# Introductory Lectures on Open Systems

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The purpose of these notes is to provide an introduction to some of the key issues involved in the study of open systems. The notes will present some proofs to illustrate certain key ideas, embedded in a larger survey of the topic.

## 1 Basic Definitions and Motivating Questions

We begin with a self-map $T : M \circlearrowleft$ of a complete metric space $M$. Let $H$ be an open subset of $M$, which we will call the 'hole' and let $\mathring{M} = M \setminus H$ denote its complement. We keep track of the iterates of a point $x \in M$ as long as they do not enter $H$. Once $T^n(x) \in H$, we say the point has fallen into the hole and no longer consider it. Alternatively, we may be interested in the dynamics near an invariant set $\Omega \subset M$ that is not an attractor, and take $\mathring{M}$ to be an open neighborhood of $\Omega$. Without specifying the geometry of $H$, these two problems are mathematically equivalent.

We denote by $\mathring{M}^n = \cap_{i=0}^n T^{-i}\mathring{M}$ the set of points which have not entered the hole by time $n$. Note $\mathring{M}^0 = \mathring{M}$. We will be interested in the open system $\mathring{T} : \mathring{M}^1 \to \mathring{M}$ and its iterates $\mathring{T}^n = T^n|\mathring{M}^n$.

### 1.1 Motivating Questions

In this section we present some motivating questions which will serve as a guide in what follows.

**Q1. (Escape rate)** Let $\mu_0$ be an initial probability measure on $M$. At what rate does $\mu_0(\mathring{M}^n)$ decay?

When we expect an exponential decay in $\mu_0(\mathring{M}^n)$, we define

$$\underline{\rho}(\mu_0; H) = \liminf_{n\to\infty} \frac{1}{n} \log \mu_0(\mathring{M}^n) \qquad \text{and} \qquad \overline{\rho}(\mu_0; H) = \limsup_{n\to\infty} \frac{1}{n} \log \mu_0(\mathring{M}^n). \tag{1.1}$$

If $\underline{\rho} = \overline{\rho} = \rho$, then $-\rho$ is the exponential *escape rate* with respect to $\mu_0$.

**Q2. (Limiting Distribution)** Define the push-forward measure $\mathring{T}^n_* \mu_0(A) = \mu_0(\mathring{T}^{-n}A) = \mu_0(T^{-n}A \cap \mathring{M}^n)$ and consider the sequence of probability measures,

$$\mu_n = \frac{\mathring{T}^n_*\mu_0}{|\mathring{T}^n_*\mu_0|} = \frac{\mathring{T}^n_*\mu_0}{\mu_0(\mathring{M}^n)}.$$

Does $\mu_n$ converge to a limiting distribution $\mu_\infty$? What are the properties of this limiting distribution? (I.e., is it absolutely continuous with respect to $\mu_0$ or is it SRB-like for hyperbolic maps?)

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**Q3. (Dependence on $\mu_0$ and $H$)** Suppose the escape rate $\rho$ and a limiting distribution exist. For what class of initial distributions $\mu_0$ is there a common escape rate $\rho$ and limiting distribution $\mu_\infty$? How do $\rho$ and $\mu_\infty$ vary with the hole? If we take a sequence of holes $H_\varepsilon$ shrinking to a point in some reasonable way, does the corresponding sequence of limiting distributions $\mu_\infty^\varepsilon$ converge to an invariant measure for $T$, the map without the hole? One can think of this as a form of stability of the invariant measure with respect to small leaks in the system.

**Q4. (Pressure on Survivor Set)** For this question, assume $T$ is a differentiable mapping and $M$ is a smooth Riemannian manifold.

Define the *survivor set* to be $\mathring{M}^\infty = \cap_{i=0}^\infty T^{-i}(\mathring{M})$ if $T$ is not invertible and $\mathring{M}^\infty = \cap_{-\infty}^\infty T^{-i}(\mathring{M})$ if $T$ is invertible. This is the $T$-invariant set of points that never enter $H$.

For $\nu$ an invariant measure on the survivor set, define the *pressure* of $\nu$ to be

$$P_\nu = h_\nu(T) - \int \chi^+ d\nu$$

where $h_\nu(T)$ denotes the Kolmogorov-Sinai entropy and $\chi^+$ denotes the sum of positive Lyapunov exponents (see Section 3.1 for definitions of these quantities).

We say the open system satisfies an *escape rate formula* if there exists a measure $\nu$, supported on the survivor set $\mathring{M}^\infty$ such that $\rho = P_\nu$. We say the system satisfies a *variational principle* if

$$\rho = \mathcal{P}_\mathcal{C} := \sup\{P_\nu : \nu \in \mathcal{C}\}$$

for some resaonable class of invariant measures $\mathcal{C}$, which will in general be system dependent. For example, for a smooth system, $\mathcal{C}$ might be the set of invariant measures for $T$ supported on $\mathring{M}^\infty$.

A variational principle for the open system can be seen as an extension of Pesin's formula for closed systems, which states that for large classes of smooth and piecewise smooth systems, $\mathcal{P}_\mathcal{C} = 0$, where $\mathcal{C}$ is the set of invariant Borel probability measures for $T$. In addition, for diffeomorphisms of compact Riemannian manifolds, the SRB measure[1] is the unique measure that attains the supremum [Y4].

## 1.2 Connection Between Limiting Distribution and Escape Rate

Define $\mathring{\mathcal{T}}_1\mu_0 = \frac{\mathring{T}_*\mu_0}{|\mathring{T}_*\mu_0|}$ and suppose

$$\lim_{n\to\infty} \mathring{\mathcal{T}}_1^n\mu_0 = \mu_\infty.$$

If $\mu_\infty$ gives 0 weight to the discontinuities of $\mathring{T}$ (note that the discontinuity set for $\mathring{T}$ includes the bounary of $H$ even if $T$ is continuous), then

$$\mathring{\mathcal{T}}_1\mu_\infty = \mathring{\mathcal{T}}_1(\lim_{n\to\infty} \mathring{\mathcal{T}}_1^n\mu_0) = \lim_{n\to\infty} \mathring{\mathcal{T}}_1^{n+1}\mu_0 = \mu_\infty,$$

so that

$$\frac{\mathring{T}_*\mu_\infty}{|\mathring{T}_*\mu_\infty|} = \mu_\infty, \tag{1.2}$$

i.e. $\mu_\infty$ is invariant under the action of pushing forward and renormalizing by the remaining mass. A measure that satisfies (1.2) is called a *conditionally invariant probability measure*. When such

[1] For (nonuniformly) hyperbolic systems, an SRB measure is an invariant measure whose conditional measures on unstable manifolds are absolutely continuous with respect to the Riemannian volume.

a measure $\mu$ is absolutely continuous with respect to a reference measure of interest (for example Lebesgue measure), we call $\mu$ an *absolutely continuous conditionally invariant measure* and abbreviate it *a.c.c.i.m.*

Set $\lambda = |\mathring{T}_*\mu_\infty| = \mu_\infty(\mathring{M}^1)$ and note that iterating (1.2) we obtain $\mu_\infty(\mathring{M}^n) = \mathring{T}_*^n \mu_\infty(\mathring{M}) = \lambda^n$, so that $-\rho(\mu_\infty) = -\log\lambda$ is the escape rate with respect to $\mu_\infty$. In addition, $\rho(\mu_0) = \log\lambda$ for any $\mu_0$ that converges to $\mu_\infty$ under $\mathring{\mathcal{T}}_1^n$. So $-\log\lambda$ represents a unified rate of escape for a class of initial distributions. $\lambda$ is often referred to as the eigenvalue of the conditionally invariant measure $\mu$ for the open system.

## 2 Conditionally Invariant Measures

The characterization of limiting distributions as conditionally invariant measures and the connection to a unified escape rate for a class of initial distributions suggests that the existence (and uniqueness?) of conditionally invariant measures, and especially *a.c.c.i.m.*, is an important first step toward understanding open systems.

### 2.1 Transfer Operator

A useful tool in the study of invariant and conditionally invariant measures is the transfer operator associated with a dynamical system.

Given a transformation $T$ on a smooth Riemannian manifold $M$ with Lebesgue measure $m$ (not necessarily invariant), let $JT$ denote the Jacobian of $T$. Then the transfer operator $\mathcal{L}$ is defined on $L^1(m)$ by,

$$\mathcal{L}f(x) = \sum_{y\in T^{-1}x} \frac{f(y)}{JT(y)}, \qquad \text{and its iterates by,} \qquad \mathcal{L}^n f(x) = \sum_{y\in T^{-n}x} \frac{f(y)}{JT^n(y)}.$$

The importance of the transfer operator stems from the fact that if $\mu$ is a measure absolutely continuous with respect to $m$ with density $f$, then the density of $T_*\mu$ is given by $\mathcal{L}f$.

Similarly, given a hole $H \subset M$, we may define the corresponding 'punctured' transfer operator for the open system by

$$\mathring{\mathcal{L}}f(x) = 1_{\mathring{M}}(x)\mathcal{L}(1_{\mathring{M}}f)(x) = \mathcal{L}(1_{\mathring{M}^1}f)(x) = \sum_{y\in\mathring{T}^{-1}x} \frac{f(y)}{JT(y)}. \tag{2.1}$$

**Problem 1.** *Prove that the iterates of $\mathring{\mathcal{L}}$ satisfy $\mathring{\mathcal{L}}^n f = \mathcal{L}^n(1_{\mathring{M}^n} f)$. Prove also that if $\mu$ is a probability measure with density $f$ with respect to $m$, then $\mathring{T}_*\mu$ has density $\mathring{\mathcal{L}}f$.*

If $\mu$ is an *a.c.c.i.m.* with eigenvaue $\lambda$ and density $f$, then for any measurable $A \subset M$,

$$\int_A \mathring{\mathcal{L}}f\, dm = \mathring{T}_*\mu(A) = \lambda\mu(A) = \lambda \int_A f\, dm,$$

so that $\mathring{\mathcal{L}}f = \lambda f$ and $f$ is an eigenvector with eigenvalue $\lambda$ for $\mathring{\mathcal{L}}$.

### 2.2 Some Simple Examples

**Example 1. Tent Map with a Hole.** Let $0 < \varepsilon < 1$ and define $T : [0,1] \circlearrowleft$ be such that

$$T(x) = \begin{cases} \frac{2}{1-\varepsilon}x, & 0 \le x \le \frac{1}{2}(1-\varepsilon) \\ \frac{2}{1-\varepsilon}(1-x), & \frac{1}{2}(1+\varepsilon) < x \le 1 \end{cases}.$$

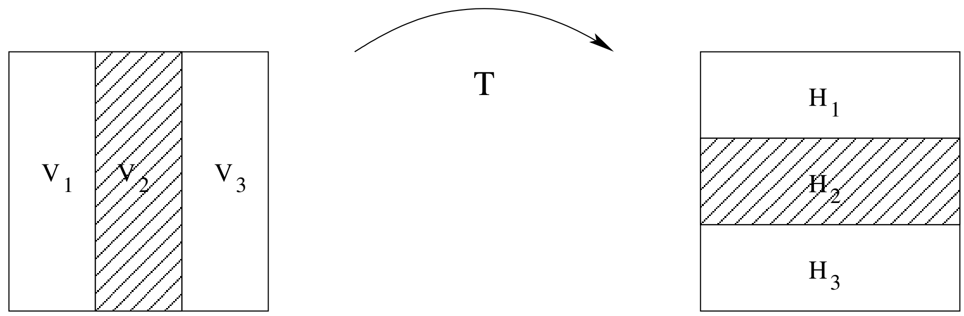


Figure 1: The triadic baker map.

