherever it is defined.
7. *One-step expansion condition.* This is a slightly more complicated version of our one-step expansion (9), but for unstable manifolds.

All these conditions are verified for the periodic Lorentz gas and dispersing billiards with corner points in [Ch1, Sections 7-9]. The only way in which our map $\hat{T}_0$ differs from $\mathscr{F}$ above is in the one-step expansion condition due to our expanded singularity set. Although the formulation of this condition in [Ch1] has a more complicated form, it is proved in [CZ, Lemma 8], that our one-step expansion condition (9) implies that the more complicated form holds for the billiard classes considered here due to the controlled accumulation of singularity curves described in our (A3)(1). This control is automatic for finite horizon billiards since the singularity sets comprise finitely many smooth compact curves; it is proved via simple geometrical estimates in the case of the infinite horizon Lorentz gas.

Thus our map $\hat{T}_0$ satisfies the assumptions of [CZ, Theorem 10] which yields a mixing tower with exponential tails for $\hat{T}_0$. We proceed to check that this tower respects the hole.

The fact that the singularity set for $\hat{T}_0$ includes ∂H implies that for each stable rectangle X_i in the horseshoe X and each $n \le R(X_i)$, we have either $\hat{T}_0^n(X_i) \subset H$ or $\hat{T}_0^n(X_i) \cap H = \emptyset$. This guarantees condition (R1).

Define for $\xi_1 > 1$ and $\delta > 0$,

$$M^{\pm}_{\xi_1,\delta}\{x \in M : \mathrm{dist}(\mathscr{F}^{\pm n}, D \cup \partial M) > \delta \xi_1^{-n}, \quad \forall n \ge 0\}.$$

The reference set X with hyperbolic product structure for the generalized horseshoe constructed in [Ch1] is defined on the positive measure Cantor set given by $M^+_{\xi_1,\delta} \cap M^-_{\xi_1,\delta}$ for fixed $\xi_1 > 1$ and $\delta > 0$ chosen sufficiently small. Since ∂H is included in the singular set for $\hat{T}_0$, this guarantees the slow approach condition (R2). Thus the constructed tower respects the hole.

The final point to check is that the hole on the tower $\pi^{-1}H$ can be made sufficiently small by making H sufficiently small in M. This follows from the slow approach condition (R2) (this argument also appears in [DWY1, DW]). If $\text{diam}_u(H)$ denotes the unstable diameter of H, then the slow approach condition (R2) implies that $\pi^{-1}H$ cannot appear in the tower below level $\ell \sim \log \text{diam}_u(H)$. Thus by decreasing the unstable diameter of H, we can make $\pi^{-1}H$ sufficiently small as long as we remain in the class $\mathscr{H}(B_0, C_0)$. This argument relies on the fact that the tail bounds on the tower depend only on the quantities introduced in items 1-7 above, which depend only on the map and the added complexity due to the hole. Thus keeping B_0 and C_0 fixed, we retain uniform tail estimates constructed for the towers respecting holes in $\mathscr{H}(B_0, C_0)$.

Now for $\pi^{-1}H$ sufficiently small in Δ, it follows from [BDM, Proposition 2.4] (see also [DWY1, Theorem 4.4]) that the transfer operator on the tower with a hole has a spectral gap. This completes the verification of the assumptions of Theorem 4 so we may apply the conclusions of that theorem to our map $\hat{T}_0$ as we did in Section 5.3.

References

[B] V. Baladi, *Positive transfer operators and decay of correlations*, Advanced Series in Nonlinear Dynamics **16**, World Scientific, 2000.

[BK] V. Baladi and G. Keller, *Zeta functions and transfer operators for piecewise monotonic transformations*, Comm. Math. Phys. **127** (1990), 459-477.

[BDM] H. Bruin, M. Demers and I. Melbourne, *Existence and convergence properties of physical measures for certain dynamical systems with holes*, Ergod. Th. and Dynam. Sys. **30** (2010), 687-728.

[BSC1] L. Bunimovich, Ya. G. Sinai and N. Chernov, *Markov partitions for two-dimensional hyperbolic billiards,* Russian Math. Surveys, **45** (1990) 105–152.

[BSC2] L. Bunimovich, Ya. G. Sinai and N. Chernov, *Statistical properties of two-dimensional hyperbolic billiards,* Russian Math. Surveys **46** (1991) 47–106.

[C1] N. N. Čencova, *A natural invariant measure on Smale's horseshoe*, Soviet Math. Dokl. **23** (1981), 87-91.

[C2] N. N. Čencova, *Statistical properties of smooth Smale horseshoes*, in Mathematical Problems of Statistical Mechanics and Dynamics, R. L. Dobrushin, ed. Reidel: Dordrecht, 1986, pp. 199-256.

[Ch1] N. Chernov, *Decay of correlations and dispersing billiards*, J. Stat. Phys. **94** (1999), 513-556.