Here $H = (\frac{1}{2}(1-\varepsilon), \frac{1}{2}(1+\varepsilon))$ is the hole. We do not specify $T$ on $H$ since that is irrelevant. It is an easy calculation to see that normalized Lebesgue measure on $\mathring{M} := [0,1]\backslash H$ is an *a.c.c.i.m.* for $\mathring{T}$ with $\lambda = 1-\varepsilon$. (For example, show $\mathring{\mathcal{L}}1_{\mathring{M}} = (1-\varepsilon)1_{\mathring{M}}$.) Note that $\mathring{M}^\infty$ is a Cantor set, so that any invariant measure $\nu$ supported on $\mathring{M}^\infty$ is necessarily singular with respect to Lebesgue.

**Example 2. Triadic Baker Map.** Let $M$ be the unit square $[0,1] \times [0,1]$ divided into three vertical strips $V_1$, $V_2$ and $V_3$ of equal width, and let $T$ be such that $T(V_i) = H_i$ where $H_1$, $H_2$ and $H_3$ are horizontal strips as shown in Figure 1. On each $V_i$, $T$ is affine and area-preserving: $V_i$ is contracted by a factor of $\frac{1}{3}$ in the vertical direction and expanded by a factor of 3 in the horizontal direction.

Now introduce the hole $H = V_2$, so that $\mathring{M} = V_1 \cup V_3$. At the first step under $\mathring{T}$, two vertical rectangles (the middle thirds of $V_1$ and $V_3$) enter $H$, so that $\mathring{M}^1$ is the union of 4 vertical rectangles of width 1/9. Continuing, we see that $\mathring{M}^n$ is the union of $2^{n+1}$ vertical rectangles of height 1 and width $3^{-n-1}$. Thus $m(\mathring{M}^n) = (2/3)^{n+1}$ and so $\rho(m) = \log(2/3)$.

Now consider an arbitrary conditionally invariant measure $\mu$ for this open system. Since $T(V_2) = H_2$, the support of $\mu$ cannot meet this set. Similarly, it cannot meet $T^2(V_2)$, which is the union of horizontal strips that are the middle thirds of $H_1$ and $H_3$. Continuing this line of argument, we see that any conditionally invariant measure must be supported on $M \setminus \cup_{n\geq 0} T^n(V_2)$, which is equal to $([0, \frac{1}{3}] \cup [\frac{2}{3}, 1]) \times \Gamma$ where $\Gamma$ is the middle thirds Cantor set in the vertical direction.

In this simple example, it is easy to see what one obtains by pushing forward Lebesgue measure $m$ and renormalizing. Starting with $m$ on $\mathring{M}$, one checks that with $\Lambda = \frac{2}{3}$, $\Lambda^{-n}\mathring{T}^n_* m$ is supported on the union of $2^n$ horizontal rectangles intersected with $\mathring{M}$ of height $3^{-n}$ each, and is uniformly distributed on it. Thus as $n$ tends to $\infty$, $\Lambda^{-n}\mathring{T}^n_* m$ converges to $\mu_\infty :=$ normalized Lebesgue measure on $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ crossed with the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure on $\Gamma$. This measure $\mu_\infty$ is, by any standard, a *natural* conditionally invariant measure.

Notice that $\mu_\infty$ is singular with respect to $m$ (even though $T$ preserves $m$), and that it has smooth conditional measures on horizontal lines, which are unstable manifolds of $T$. This is the typical structure for conditionally invariant measures for invertible maps: they are supported on the (singular) set $M \setminus \cup_{n\geq 0} T^n(H)$. By contrast, the survivor set $\mathring{M}^\infty$ is smaller still, $\mathring{M}^\infty = \Gamma \times \Gamma$, the direct product of two Cantor sets.[2]

**Remark 2.1.** *Example 2 suggests a strong resemblance between a.c.c.i.m. in open systems and SRB measures in closed systems. For invertible maps, conditionally invariant measures with absolutely*

[2]The open baker map with $H = V_2$ presented here is equivalent to a linear Smale horseshoe. For more general horseshoes studied from this point of view, see [C1, C2].

*continuous conditional measures on unstable manifolds are natural replacements for* a.c.c.i.m. *To avoid cumbersome language, we use* a.c.c.i.m. *as an abbreviation for these measures as well.*

### 2.3 Construction of many *a.c.c.i.m.* with overlapping supports

The two examples above paint a simple, unambiguous picture for *a.c.c.i.m.* and their relation to SRB measures, but unless more conditions are imposed on this class of measures, the actual situation cannot be more different. In this section we demonstrate for non-invertible maps how in quite general settings one can construct uncountably many *a.c.c.i.m.* with overlapping supports for *any* given $\lambda \in [0, 1)$. Similar constructions of conditionally invariant measures smooth on unstable manifolds are easily carried out for Anosov diffeomorphisms.

For this section, assume that $M$ is a Riemmanian manifold, possibly with boundary, and that on an open subset $U$ of $M$ of full Lebesgue measure, $T$ is locally invertible and nonsingular with respect to $m$ with Jacobian $JT > 0$. We assume further that for each $x \in U$, $\sum_{z\in T^{-1}x}(JT(z))^{-1} < \infty$. In the language of transfer operators, this assumption is $\mathcal{L}1 < \infty$.

Let $H \subset M$ be an open set. We will proceed to identify a disjoint sequence of sets which march progressively toward the hole and eventually fall in. Additionally, these sets should consist of points with good pre-images. More precisely, for each $n \geq 1$, let $E^n = \mathring{M}^{n-1} \setminus \mathring{M}^n$ denote the set of points that enters $H$ for the first time at precisely time $n$. Let $G := \{x \in M : \text{for every } n \geq 1, \exists z_n \in \mathring{M}^n$ such that $\mathring{T}^n(z_n) = x\}$ denote the set of points that have at least one preimage under each iterate of $\mathring{T}$.

**Theorem 2.2.** *Let $(T, M, H)$ be as above. We assume*

$$m(E^1 \cap G) > 0.$$

*Then given any $\lambda$, $0 \leq \lambda < 1$, $\mathring{T}$ admits uncountably many a.c.c.i.m. with escape rate $-\log \lambda$.*

*Proof.* Let $G^n := E^n \cap G$. We claim that

(i) for $n = 1, 2, \ldots$, the sets $G^n$ are pairwise disjoint with $m(G^n) > 0$;
(ii) for each $n > 1$, $\mathring{T}(G^n) = G^{n-1}$ and $T^{-1}(G^{n-1}) \cap G = G^n$.

To prove (ii), note that $x \in G^n$ implies $\mathring{T}(x) \in E^{n-1}$ and if $\{z_i\}_{i=1}^{\infty}$ is a sequence of preimages with $z_i \in \mathring{M}^i$ and $\mathring{T}^i(z_i) = x$, then $\{\mathring{T}(z_i)\}_{i=1}^{\infty} \subset \mathring{M}$ is the required sequence of primages of $\mathring{T}(x)$ so that $\mathring{T}(x) \in G^{n-1}$. This implies that $T(G^n) \subset G^{n-1}$ and since the above argument holds for any $y \in T^{-1}(\mathring{T}x) \cap G$, also $G^n \subset T^{-1}(G^{n-1}) \cap G$.

Similarly, consider $x \in G^{n-1}$ and an associated sequence of preimages $\{z_i\}_{i=1}^{\infty}$. Then $z_i \in G$ as well using the same sequence. In particular, $z_1 \in G^n$ so that $x \in T(G^n)$, proving both reverse inequalities. This proves (ii). As for (i), the sets $G^n$ are pairwise disjoint because the $E^n$ are, and the fact that they have positive measure follows inductively from $m(G^1) > 0$, $\mathring{T}(G^n) = G^{n-1}$ and $JT > 0$.

Now fix $\lambda$ such that $0 \leq \lambda < 1$, and let $\psi \geq 0$ be an integrable function with $\int_{G^1} \psi dm = 1$. On $G^1$, define

$$f(x) = (1 - \lambda)\psi(x).$$

Note that $\int_{G^1} f dm = 1 - \lambda$. This is the amount that falls in the hole at time 1.

Proceeding inductively, suppose $f$ has been defined on $G^{n-1}$. For $y \in G^n$, we let $\mathring{T}(y) = x$, and define

$$f(y) = \frac{\lambda f(x)}{\displaystyle\sum_{z\in \mathring{T}^{-1}x\cap G} \frac{1}{JT(z)}} = \frac{\lambda f(x)}{\mathring{\mathcal{L}}(1_G)}, \tag{2.2}$$

so that $f$ is constant on $\mathring{T}^{-1}x \cap G$. Set $f = 0$ on $M \setminus \cup_n G^n$.

To prove conditional invariance with escape rate $-\log\lambda$, it suffices, in light of (i), (ii) and the definition of $f$ on $G^1$, to check that for each $n > 1$,

$$\mathcal{L}(f|G^n) = \lambda\ f|G^{n-1}. \tag{2.3}$$

This follows from the way we have defined $f$ in (2.2).

Since $G \subset T^{-1}G$, (ii) implies inductively that $G^n = \mathring{T}^{-n+1}G^1 \cap G$. Using this together with (2.3), we conclude

$$\int_M f dm \;=\; \sum_{n=1}^{\infty}\int_{G^n} f dm \;=\; \sum_{n=1}^{\infty}\int_{G^1}\mathring{\mathcal{L}}^n f\, dm \;=\; \sum_{n=1}^{\infty}\lambda^{n-1}\int_{G^1} f dm \;=\; \sum_{n=0}^{\infty}\lambda^n(1-\lambda) \;=\; 1,$$

so that $f$ as defined is a probability density. In the second equality we have used the fact that $f = 0$ on $(\mathring{T}^{-n+1}G^1) \setminus G$. Since the choice of $\psi$ is entirely arbitrary, the construction above gives uncountably many *a.c.c.i.m.* as desired. □

**Remark 2.3.** *As the proof above shows, a.c.c.i.m. can be constructed quite arbitrarily – with overlapping supports if one so desires – once a suitable sequence of sets in $M$ is located. Analogously, one can construct arbitrary* a.c.c.i.m. *for invertible systems supported on the singular sets $M \setminus \cup_{i=0}^{\infty} T^i(H)$, as in Example 2.*

We conclude that absolute continuity with respect to $m$ alone is not a sufficient condition for identifying a meaningful class of conditionally invariant measures (contrast this with the fact that ergodic invariant measures for closed systems cannot have overlapping supports). This result highlights the importance of **(Q2)** from Section 1.1: Characterizing physically relevant *a.c.c.i.m.* as limiting distributions for a class of initial measures.

### 2.4 Examples

We revisit Example 1 in Sect. 2.2, the tent map with a hole. Observe that in this example, $G = [0,1] \setminus H$, and for each $n \geq 1$, $G^n = E^n$ is the union of $2^n$ disjoint intervals of length $(\frac{1-\varepsilon}{2})^n$ each. The following are three examples of *a.c.c.i.m.* obtained from the constructions in the proof of Theorem 2.2.

(i) $\lambda = 1-\varepsilon$ and $\psi = \frac{1}{m(E^1)}$. The construction in Theorem 2.2 gives $\mu$ = normalized Lebesgue measure on $[0,1]\setminus H$. It is easy to check that $\lambda = \Lambda$ where $-\log\Lambda$ is the escape rate of Lebesgue measure.

(ii) Let $E^1 =: [a_1, b_1] \cup [a_2, b_2]$. On $[a_1, b_1]$, we define $\psi(x) = \frac{1}{m(E^1)}\left(1 + c\sin\frac{2\pi(x-a_1)}{b_1-a_1}\right)$; $\psi$ is defined analogously on $[a_2, b_2]$. Again choose $\lambda = 1-\varepsilon$. For $|c| < 1$, the construction in Theorem 2.2 yields an *a.c.c.i.m.* with density bounded away from zero and infinity (so it follows that $\lambda = \Lambda$, independently of (i)). The density, however, is not of bounded variation.[3]

(iii) Choose $\alpha < \frac{1}{2}$ and set $f = \alpha^n$ on $E^n$ for $n \geq 1$. The resulting conditionally invariant measure has a density of bounded variation and $\lambda = \alpha(1-\varepsilon)$ so that its escape rate is strictly greater than $-\log\Lambda$. (This construction is due to Chernov and van den Bedem; see [BC].)

[3]We mention the idea of bounded variation because it is well known that densities in this class are preserved by piecewise expanding maps (without holes). The connection will become clear in Section 4.

**Problem 2.** *Consider the triadic baker map with $H = V_2$. Using the ideas of Theorem 2.2, for any $0 \le \lambda < 1$, construct a family of conditionally invariant measures with eigenvalue $\lambda$ and having absolutely continuous conditional measures on unstable curves. This illustrates that the ideas of Theorem 2.2 apply to invertible systems as well, even though* a.c.c.i.m. *for such systems are necessarily singular with respect to Lebesgue.*

# 3 Connection Between Escape Rate and Pressure: Holes of Any Size

In this section we focus on inequalities and equalities relating escape rates and pressure and address primarily motivating questions **(Q3)** and **(Q4)**. We do not assume the holes are small and so do not take a perturbative view of the systems in question.

## 3.1 Entropy and Lyapunov Exponents

We begin by recalling some essential definitions needed for the discussion that follows. A more complete treatment of these topics can be found, for example, in [W, KH].

**I. Entropy.** Suppose $(X, \mathcal{B}, \mu)$ is a probability space. A *partition* of $X$ is a pairwise disjoint collection of elements of $\mathcal{B}$ whose union is $X$.

Given two partitions $\alpha$ and $\beta$, the *join* of $\alpha$ and $\beta$ is defined by,

$$\alpha \vee \beta = \{A_i \cap B_j : A_i \in \alpha, B_j \in \beta\}.$$

This definition can be extended to the join of multiple partitions $\alpha \vee \beta \vee \gamma$ in the obvious way.

The *entropy of a partition* $\alpha$ is given by

$$H(\alpha) = -\sum_{A_i \in \alpha} \mu(A_i) \log \mu(A_i).$$

If $T$ is a measure-preserving transformation of $X$, then $T^{-n}\alpha$ denotes the partition $\{T^{-n}A_i : A_i \in \alpha\}$. The *entropy of $T$ with respect to $\alpha$* is defined by

$$h_\mu(T, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right),$$

while the *entropy of $T$* is defined as,

$$h_\mu(T) = \sup\{h_\mu(T, \alpha) : \alpha \text{ is a finite partition of } X\}.$$

Intuitively, one can think of the entropy as capturing the rate of increase in complexity under the dynamics of $T$. Thus if the process of dynamically refining a partition $\alpha$, is 'chopping' $X$ into roughly $e^{nh_\mu(T)}$ pieces, one might expect that the measure of a typical pieces is roughly $e^{-nh_\mu(T)}$. This intuition is formalized by the Shannon-MacMillan-Breiman theorem. We do not state this theorem, but an extension of it that we will use.

Given a transformation $T$ of a compact metric space $X$ and a function $g$ on $X$, define the dynamical (Bowen) balls

$$B(x, n, g) = \{y \in M : d(T^i x, T^i y) < g(T^i x), 0 \le i \le n\}.$$

**Theorem 3.1.** *Let $T : X \circlearrowleft$ be a measurable transformation of a compact metric space of finite capacity,*[4] *and let $\mu$ be an ergodic invariant measure for $T$. Let $\hat{g}_\varepsilon : X \to \mathbb{R}$ be a family of functions satisfying $|\hat{g}_\varepsilon|_\infty \le C\varepsilon$ and $\int_X -\log \hat{g}_\varepsilon \, d\mu < \infty$, for some $C > 0$. Then for $\mu$-a.e. $x$,*

$$\lim_{\varepsilon\to 0^+} \liminf_{n\to\infty} -\frac{1}{n}\log \mu(B(x,n,\hat{g}_\varepsilon)) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} -\frac{1}{n}\log \mu(B(x,n,\hat{g}_\varepsilon)) = h_\mu(T).$$

The above theorem follows from [M, Lemma 2] and [BrK, Main Theorem].

**II. Lyapunov exponents.**

Given an invertible (piecewise) differentiable[5] map $T$ of a manifold $M$ of dimension $d$, one defines,

$$\overline{\lambda}_+(x,v) = \limsup_{n\to\infty} \frac{1}{n}\log \|DT^n(x)v\|, \text{ and } \underline{\lambda}_+(x,v) = \liminf_{n\to\infty} \frac{1}{n}\log \|DT^n(x)v\|,$$

for any $v \in \mathbb{R}^d$ and $x \in M$. When $\overline{\lambda}_+ = \underline{\lambda}_+$, we define the common value to be $\lambda_+$. Similar definitions hold for $\overline{\lambda}_-$, $\underline{\lambda}_-$ and $\lambda_-$ with $DT^n$ replaced by $DT^{-n}$.

**Theorem 3.2** (Oseledec's Theorem, invertible version [O])**.** *Let $T : M \circlearrowleft$ be as above and let $\mu$ be an invariant measure for $T$. At $\mu$-a.e. $x \in M$, there exist numbers $\lambda_1(x) < \cdots < \lambda_{r(x)}(x)$ and subspaces $E_1(x), \ldots, E_{r(x)}(x)$, such that*

*(i)* $\mathbb{R}^d = E_1(x) \oplus \cdots \oplus E_{r(x)}(x)$;

*(ii)* *for all* $v \in E_i(x)$, $\lambda_+(x,v) = -\lambda_-(x,v) = \lambda_i(x)$;

*(iii)* *for* $i \ne j$, $\lim_{n\to\infty} \frac{1}{n}\log|\sin\measuredangle(DT^{\pm n}E_i(x), DT^{\pm n}E_j(x))| = 0$.

*Moreover, the functions $r$, $\lambda_i$ and $E_i$ are measureable.*

The numbers $\lambda_i(x)$ are called the *Lyapunov exponents* of $(T,\mu)$. They measure the exponential rate of contraction and expansion in dynamical directions preserved by $T$. The multiplicity of $\lambda_i$ is the dimension of $E_i$. A quantity of interest will be the sum of positive Lyapunov exponents for $(T,\mu)$ counted with multiplicity, which we will denote by $\chi^+(x)$. When $\mu$ is ergodic, the functions $r(x)$ and $\lambda_i(x)$ are constant for $\mu$-a.e. $x$ and we define $\chi^+_\mu$ to be the sum of positive Lyapunov exponents, constanst almost everywhere.

## 3.2 Relation to Escape

**Example 3. Baker Map Revisited.** Consider the triadic baker map of Example 2 in Section 2.2 with hole $H = V_2$. As mentioned there, $\mathring{M}^n$ is a union of $2^{n+1}$ vertical rectangles each of height 1 and width $3^{-n-1}$. Thus $m(\mathring{M}^n) = (2/3)^{n+1}$ so that

$$\rho(m) = \log(2/3) = \log 2 - \log 3.$$

The survivor set for this example is $\mathring{M}^\infty = \Gamma \times \Gamma$, where $\Gamma$ is the middle thirds Cantor set. The dynamics on this set is equivalent to the full two-sided shift on two symbols (simply identify $x$ with the sequence $(i_j)_{j=-\infty}^{\infty}$, $i_j \in \{1,3\}$, where $T^j(x) \in V_{i_j}$ for $j \in \mathbb{N}$ and $T^{-j}(x) \in H_{i_j}$ for

[4]Finite capacity means there exists $d < \infty$ such that $\limsup_{r\to 0} \frac{\log C(r)}{-\log r} = d$ where $C(r)$ is the minimum cardinality of a covering of $X$ by open balls of radius $r$.

[5]For maps with singularities, one requires that $\int \log^+ \|DT\| d\mu < \infty$ and $\int \log^+ \|DT^{-1}\| d\mu < \infty$, where $\mu$ is an invariant measure of interest.

$j \in \mathbb{N}^+$). In terms of the formula given above, $\log 2$ is the topological entropy of the full shift on two symbols (also the measure-theoretic entropy of the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift), while $\log 3$ is the positive Lyapunov exponent for each $x \in \mathring{M}^\infty$. Thus one sees very easily in this example why $\rho(m) = \mathcal{P}_{\mathcal{I}_H}$, where $\mathcal{I}_H$ is the set of invariant measures on $\mathring{M}^\infty$.

**Problem 3.** *Consider a linear toral automorphism $T : \mathbb{T}^2 \circlearrowleft$ whose stable and unstable directions are mutually perpendicular. Let $\mathcal{R} = \{R_1, R_2, \dots R_q\}$ be a Markov partition for $T$ whose elements are (real) rectangles. Let $H = R_q$.*

*Define $Q = (q_{i,j})$ to be the adjacency matrix for $T$: $q_{i,j} = 1$ if $m(R_i \cap T^{-1}R_j) > 0$; $q_{i,j} = 0$ otherwise. Let $\mathring{Q}$ denote the $(q-1) \times (q-1)$ matrix obtained by deleting the $q$th row and column of $Q$. Assume that $\mathring{Q}$ is irreducible and aperiodic.*

*Prove that $\frac{\mathring{T}_*^n m}{m(\mathring{M}^n)} \to \mu$ as $n \to \infty$, where on each $R_k$, $\mu$ is the product of Lebesgue measure on unstable segments and a measure supported on a Cantor set in the stable direction.*

*Show also that the escape rate with respect to Lebesgue, $-\log \Lambda$, exists and equals $\beta\alpha^{-1}$, where $\beta$ is the largest eigenvalue of $\mathring{Q}$ and $\alpha > 1$ is the derivative of $T$ in the unstable direction.*

Now let $T$ be a diffeomorhphism of a smooth compact, Riemannian manifold $M$, and let $H \subset M$ be an open set with piecewise smooth boundary. Let $\mathcal{E}_H$ denote those ergodic invariant measures supported on $\mathring{M}^\infty$. Define

$$\mathcal{G}_H = \{\nu \in \mathcal{E}_H : \exists C, \gamma > 0 \text{ such that } \forall \varepsilon > 0,\, \nu(N_\varepsilon(\partial H)) \le C\varepsilon^\gamma\}, \tag{3.1}$$

where $N_\varepsilon(\cdot)$ denotes the $\varepsilon$-neighborhood of a set. $\mathcal{G}_H$ consists of invariant measures supported on $\mathring{M}^\infty$ that do not give too much weight to the boundary of $H$. This restriction essentially treats $\partial H$ as part of the singularity set for $T$ and the restriction is similar to that used to obtain pressure results for maps with singularities (and no holes). See, for example, [KS].