[CM1] N. Chernov and R. Markarian, *Ergodic properties of Anosov maps with rectangular holes*, Bol. Soc. Bras. Mat. **28** (1997), 271-314.

[CM2] N. Chernov and R. Markarian, *Anosov maps with rectangular holes. Nonergodic cases*, Bol. Soc. Bras. Mat. **28** (1997), 315-342.

[CM3] N. Chernov and R. Markarian, *Chaotic Billiards*, Mathematical Surveys and Monographs, **127**, AMS, Providence, RI, 2006.

[CV] N. Chernov and H. van den Bedem, *Expanding maps of an interval with holes*, Ergod. Th. and Dynam. Sys. **22** (2002), 637-654.

[CZ] N. Chernov and H.-K. Zhang, *Billiards with polynomial mixing rates*, Nonlinearity **18** (2005), 1527-1553.

[CMT1] N. Chernov, R. Markarian and S. Troubetskoy, *Conditionally invariant measures for Anosov maps with small holes*, Ergod. Th. and Dynam. Sys. **18** (1998), 1049-1073.

[CMT2] N. Chernov, R. Markarian and S. Troubetskoy, *Invariant measures for Anosov maps with small holes*, Ergod. Th. and Dynam. Sys. **20** (2000), 1007-1044.

[CMS1] P. Collet, S. Martínez and B. Schmitt, *The Yorke-Pianigiani measure and the asymptotic law on the limit Cantor set of expanding systems*, Nonlinearity **7** (1994), 1437-1443.

[CMS2] P. Collet, S. Martínez and B. Schmitt, *Quasi-stationary distribution and Gibbs measure of expanding systems*, in Instabilities and Nonequilibrium Structures V, E. Tirapegui and W. Zeller, eds. Kluwer: Dordrecht, 1996, pp. 205-19.

[D1] M.F. Demers, *Markov Extensions for Dynamical Systems with Holes: An Application to Expanding Maps of the Interval*, Israel J. of Math. **146** (2005), 189-221.

[D2] M.F. Demers, *Markov Extensions and Conditionally Invariant Measures for Certain Logistic Maps with Small Holes*, Ergod. Th. and Dynam. Sys. **25**:4 (2005), 1139-1171.

[DW] M.F. Demers and P. Wright, *Behavior of the escape rate function in hyperbolic dynamical systems*, Nonlinearity **25** (2012), 2133-2150.

[DY] M.F. Demers and L.-S. Young, *Escape rates and conditionally invariant measures*, Nonlinearity **19** (2006), 377-397.

[DZ1] M.F. Demers and H.-K. Zhang, *Spectral analysis of the transfer operator for the Lorentz gas,* J. Modern Dynamics **5**:4 (2011), 665-709.

[DZ2] M.F. Demers and H.-K. Zhang, *Spectral analysis for hyperbolic systems with singularities*, submitted.

[DWY1] M.F. Demers, P. Wright and L.-S. Young, *Escape rates and physically relevant measures for billiards with small holes,* Commun. Math. Phys. **294**:2 (2010), 353-388.

[DWY2] M.F. Demers, P. Wright and L.-S. Young, *Entropy, Lyapunov exponents and escape rates in open systems,* Ergod. Th. and Dynam. Sys. **32**:4 (2012), 1270-1301.

[HH] H. Hennion, L. Hevré, *Limit Theorems for Markov Chains and Stochastic Properties of Dynamical Systems by Quasi-Compactness*, Lecture Notes in Mathematics **1766**, Springer-Verlag, Berlin, 2001.

[HY] A. Homburg and T. Young, *Intermittency in families of unimodal maps*, Ergod. Th. and Dynam. Sys. **22**:1 (2002), 203-225.

[KS] A. Katok and J.-M. Strelcyn, with the collaboration of F. Ledrappier and F. Przytycki, *Invariant Manifolds, Entropy and Billiards; Smooth Maps with Singularities*, Lecture Notes in Math. **1222**, Springer-Verlag: Berlin, 283 pp, 1986.

[KL] G. Keller and C. Liverani, *Stability of the spectrum for transfer operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), 141-152.

[LiM] C. Liverani and V. Maume-Deschamps, *Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set*, Annales de l'Institut Henri Poincaré Probability and Statistics, **39** (2003), 385-412.

[LM] A. Lopes and R. Markarian, *Open Billiards: Cantor sets, invariant and conditionally invariant probabilities*, SIAM J. Appl. Math. **56** (1996), 651-680.

[PY] G. Pianigiani and J. Yorke, *Expanding maps on sets which are almost invariant: decay and chaos*, Trans. Amer. Math. Soc. **252** (1979), 351-366.

[R] P.A. Richardson, Jr., *Natural Measures on the Unstable and Invariant Manifolds of Open Billiard Dynamical Systems*, Doctoral Dissertation, Department of Mathematics, University of North Texas, 1999.

[Y] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. Math. **147** (1998), 585-650.