The following theorem generalizes earlier results of Bowen [Bo2] and Young [Y1] regarding escape from neighborhoods of uniformly hyperbolic invariant sets.

**Theorem 3.3** ([DWY2]). *Let $T$ and $H$ be as above and let $m$ denote Lebesgue measure. Then*

$$\underline{\rho}(m) \ge \mathcal{P}_{\mathcal{G}_H} = \sup_{\nu \in \mathcal{G}_H} \{h_\nu(T) - \chi_\nu^+\}.$$

Setting $H = \emptyset$, one has $0 \le \mathcal{P}_{\mathcal{E}_H}$, which holds in great generality for closed systems. For (piecewise) hyperbolic systems, the equality $0 = \mathcal{P}_{\mathcal{E}_H}$ is known as Pesin's formula.

*Proof of Theorem 3.3.* We give an idea of the proof, which uses Theorem 3.1 in addition to the following volume lemma.

**Proposition 3.4.** *Let $g_\varepsilon = \frac{1}{3}\min\{\varepsilon, d(x, \partial H)\}$. Given $\nu \in \mathcal{G}_H$, there exists a measurable set $E \subset \mathring{M}^\infty$ with $\nu(E) > 0$ such that for every $x \in E$,*

$$\sup_{\varepsilon > 0} \limsup_{n \to \infty} -\frac{1}{n} \log m(B(x, n, g_\varepsilon)) \le \chi_\nu^+.$$

This volume lemma holds in a number of contexts, including maps with singularities and diffeomorphisms of manifolds, even allowing for 0 Lyapunov exponents. For an example of its proof in several contexts, see [DWY2, Section 4]. In fact, the inequality is often an equality, but we do not need the reverse inequality for the present theorem.

To proceed with the proof of the theorem, we fix $\nu \in \mathcal{G}_H$ and show that $\hat{g}_\varepsilon = 3g_\varepsilon$ satisfies the hypotheses of Theorem 3.1. Using the fact that $\nu(N_\varepsilon(\partial H)) \le C\varepsilon^\gamma$ for some $C, \gamma > 0$ and all $\varepsilon > 0$, we write,

$$\begin{aligned}\int_M -\log \hat{g}_\varepsilon \, d\nu &\le -\log\varepsilon + \sum_{n=0}^{\infty} \nu(N_{\varepsilon e^{-n}}(\partial H) \setminus N_{\varepsilon e^{-(n+1)}}(\partial H))(n+1-\log\varepsilon) \\ &\le -\log\varepsilon + \sum_{n=0}^{\infty} C\varepsilon^\gamma e^{-\gamma n}(n+1-\log\varepsilon) < \infty.\end{aligned}$$

Next fix $\delta > 0$, and let $\sigma := \nu(E)$ where $E$ is given by Proposition 3.4. By Theorem 3.1, we may choose first $\varepsilon > 0$ sufficiently small, and then $n_0 = n_0(\delta, \varepsilon) \in \mathbb{Z}^+$ sufficiently large and a measurable set $E' \subset E$ with $\nu(E') \ge \sigma/2$ such that for every $x \in E'$,

(i) $\nu(B(x, n, 3g_\varepsilon)) \le e^{-n(h_\nu - \delta)}$ for all $n \ge n_0$;

(ii) $m(B(x, n, g_\varepsilon)) \ge e^{-n(\chi_\nu^+ + \delta)}$ for all $n \ge n_0$.

For $n \ge n_0$, let $\mathcal{C}_n \subset E'$ be a maximal set of points such that $B(x_i, n, g_\varepsilon) \cap B(x_j, n, g_\varepsilon) = \emptyset$ whenever $x_i, x_j \in \mathcal{C}_n$, $x_i \neq x_j$. By the maximality of $\mathcal{C}_n$, for every $y \in E'$, there exists $x_i \in \mathcal{C}_n$ such that $B(y, n, g_\varepsilon) \cap B(x_i, n, g_\varepsilon) \neq \emptyset$. We will show that $y \in B(x_i, n, 3g_\varepsilon)$. This will imply $E' \subset \cup_{x_i \in \mathcal{C}_n} B(x_i, n, 3g_\varepsilon)$, and hence $\#\mathcal{C}_n \ge \frac{\sigma}{2} e^{n(h_\mu - \delta)}$ by (i), where $\#\mathcal{C}_n$ denotes the cardinality of the set.

To show $y \in B(x_i, n, 3g_\varepsilon)$, it suffices to show $d(f^k y, f^k x_i) < 3g_\varepsilon(f^k x_i)$ $\forall k \le n$, since $y \in E' \subset \mathring{M}^n$. Now $B(y, n, g_\varepsilon) \cap B(x_i, n, g_\varepsilon) \neq \emptyset$ means there exists $z \in M$ such that $d(f^k x_i, f^k z) \le g_\varepsilon(f^k x_i)$ and $d(f^k z, f^k y) \le g_\varepsilon(f^k y)$ for all $0 \le k \le n$. Thus the assertion above boils down to the following lemma.

**Lemma 3.5.** *For any $x, y \in M$, if there exists $z \in M$ with $d(x, z) \le g_\varepsilon(x)$ and $d(z, y) \le g_\varepsilon(y)$, then $d(x, y) \le 3g_\varepsilon(x)$.*

*Proof of Lemma.* It suffices to show $g_\varepsilon(y) \le 2g_\varepsilon(x)$, for that will imply $d(x, y) \le d(x, z) + d(z, y) \le g_\varepsilon(x) + g_\varepsilon(y) \le 3g_\varepsilon(x)$, proving the lemma. Observe that

$$\begin{aligned}d(y, \partial H) &\le d(y, z) + d(z, x) + d(x, \partial H) \\ &\le g_\varepsilon(y) + g_\varepsilon(x) + d(x, \partial H) \;\le\; \tfrac{1}{3} d(y, \partial H) + \tfrac{4}{3} d(x, \partial H),\end{aligned}$$

the last inequality following from $g_\varepsilon(\cdot) \le \frac{1}{3} d(\cdot, \partial H)$. Altogether, this gives $d(y, \partial H) \le 2d(x, \partial H)$.

To finish, consider the following two cases:

Case 1: $d(x, \partial H) > \varepsilon$. With $g_\varepsilon(x) = \frac{1}{3}\varepsilon$, $g_\varepsilon(y)$ is automatically $< 2g_\varepsilon(x)$ since it is $\le \frac{1}{3}\varepsilon$.

Case 2: $d(x, \partial H) \le \varepsilon$. In this case $g_\varepsilon(y) \le \frac{1}{3} d(y, \partial H) \le \frac{2}{3} d(x, \partial H) = 2g_\varepsilon(x)$. □

For each $x \in E'$, we have $B(x, n, g_\varepsilon) \subset \mathring{M}^n$ by definition of $g_\varepsilon$. Since the $B(x_i, n, g_\varepsilon)$ are disjoint, we may estimate $m(\mathring{M}^n)$ by

$$m(\mathring{M}^n) \;\ge\; \sum_{x_i \in \mathcal{C}_n} m(B(x_i, n, g_\varepsilon)) \;\ge\; \#\mathcal{C}_n \cdot \min_{x_i \in \mathcal{C}_n} m(B(x_i, n, g_\varepsilon)) \;\ge\; \frac{\sigma}{2} e^{n(h_\nu - \delta)} e^{-n(\chi_\nu^+ + \delta)}.$$

This yields

$$\liminf_{n \to \infty} \frac{1}{n} \log m(\mathring{M}^n) \ge h_\nu(T) - \chi_\nu^+ - 2\delta.$$

The theorem is proved since $\delta$ was chosen arbitrarily. □

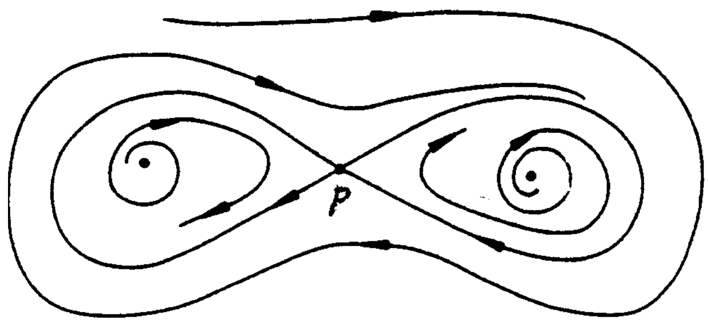


Figure 2: The Figure 8 attractor. The only invariant measure is $\delta_p$, where $p$ is the saddle point.

Theorem 3.3 can be generalized in a number of ways. For example, one can vary the initial distribution. For $\varphi \in L^1(m)$, $\varphi \geq 0$, define $\mu_\varphi := \varphi m$. Define

$$\mathcal{G}_\varphi = \{\nu \in \mathcal{E}_H \text{ for which } \exists c_\nu > 0 \text{ and an open set } O \text{ such that } \nu(O) > 0 \text{ and } \varphi|_O \geq c_\nu\}.$$

Then, $\underline{\rho}(\mu_\varphi) \geq \mathcal{P}_{\mathcal{G}_H \cap \mathcal{G}_\varphi}$ ([DWY2, Theorem A]).

Alternatively, for a map with a singularity set $\mathcal{S}$, one can define $\mathcal{G}_\mathcal{S}$ as in (3.1), but with $N_\varepsilon(\partial H)$ repleced by $N_\varepsilon(\mathcal{S})$. One can then show, $\underline{\rho}(\mu_\varphi) \geq \mathcal{P}_{\mathcal{G}_H \cap \mathcal{G}_\varphi \cap \mathcal{G}_\mathcal{S}}$ ([DWY2, Theorem C]).

Similarly, if one is interested in escape rates with respect to SRB measures that are singular with respect to Lebesgue, one can adopt suitable restrictions on $\mathcal{G}$ to measures which 'see' the SRB measure. For details, see [DWY2, Theorem B].

**Remark 3.6.** *Upper bounds on $\rho(m)$ do not hold in as great generality as the results discussed above. Consider the case when $\Omega \subset M$ is an attractor and assume there exists an open neighborhood $O$ of $\Omega$ such that $T(\overline{O}) \subset O$ and $\Omega = \cap_{n \geq 0} T^n(O)$. Let $H = M \setminus O$. Then immediately, $\rho(m) = 0$. Since $h_\nu(T) \leq \chi_\nu^+$ for all $\nu \in \mathcal{E}_H$ [R], showing that $\rho(m) \leq \mathcal{P}_\mathcal{G}$ for some class of invariant measures $\mathcal{G}$ in this case is equivalent to proving $0 = \mathcal{P}_\mathcal{E}$, which is false in general. The Figure 8 attractor is an example for which $\mathcal{P}_\mathcal{E} < 0$ while $\rho(m) = 0$ if the open set $O$ is some neighborhood of the attractor (see Figure 2). For a more extensive example, see [BBS].*

### 3.3 Special case: Anosov diffeomorphisms

With Remark 3.6 in mind, we next consider both upper and lower bounds for the escape rate in a uniformly hyperbolic setting. In this section, $T$ is a $C^{1+\varepsilon}$ Anosov diffeomorphism of a compact manifold $M$ and $H$ is an open set with finitely many components. Let $\mathcal{H}$ denote the collection of such holes in $M$ endowed with the topology induced by the Hausdorff metric. The sets $\mathcal{G}_H$ and $\mathcal{E}_H$ are defined as in the previous section.

Anosov diffeomorphisms with holes that are elements of a Markov partition were considered in [CM1, CM2] and later extended to small non-Markov holes in [CMT1, CMT2] by approximation. In the current setting, we do not assume the holes are elements of a Markov partition and we do not assume they are small.

**Theorem 3.7** ([DW])**.** *Let $T : M \circlearrowleft$ be a $C^{1+\varepsilon}$ Anosov diffeomorphism with hole $H \in \mathcal{H}$. Then*

$$\mathcal{P}_{\mathcal{G}_H} \leq \underline{\rho}(m) \leq \overline{\rho}(m) \leq \mathcal{P}_{\mathcal{E}_H}.$$

*If in addition $\partial H \cap \mathring{M}^\infty = \emptyset$, then $\rho(m)$ is well-defined and equals $\mathcal{P}_{\mathcal{G}_H} = \mathcal{P}_{\mathcal{E}_H}$.*

*Proof.* The lower bound follows from Theorem 3.3. The upper bound follows from approximating $H \in \mathcal{H}$ by an increasing sequence of 'Markov' holes $H_k \subset H$, i.e. each $H_k$ is a finite union of elements of a Markov partition for $T$. Since $T$ admits Markov partitions with arbitrarily fine diameter, we may choose $H_k$ so that $\cup_{k\geq 1} H_k = H$.

Set $\mathring{M}_k = M \backslash H_k$ and in general denote by the subscript $k$ objects associated with $H_k$. Now $\mathring{M}_k$ is a decreasing sequence of sets with $\cap_{k\geq 1}\mathring{M}_k = \mathring{M}$ and similarly for the survivor sets $\mathring{M}^\infty = \cap_{k\geq 1}\mathring{M}_k^\infty$. Since $H_k$ is a Markov hole, the results of [CM2] imply that there exists an invariant measure $\nu_k$ supported on $\mathring{M}_k^\infty$ such that $\rho(m) = P_{\nu_k}$.

Let $\nu$ be a limit point of the $\nu_k$. Then $\nu$ is an invariant measure supported on $\mathring{M}^\infty$. Moreover, letting $E^u(x)$ denote the unstable subspace for $T$ at $x$, since $\log|\det(DT|_{E^u})(x)|$ is continuous, we have $\lim_{k\to\infty}\int \chi^+ d\nu_k = \int \chi^+ d\nu$.

In addition, $h_\nu(T) \geq \limsup_{k\to\infty} h_{\nu_k}(T)$ due to the expansiveness[6] of $T$ [Bo1]. The brief proof is included here for convenience. Since $T$ is expansive, there exists $\varepsilon > 0$ such that if $\alpha$ is a finite measurable partition of $M$ with $\mathrm{diam}(\alpha) < \varepsilon$, then $h_\eta(f,\alpha) = h_\eta(f)$ for any invariant Borel measure $\eta$. Fix such a partition $\alpha$ with $\nu(\partial\alpha) = 0$. Let $H_\eta(\alpha_n)$ denote the entropy of the partition $\alpha_n = \bigvee_{i=-n}^n T^i\alpha$ with respect to a measure $\eta$, and for $\delta > 0$ choose $n$ such that $\frac{1}{n}H_\nu(\alpha_n) \leq h_\nu(T)+\delta$. Then since $\frac{1}{n}H_\eta(\alpha_n)$ is a decreasing function of $n$ for any $\eta$, we have

$$\limsup_{k\to\infty} h_{\nu_k}(T) = \limsup_{k\to\infty} h_{\nu_k}(T,\alpha) \leq \limsup_{k\to\infty}\frac{1}{n}H_{\nu_k}(\alpha_n) = \lim_{k\to\infty}\frac{1}{n}H_\nu(\alpha_n) \leq h_\nu(T) + \delta,$$

proving the claim, since $\delta > 0$ is arbitrary.

We have shown that

$$P_\nu \geq \limsup_{k\to\infty} P_{\nu_k} = \limsup_{k\to\infty}\rho_k(m) \geq \overline{\rho}(m),$$

where the last inequality is true by monotonicity: $\mathring{M}_k \supset \mathring{M}$ for each $k$. By the ergodic decomposition, there exists a measure $\pi_\nu$ on the set $\mathcal{I}(\mathring{M}^\infty)$ of invariant Borel probability measures on $\mathring{M}^\infty$ such that $\nu = \int_{\mathcal{I}} \eta\, d\pi_\nu(\eta)$. In fact, since $T$ and $\log|\det(DT|_{E^u})|$ are continuous, we have $h_\nu(T) - \int\chi^+ d\nu = \int_{\mathcal{I}}(h_\eta(T) - \int\chi^+ d\eta)\, d\pi_\nu(\eta)$ (see for example [W, Theorem 8.4]), so there must exist an ergodic measure $\eta \in \mathcal{E}_H$ such that $P_\eta \geq \overline{\rho}(m)$.

To prove the last statement of the theorem, notice that since $\partial H$ and $\mathring{M}^\infty$ are compact, the requirement $\partial H \cap \mathring{M}^\infty = \emptyset$ is equivalent to $d(\partial H \cap \mathring{M}^\infty) > 0$. Thus $\mathcal{G}_H = \mathcal{E}_H$ in this case, completing the proof of the theorem. $\square$

Although simple and quite general, Theorem 3.7 gives a surprising amount of information about the structure of the escape rate as a function of the hole $H$. Our next set of results are taken from [DW] and illustrate the essential consequences of this connection between pressure and escape. For simplicity, we will restrict ourselves to the case when $M$ is two-dimensional. In two dimensions, we call a hole $H \in \mathcal{H}$ *regular* if its boundary comprises a finite union of stable and unstable manifolds. The main ideas are a consequence of the following observations.

- The condition $\partial H \cap \mathring{M}^\infty = \emptyset$ is typically satisfied: If $T$ is topologically transitive, it holds for an open and dense set of holes and for a full measure set of parameters along sequences of regular holes.

- Exceptional situations do occur; indeed, these exceptions cause the escape rate to vary - otherwise, it remains locally constant.

[6] Expansive means there exists $\varepsilon_0 > 0$ such that for any $x, y \in M$, $x \neq y$, there exists $i = i(x,y) \in \mathbb{Z}$ such that $d(T^i x, T^i y) > \varepsilon_0$.

Observe that while the assumption that $H$ be regular places a strong restriction on the boundary, this condition is satisfied by any hole that is a union of elements of a Markov partition. Thus this class of holes contains all holes compatible with any Markov coding that can be used to study the system.

**Proposition 3.8.** *Let $\dim(M) = 2$, $T$ be topologically transitive and suppose $\{H_t\}_{t \in I} \subset \mathcal{H}$ is a sequence of regular holes such that the number of smooth components of $\partial H_t$ is uniformly bounded on $I$ and $t \mapsto \partial H_t$ is continuous. Assume,*

> *For any subinterval $J \subseteq I$ and any curve $\gamma$ locally transverse to $\{\partial H_t\}_{t \in J}$, if $E \subset J$ has positive Lebesgue measure in I, then $\{\gamma \cap \partial H_t\}_{t \in E}$ has positive Lebesgue measure on $\gamma$.*

*Let $-\underline{\rho}(t)$ and $-\overline{\rho}(t)$ denote the upper and lower escape rates from $M \setminus H_t$ with respect to Lebesgue. Then*

*(a) $\mathring{M}^\infty(H_t) \cap \partial H_t = \emptyset$ for an open and dense set of $t \in I$ and the exceptional set has zero Lebesgue measure in $I$. As a consequence, $\rho(t)$ exists and is locally constant on an open and full measure set of $t$.*

*Now assume that $\{H_t\}_{t \in I}$ is monotonically increasing. Then*

*(b) the functions $t \mapsto \underline{\rho}(t)$ and $\overline{\rho}(t)$ are monotonically decreasing and each forms a devil's staircase, possibly with jumps;*

*(c) $\underline{\rho}(\cdot)$ and $\overline{\rho}(\cdot)$ are in general neither upper nor lower semi-continuous once they are out of the small hole regime;*

*(d) if $\overline{\rho}(\cdot)$ is lower semi-continuous at $t$, then $\rho(t)$ exists;*

*(e) if $\underline{\rho}(\cdot)$ is upper semi-continuous at $t$, then $\rho(t)$ exists.*

Note that statement (a), the sequence $\{H_t\}_{t \in I}$ is neither assumed to converge to a point nor to be monotonic. Also, statements (d) and (e) imply that $\rho(t)$ typically exists even when $\partial H_t \cap \mathring{M}^\infty(H_t) \neq \emptyset$. The only values of $t$ at which $\rho(t)$ may not exist are those at which $\overline{\rho}(t)$ and $\underline{\rho}(t)$ jump and fail to be lower and upper semi-continuous, respectively. This can occur at most countably many times along the sequence.

**Remark 3.9.** *If one considers the recent results in [BY, KL2, FP] regarding the existence of the derivative of $\rho(t)$ in the zero hole limit (i.e. as $H_t$ shrinks to a point) in a number of hyperbolic settings, the picture of $\rho(t)$ that emerges from Proposition 3.8 is rather surprising. It indicates that along sequences of regular holes, $\rho(t)$ cannot be smooth on any interval containing $0$: Indeed $\rho(t)$ cannot even be absolutely continuous on any interval on which it is not constant.*

**Proposition 3.10.** *There are examples of Anosov diffeomorphisms with regular holes where*

*(a) $\mathcal{P}_{\mathcal{G}_H} < \rho(m) = \mathcal{P}_{\mathcal{E}_H}$; (b) $\mathcal{P}_{\mathcal{G}_H} = \rho(m) < \mathcal{P}_{\mathcal{E}_H}$; and (c) $\mathcal{P}_{\mathcal{G}_H} < \underline{\rho}(m) \le \overline{\rho}(m) < \mathcal{P}_{\mathcal{E}_H}$.*

**Problem 4.** *Construct examples of Anosov diffeomorphisms and holes $H$ to illustrate each of the items of Proposition 3.10.*

**Remark 3.11.** *We conclude with an observation on large versus small holes in hyperbolic systems. It follows from other techniques not presented here that for a large class of small holes, $t \mapsto \rho(t)$ is Hölder continuous. Evidently then, no invariant measure with pressure close enough to $0$ can be too concentrated near $\partial H$ for this class of holes. On the other hand, for larger holes invariant measures which maximize pressure can live on $\partial H$ and create jumps in the escape rate as demonstrated by Propositions 3.8 and 3.10.*

# 4 Small Holes: Spectral Methods

In this section we will consider small holes in uniformly hyperbolic systems taking the point of view that the open system is a perturbation of the closed system, in a sense that we shall make precise. For definiteness, we specialize our discussion to 1-dimensional piecewise expanding maps.

Let $I = [0, 1]$ and let $T : I \circlearrowleft$ be piecewise expanding, i.e. we assume there exists a finite partition of $I$ such that $T$ extends to a $C^2$ map on the closure of each partition element. Moreover, $|T'| \geq \sigma^{-1} > 2$. We assume for simplicity that $\sigma^{-1} > 2$. If it is not, one can always choose $n_0 \in \mathbb{N}$ such that $\sigma^{-n_0} > 2$ and then work with $T^{n_0}$.

Let $m$ denote Lebesgue measure on $I$. For an interval $J \subseteq I$, we define the *variation* of a function $f \in L^1(m)$ on $J = [a, b]$ by

$$\bigvee_J f = \sup_{\varphi \in \mathcal{K}(J)} \int f \varphi' \, dm,$$

where $\mathcal{K}(J) = \{\varphi \in C^1(J), |\varphi|_{\mathcal{C}^0(J)} \leq 1, \varphi(a) = \varphi(b) = 0\}$. Define the variation norm of $f$ to be $\|f\|_{BV} = \bigvee_I f + |f|_{L^1(m)}$, and let $BV = \{f \in L^1(m) : \|f\|_{BV} < \infty\}$ denote the functions of bounded variation on $I$.

It is a classical result that for this class of maps, the associated transfer operator $\mathcal{L}$ acting on functions of bounded variation is quasi-compact. This result relies on the following fundamental inequalities [LY]: There exists $C > 0$ such that for all $f \in BV$ and all $n \geq 0$,

$$\|\mathcal{L}^n f\|_{BV} \leq C\sigma^n \|f\|_{BV} + C|f|_{L^1(m)}, \qquad |\mathcal{L}^n f|_{L^1(m)} \leq |f|_{L^1(m)}. \tag{4.1}$$

The first inequality above is called a *Lasota-Yorke* or *Doeblin-Fortet* inequality. It says that due to the expansion of the map, the action of the transfer operator decreases the variation of a function, up to a 'weak' term depending on the nonlinearities and discontinuities of the map. The second inequality is a simple consequence of the fact that $\mathcal{L}$ gives the change of variables with respect to Lebesgue measure, $\int \mathcal{L} f \, dm = \int f \, dm$.

The contraction given by (4.1) together with the compactness of the unit ball of $BV$ in $L^1(m)$ yield the following theorem.

**Theorem 4.1** ([K])**.** *$\mathcal{L}$ acting on $BV$ is quasi-compact. It has spectral radius 1 and essential spectral radius $\leq \sigma$. The spectrum of $\mathcal{L}$ outside any disk of radius $\sigma' > \sigma$ consists of finitely many eigenvalues, each of finite multiplicity.*

*If $T$ is topologically mixing,[7] then $\mathcal{L}$ has a spectral gap: 1 is a simple eigenvalue and the rest of the spectrum is contained inside a disk of radius $\tau < 1$.*

*Comments about the proof.* The fact that the spectral radius is at most 1 is due to (4.1) together with the bound $|f|_{L^1(m)} \leq \|f\|_{BV}$. That it is at least 1 follows from, $\int \mathcal{L}^n 1 \, dm = \int 1 \, dm = 1$ $\forall n \in \mathbb{N}$. The bound on the essential spectral radius and resulting quasi-compactness follows from a general result of Hennion [H] using a formula for the essential spectral radius due to Nussbaum [N].

Once quasi-compactness is established, a spectral gap follows once one proves that 1 is simple and eliminates other eigenvalues on the unit circle. The mixing assumption is sufficient for this. $\square$

[7] This is equivalent to the *covering property*: For each interval $J \subset I$, $\exists n = n(J)$ such that $T^n(J) = I$ (mod 0) [L2].

A consequence of quasi-compactness is the following decomposition of $\mathcal{L}$. For each $\sigma' > \sigma$, there exists a number $N(\sigma')$ and mutually orthogonal projections $\Pi_i$, $i = 1, \dots N(\sigma')$, and $\mathcal{R}$, commuting with $\mathcal{L}$, such that $\mathcal{R} + \sum_{i=1}^{N(\sigma')} \Pi_i = \mathrm{Id}$ and $\mathrm{Rank}(\Pi_i) < \infty$. Then

$$\mathcal{L} = \sum_{i=1}^{N(\sigma')} \lambda_i \Pi_i \Lambda_i + \mathcal{R}\mathcal{L}, \tag{4.2}$$

where $\|\mathcal{R}\mathcal{L}^n\|_{BV} \le C\tau^n$, $\tau < |\lambda_i| \le 1$ and $\Lambda_i = \Pi_i + N_i$, where $N_i$ is nilpotent and $\Pi_i N_i = N_i \Pi_i = N_i$.

## 4.1 Piecewise Expanding Maps with Holes

With the above spectral picture in mind and recalling the discussion in Section 2.1, that the density of an *a.c.c.i.m.* is an eigenvector for the punctured transfer operator $\mathring{\mathcal{L}}$, the goal at this point will be to show that the nice spectral picture persists for the punctured transfer operator $\mathring{\mathcal{L}}$ for sufficiently small holes.

Unfortunately, in both $L^1(m)$ and $BV$, arbitrarily small holes are order 1 perturbations, thus standard perturbation theory will not apply.

**Lemma 4.2.** *Let $H \subset I$ be a finite union of intervals such that $m(I \setminus \mathring{I}^1) > 0$. Then $\|\mathcal{L} - \mathring{\mathcal{L}}\|_{BV} = O(1)$ and $|\mathcal{L} - \mathring{\mathcal{L}}|_{L^1(m)} = 1$.*

*Proof.* To prove the second claim, choose $f \in L^1(m)$, $f \ge 0$ so that $\mathrm{supp}(f) \subset I \setminus \mathring{I}^1$. Then using (2.1),

$$\int |(\mathcal{L} - \mathring{\mathcal{L}})f| \, dm = \int \mathcal{L}(1_{I \setminus \mathring{I}^1} f) \, dm = \int 1_{I \setminus \mathring{I}^1} f \, dm = |f|_{L^1(m)},$$

so that $|\mathcal{L} - \mathring{\mathcal{L}}|_{L^1(m)} = 1$. (Note that $|(\mathcal{L} - \mathring{\mathcal{L}})f|_{L^1(m)} \le |f|_{L^1(m)}$ so the norm cannot be more than 1.)

To prove the first claim, for $f \in BV$ and $\varphi \in \mathcal{K}(I)$ we write,

$$\int \varphi' (\mathcal{L} - \mathring{\mathcal{L}})(f) \, dm = \int \varphi' \mathcal{L}(1_{I \setminus \mathring{I}^1} f) \, dm = \int \varphi' \mathcal{L}(1_{H \cup T^{-1}H} f) \, dm \, .$$

Now $H \cup T^{-1}H$ is a finite union of intervals and if $H$ is small, then $\mathcal{L}(1_{H \cup T^{-1}H})$ will not be constant and indeed will have a jump of at least $\frac{1}{\max |T'|}$. So choosing $\varphi$ so that $\varphi'$ is large in a neighborhood of such a jump gives a lower bound on $\|\mathcal{L} - \mathring{\mathcal{L}}\|_{BV}$ of at least $\frac{1}{\max |T'|}$. Note however, that $\|\mathcal{L} - \mathring{\mathcal{L}}\|_{BV} < \infty$ since

$$\|(\mathcal{L} - \mathring{\mathcal{L}})f\|_{BV} \le \|\mathcal{L}(1_{H \cup T^{-1}H} f)\|_{BV} \le 2\|\mathcal{L}\|_{BV} \|1_{H \cup T^{-1}H}\|_{BV} \|f\|_{BV},$$

and $1_{H \cup T^{-1}H}$ has finite variation while $\|\mathcal{L}\|_{BV}$ is finite by (4.1).

In very special cases, if $H$ is very large, it may be that $\mathcal{L}(1_{H \cup T^{-1}H})$ is constant in $I$. In this case, we may choose $f$ so that $f$ has a jump of order 1 in the interior of $H \cup T^{-1}H$. Then $\mathcal{L}(1_{H \cup T^{-1}H} f)$ will have a jump of at least $\frac{1}{\max |T'|}$ which again gives a lower bound on the variation. $\square$

For an alternative perturbative approach, we turn to a framework created by Keller and Liverani [KL1], which allows one to consider a wider class of perturbations by viewing $\mathring{\mathcal{L}}$ as an operator from $BV$ to $L^1(m)$. The essential ingredients are as follows:[8]

[8] In fact, the requirements given in [KL1] are more general than those listed here, but we have simplified the requirements to clarify the exposition.

(1) (Compact embedding) A pair of Banach spaces, $(\mathcal{B}_s, \|\cdot\|_s)$ (strong) and $(\mathcal{B}_w, |\cdot|_w$ (weak), such that $|f|_w \le C\|f\|_s$ for all $f \in \mathcal{B}_s$ and the unit ball of $\mathcal{B}_s$ is compactly embedded in $\mathcal{B}_w$.

(2) (Uniform Lasota-Yorke inequlities) A family of bounded, linear operators $\{L_\varepsilon\}_{\varepsilon \ge 0}$ on $\mathcal{B}_s$ for which there exist constants $C > 0$ and $\varsigma < 1$ such that for all $n \ge 0$,

$$\|L_\varepsilon^n f\|_s \le C\varsigma^n \|f\|_s + C|f|_w$$
$$|L_\varepsilon^n f|_w \le C|f|_w \qquad \forall f \in \mathcal{B}_s$$

(3) (Closeness of the operators in a weak sense) There exists a monotone increasing function $\psi(\varepsilon)$, such that $|||L_0 - L_\varepsilon||| \le \psi(\varepsilon) \to 0$ as $\varepsilon \to 0$ where

$$|||L||| := \sup\{|Lf|_w : f \in \mathcal{B}_s, \|f\|_s \le 1\}.$$

**Theorem 4.3** ([KL1])**.** *Under conditions (1)–(3) listed above, the essential spectrum of all operators $L_\varepsilon$ is bounded by $\varsigma < 1$. Moreover, for $\varepsilon$ sufficiently small, the spectrum and spectral projectors of $L_\varepsilon$ outside the disk of radius $\varsigma$ vary as a Hölder continuous function of $\psi(\varepsilon)$ in the weak norm.*

**Remark 4.4.** *The theorem implies that if $\lambda_0$ is an isolated eigenvalue of $L_0$ with $|\lambda_0| > \varsigma$, then $\lambda_0$ has finite rank and for $\delta > 0$ sufficiently small, there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that for $\varepsilon < \varepsilon_0$, the spectrum of $L_\varepsilon$ has finitely many eigenvalues $\lambda_\varepsilon^{(j)}$ inside the disk of radius $\delta$ around $\lambda_0$ and the sum of the multiplicities of the $\lambda_\varepsilon^{(j)}$ equals the multiplicity of $\lambda_0$.*

*In particular then, if $L_0$ has a spectral gap, then so does $L_\varepsilon$ for $\varepsilon > 0$ sufficiently small.*

The importance of this result lies in the fact that closeness in the norm $|||\cdot|||$ is much weaker than closeness in $\|\cdot\|_s$ or $|\cdot|_w$, and so allows the development of perturbation theory for a much wider class of perturbations. In what follows, we define this norm with respect to BV and $L^1(m)$, i.e.

$$|||L||| := \sup\{|Lf|_{L^1(m)} : f \in BV, \|f\|_{BV} \le 1\}.$$

**Lemma 4.5.** *Let $H \subset I$ be a finite union of intervals. Then,*

$$|||\mathcal{L} - \mathring{\mathcal{L}}||| \le m(H \cup T^{-1}H).$$

*Proof.* Let $f \in BV$ with $\|f\|_{BV} \le 1$. Now

$$|(\mathcal{L} - \mathring{\mathcal{L}})f|_{L^1(m)} = \int \mathcal{L}(1_{I \setminus \mathring{I}^1} f)\, dm = \int 1_{H \cup T^{-1}H} f\, dm \le \|f\|_{BV} m(H \cup T^{-1}H),$$

using the fact that $|f|_\infty \le \|f\|_{BV}$. Taking the appropriate supremum completes the proof of the lemma. □

We consider families of holes $\{H_\varepsilon\}_{\varepsilon \ge 0}$ satisfying a certain uniform property. Let $\mathring{T}_\varepsilon$ denote the map corresponding to the hole $H_\varepsilon$ and $\mathring{\mathcal{L}}_\varepsilon$ denote the associated transfer operator for $\mathring{T}_\varepsilon$. We denote by $\mathcal{J}_\varepsilon$ the maximal partition of $\mathring{I}_\varepsilon$ into intervals on which $\mathring{T}_\varepsilon$ is smooth. We assume the following property of the family $\{H_\varepsilon\}$.

**(H)** $$\inf_{\varepsilon \ge 0} \inf_{J \in \mathcal{J}_\varepsilon} m(\mathring{T}_\varepsilon(J)) > 0.$$

**Corollary 4.6.** *Suppose $T$ is a piecewise expanding map of the interval such that $\mathcal{L}$ has a spectral gap on $BV$. Let $\{H_\varepsilon\}_{\varepsilon \geq 0}$ be a family of holes, each comprising finitely many intervals, with $m(H_\varepsilon) \to 0$ as $\varepsilon \to 0$ and satisfying property (H).*

*Then for $\varepsilon$ sufficiently small, $\mathring{\mathcal{L}}_\varepsilon$ has a spectral gap on $BV$.*

*Proof.* The corollary follows from Theorem 4.3 once we establish (1)-(3), with $L_\varepsilon = \mathring{\mathcal{L}}_\varepsilon$. Note that (1) holds for $\mathcal{B}_s = BV$ and $\mathcal{B}_w = L^1(m)$ and (3) follows from Lemma 4.5. It remains to verify (2).

The second inequality in (2) is immediate with $C = 1$. The first inequality also follows from standard estimates, yet it is important to note the uniformly of the constants in $\varepsilon$. Standard estimates show (see for example, [L2, eq. (2.1)]) that for $f \in BV$,

$$\bigvee_I \mathring{\mathcal{L}}_\varepsilon f \leq 2\sigma \bigvee_I f + \left( \sup_{x \in I} \frac{|D^2T(x)|}{|DT(x)|^2} + 2 \sup_{J \in \mathcal{J}_\varepsilon} \frac{1}{m(\mathring{T}_\varepsilon(J))} \Big( 1 + \frac{\sup_{x \in J} |D^2T(x)|}{\inf_{x \in J} |DT(x)|} \Big) \right) \int_I |f| \, dm. \quad (4.3)$$

Note that the only constant above that depends on $\varepsilon$ is $\sup_{J \in \mathcal{J}_\varepsilon} \frac{1}{m(\mathring{T}_\varepsilon(J))}$, which is bounded uniformly by assumption (H). Thus there exists $C_0 > 0$, independent of $\varepsilon$, such that the second term in (4.3) is bounded by $C_0 |f|_{L^1(m)}$. The inequality can then be iterated for any $n$, proving (2)

$$\bigvee_I \mathring{\mathcal{L}}^n_\varepsilon f \leq (2\sigma)^n \bigvee_I f + \frac{C_0}{1 - 2\sigma} |f|_{L^1(m)}.$$

□

The consequences of $\mathring{\mathcal{L}}_\varepsilon$ having a spectral gap are numerous and allow us to answer many of the motivating questions posed in Section 1.1 regarding limiting distributions and escape rates.

**Corollary 4.7.** *Suppose $H$ is a finite union of intervals and that $\mathring{\mathcal{L}}$ has a spectral gap. Let $\lambda < 1$ be the largest eigenvalue of $\mathring{\mathcal{L}}$. Then there exists $g \in BV$, with $g \geq 0$ and $|g|_{L^1(m)} = 1$ such that $\mathring{\mathcal{L}} g = \lambda g$. Moreover,*

*(a) There exists $\varsigma \in (0,1)$ and $C > 0$ such that for all $n \geq 0$ and $f \in BV$,*

$$\| \lambda^{-n} \mathring{\mathcal{L}}^n f - c(f) g \|_{BV} \leq C \varsigma^n \|f\|_{BV}$$

*for some constant $c(f)$.*

*(b) If $f \in BV$ has $c(f) > 0$, then $\rho(\mu_f) = \log \lambda$, where $d\mu_f = f dm$.*

*(c) $c(f) > 0$ if and only if*

$$\lim_{n \to \infty} \left\| \frac{\mathring{\mathcal{L}}^n f}{|\mathring{\mathcal{L}}^n f|_{L^1(m)}} - g \right\|_{BV} \leq C \varsigma^n \|f\|_{BV},$$

*for some constant $C$ independent of $f \in BV$.*

*(d) Suppose $\{H_\varepsilon\}$ is a family of holes as in Corollary 4.6. Let $\lambda_\varepsilon$ be the largest eigenvalue of $\mathring{\mathcal{L}}_\varepsilon$ and let $g_\varepsilon$ be its associated (normalized) eigenvector. Then*

$$|g_\varepsilon - g_0|_{L^1(m)} \to 0 \text{ and } \lambda_\varepsilon \to 1 \text{ as } \varepsilon \to 0,$$

*where $g_0$ is the density of the unique absolutely continuous invariant measure for $T$.*

*Proof.* (a) $\lambda^{-1}\mathring{\mathcal{L}}$ has spectral radius 1, essential spectral radius at most $2\sigma\lambda^{-1} < 1$ and a decomposition via spectral projectors analogous to the one given in (4.2). The conclusion of (a) follows for any $\varsigma$ greater than the second largest eigenvalue of $\lambda^{-1}\mathring{\mathcal{L}}$ or $2\sigma\lambda^{-1}$, whichever is larger.

(b) By (a), since convergence in BV implies convergence in $L^1(m)$,

$$\lambda^{-n}\int_{\mathring{I}^n} f\,dm = \lambda^{-n}\int \mathring{\mathcal{L}}^n f\,dm \xrightarrow{n\to\infty} c(f).$$

Then if $c(f) > 0$, $\rho(\mu_f) = \log\lambda$.

(c) Assume $c(f) > 0$. Then using the same calculation as in part (b),

$$\lim_{n\to\infty}\frac{\mathring{\mathcal{L}}^n f}{|\mathring{\mathcal{L}}^n f|_{L^1(m)}} = \lim_{n\to\infty}\frac{\mathring{\mathcal{L}}^n f}{\lambda^n}\frac{\lambda^n}{|\mathring{\mathcal{L}}^n f|_{L^1(m)}} = c(f)g\cdot\tfrac{1}{c(f)} = g.$$

The converse follows from the linear structure of $\mathring{\mathcal{L}}$. Let $\mathbb{V}$ be the eigenspace corresponding to $\lambda$, spanned by $g$. Then we can write $BV = \mathbb{V}\oplus\mathbb{W}$ where $\mathbb{W} = \{f\in BV : c(f) = 0\}$.

(d) The convergence of $g_\varepsilon$ to $g_0$ and $\lambda_\varepsilon$ to 1 follows from the Hölder continuity of the spectrum and spectral projectors in the weak norm outside the disk of radius $2\sigma$ guaranteed by Theorem 4.3. □

**Remark 4.8.** *We have described the application of this perturbation theory in the setting of piecewise expanding maps to keep the exposition simple; however, it is general enough to apply to hyperbolic maps as well, as soon as sutiable Banach spaces have been constructed on which the transfer operator for the closed system is quasi-compact. To date, this has been accomplished for piecewise hyperbolic maps with holes [DL], and dispersing billiards with holes [D3], including holes which interact with infinite horizon corridors [D4].*

There is one last question we have not yet addressed using spectral theory, which is **(Q4)**, connecting pressure on the survivor set to the escape rate. Indeed, given a limiting distribution with a strong convergence property as in Corollary 4.7(b), there is a standard way to construct a physically relevant invariant measure on the survivor set. (See [CMS1] for example.)

For any $\psi\in BV$, define

$$Q(\psi) = \lim_{n\to\infty}\lambda^{-n}\int_{\mathring{I}^n}\psi g\,dm, \tag{4.4}$$

where $g$ is the (normalized) eigenfunction of $\mathring{\mathcal{L}}$ corresponding to $\lambda$. The limit exists by Corollary 4.7; indeed, $Q(\psi) = c(\psi g)$. Since $Q(1) = 1$ and $|Q(\psi)| \le |\psi|_\infty$, $Q$ extends to a bounded linear functional on $C^0(I)$. By the Riesz representation theorem, there exists a Borel probability measure $\nu$ such that $\nu(\psi) = Q(\psi)$ for each $\psi\in C^0(I)$. It is immediate from (4.4) that $\nu$ is supported on $\mathring{I}^\infty$. Moreover, $\nu$ is a $T$ invariant measure since,

$$\nu(\psi\circ T) = \lim_{n\to\infty}\lambda^{-n}\int_{\mathring{I}^n}\psi\circ T\,g\,dm = \lim_{n\to\infty}\lambda^{-n}\int_{\mathring{I}^{n-1}}\psi\mathring{\mathcal{L}}g\,dm = \lim_{n\to\infty}\lambda^{1-n}\int_{\mathring{I}^{n-1}}\psi\,g\,dm = \nu(\psi),$$

for each $\psi\in C^0(I)$.

The question of whether $\nu$ satisfies an escape rate formula and achieves the supremum of pressures over some class of invariant measures supported on $\mathring{I}^\infty$ has so far not been established using the spectral gap for $\mathring{\mathcal{L}}$ on BV. Rather, typically one uses a Markov extension for the open system to bring to bear the theory of Gibbs measures for countable Markov shifts to establish this fact. That is the content of the next section.

# 5 Markov Extensions for Open Systems

Briefly, a Markov extension of a dynamical system $(T, M)$ is a system $(F, \Delta)$ admitting a finite or countable Markov partition for which $(T, M)$ is a quotient. What one gains by passing to a Markov extension is the ability to code the system and use the powerful results available to finite and countable state Markov shifts. What one loses is compactness ($\Delta$ is typically not compact even when $M$ is) and some control of objects that do not 'lift' to the extension.

We sketch the main ideas in the construction of a certain type of Markov extension, known as a *Young tower*. Given a map $T : M \circlearrowleft$, choose a reference set $\Lambda \subset M$ and consider iterates of the form $T^i(\Lambda)$. We wait for a part of $T^i(\Lambda)$ to make a 'good' return to $\Lambda$.[9] A stopping time is then declared on the piece of $\Lambda$ which has made a good return, and we continue to iterate the rest of $\Lambda$ until another good return is made. In this way, one obtains a countable partition $\{\Lambda_i\}$ and a stopping time $\tau : \Lambda \to \mathbb{N}$ such that $\tau$ is constant on each $\Lambda_i$ and $T^\tau(\Lambda_i)$ makes a good return to $\Lambda$. The induced map $T^\tau : \Lambda \to \Lambda$ can be viewed as a generalized horseshoe with variable return times and countably many branches.

Define $\Delta_0 = \Lambda$ and

$$\Delta = \{(x, n) \in \Delta_0 \times \mathbb{N} : n < \tau(x)\}$$

to be the tower. We denote by $\Delta_\ell$ the set of points $(x, n)$, with $n = \ell$. The tower map is defined by $F(x, \ell) = (x, \ell+1)$ if $\ell < \tau(x)-1$ and $F(x, \tau(x)-1) = (T^\tau(x), 0)$. This defines a natural projection $\pi : \Delta \to M$ such that $\pi \circ F = T \circ \pi$ and an identification of $\Delta_\ell$ with those points in $\Lambda$ which have not made a good return to $\Lambda$ by time $n$.

Pushing the partition $\{\Lambda_i\}$ up the levels of the tower induces a countable Markov partition on $\Delta$. Alternatively, one sometimes constructs a dynamical partition $\{\Delta_{\ell,j}\}$ defined inductively during the construction of the tower which is coarser than that induced by $\{\Lambda_i\}$. In any case, one requires some bounded distortion property on partition elements at return times, such as,

$$\left| \frac{DT^\tau(x)}{DT^\tau(y)} - 1 \right| \le C_d d(T^\tau(x), T^\tau(y)) \tag{5.1}$$

for all $x, y \in \Lambda_i$ and some constant $C_d > 0$, independent of $i$.

Important statistical properties of the system are reflected in the rate of decay in the tail of the return time: $m(\tau > n)$. If this quantity decays exponentially and g.c.d.$\{\tau\} = 1$, then the rate of decay of correlations is exponential; while if the rate of decay is polynomial, so is the decay of correlations.

## 5.1 Young Towers for Expanding Maps with Holes

We describe the construction of a Young tower more explicitly in the setting of piecewise expanding maps fo the interval with holes. Unfortunately, even if a tower has been constructed in a system without a hole, a new tower must be constructed after the introduction of the hole. This is because to preserve the Markov property for the open system, the hole should lift to a countable union of partition elements in $\Delta$. This means that the boundary of the hole must be considered as part of the discontinuity set of the map as the reference set is iterated. These new discontinuities will affect return times in unbounded ways and there is no monotonicity: some intervals may make a good return earlier and some may make a good return later than they would have had the hole not been present. When $H$ lifts to a countable union of partition elements, we say the tower *respects the hole.*

[9] In the context of expanding systems, a good return is one which completely covers $\Lambda$; for hyperbolic systems, a good return crosses $\Lambda$ in the unstable directions while lying strictly inside it in the stable directions.

In this section, $T$ is a piecewise expanding map as in Section 4 and $H$ is a finite union of open intervals. Let $\{I_j\}$ denote the finite collection of intervals on which $\mathring{T}$ is monotonic and smooth. In this one-dimensional setting, $\Lambda$ will be an interval contained inside one of the $I_j$. First we prove a preliminary growth lemma.

**Lemma 5.1.** *Fix a length scale $d \le \min_j m(I_j)$. Let $J$ be any interval of length at least $d$ lying entirely in one of the $I_j$. There exists a countable partition $\mathcal{Z}$ of $J$ into intervals, and a stopping time $t : J \to \mathbb{N}$, constant on each $Z \in \mathcal{Z}$, such that*

*(a) $T^t$ is $C^2$ on each $Z \in \mathcal{Z}$ and satisfies bounded distortion,* (5.1)*;*

*(b) for each $Z \in \mathcal{Z}$, either $Z \subset \mathring{I}^t$ and $T^t(Z) = I_j$ for some $I_j$, or $Z \subset \mathring{I}^{t-1}$ and $T^t(Z) \subset H$;*

*(c) $m(t > n) \le (2\sigma)^n$;*

*(d) $m(x \in J : T^t(x) \in H) \le \frac{\sigma m(H)}{1-2\sigma}$.*

*Proof.* We define a partition on $J$ inductively. Let $\Omega_0 = \{J\}$ and let $\Omega_{n-1} \subseteq J$ denote the set of points on which $t$ has not been defined by time $n-1$. $\Omega_{n-1}$ consists of a finite number of intervals.

For $\omega \in \Omega_{n-1}$, consider $T^n\omega$. If $T^n\omega$ does not cover any $I_j$, then it can intersect at most one component of $H$ and $T^{n-1}\omega$ contains at most one singularity point of $T$. On the at most one component of $\omega \cap T^{-n}H$, we define $t = n$ and place this interval into $\mathcal{Z}$; we place the at most two components of $\omega$ on which $T^n$ is smooth into $\Omega_n$.

If, on the other hand, $T^n\omega$ contains one or more of the $I_j$, we define $t = n$ on any components $\omega'$ for which $T^n\omega' \subset H$ or $T^n\omega' = I_j$; each of these intervals is included as a element of $\mathcal{Z}$. The at most two components of $\omega$ on which $T^n$ is smooth and on which $S$ has not yet been defined are placed into $\Omega_n$.

From this construction of $t$ and $\mathcal{Z}$, (a) and (b) are automatic. (c) follows since $\Omega_n$ consists of at most $2^n$ intervals of length at most $D\sigma^n$, where $D$ is the maximum length of one of the $I_j$. Thus $m(S > n) = m(\Omega_n) \le D(2\sigma)^n$.

Similarly, from each component of $\Omega_{n-1}$, at most $m(H)\sigma^n$ measure can enter $H$ at time $n$. Since there are at most $2^{n-1}$ such components, the set of points that can fall into $H$ at time $n$ is at most $m(H)\sigma^n 2^{n-1}$. Summing this over $n$ yields (e). □

From this growth lemma, the tower can easily be constructed, using some combinatorial condition of the form: For each $I_j$, there exists $n_j$ such that

$$\mathring{T}^{n_j}(I_j \cap \mathring{I}^{n_j}) = \mathring{I}. \tag{5.2}$$

This condition can be viewed as a property for the open system similar to the covering property for closed systems.

For each $j$, $I_j \cap \mathring{I}^{n_j}$ is a finite collection of intervals on which $\mathring{T}^{n_j}$ is injective. The images of these intervals cover $\mathring{I}$ by (5.2) and the endpoints of the intervals induce a finite partition $Q_j$ of $\mathring{I}$. Let $Q = \vee_j Q_j$ and note that $Q$ is still a finite partition of $\mathring{I}$.

Choose $\Lambda$ to be an element of $Q$. Apply Lemma 5.1 to each element of $Q$ with $d$ fixed as the length of the shortest element of $Q$. This gives a stopping time according to which every interval has either entered $H$ or grown to cover at least one $I_j$. Choose $\Lambda$ to be an element of $Q$ covered by at least one of the intervals in $\mathring{T}^{n_j}(I_j \cap \mathring{I}^n)$. The combinatorial condition in (5.2) implies that every interval of fixed length must have a positive fraction of its measure, call it $\delta$, make a good return at least every $n_0$ iterates, where $n_0 = \max_j n_j$. This yields the desired return time $\tau$ with decay rate,

$$m(x \in \Lambda : \tau(x) > n\} \le C(2\sigma)^n + C(1-\delta)^{n/n_0}.$$

## 5.2 The Open Tower: Spectral Arguments

With the tower constructed respecting the hole, we now wish to leverage the Markov structure of the tower to analyze the open system. First notice that the Lebesgue measure $m$ lifts easily to a reference measure $\overline{m}$ on the tower simply by $\overline{m}|_{\Delta_0} = m|_\Lambda$, and $\overline{m}|_{\Delta_\ell} = F_*\overline{m}|_{\Delta_{\ell-1}}$ for $\ell \geq 1$. Since $JF \equiv 1$ except at return times, this immediately implies that defining the associated transfer operator by,

$$\mathcal{L}_\Delta f = \sum_{y \in F^{-1}x} \frac{f(y)}{JF(y)}, \qquad \text{for } f \in L^1(\overline{m}),$$

one immediately has $\int_\Delta \mathcal{L}_\Delta f \, d\overline{m} = \int_\Delta f \, dm$. And for the open system, $\int_{\mathring{\Delta}} \mathring{\mathcal{L}}_\Delta f \, dm = \int_{\mathring{\Delta}^1} f \, dm$, with the usual definition of $\mathring{\mathcal{L}}_\Delta$ and $\mathring{\Delta}^n$. Note, however, that $\pi_*\overline{m} \neq m$.

The inductive construction of $\Omega_n$ in the proof of Lemma 5.1 induces a natural Markov partition $\{\Delta_{\ell,j}\}$ on the tower. In this simple setting, there are only finitely many elements on each level of the tower, although that need not be the case for more general systems. We assume there exist constants $C, \alpha > 0$ such that $\overline{m}(\Delta_\ell) = m(\tau > \ell) \leq Ce^{-\alpha\ell}$.

The usual mixing condition for $F$ is that g.c.d.$\{\tau\} = 1$, which we can impose on our construction using the transitivity condition (5.2).

Define the separation time $s(x, y) = \inf\{n \in \mathbb{N} : F^n(x), F^n(y) \text{ lie in different } \Delta_{\ell,j}\}$. Choose $\beta \in (0, \alpha)$ and define a separation time metric $d_\beta(x, y) = e^{-\beta s(x,y)}$.

The function space introduced in [Y2] and used since in a variety of settings is the weighted function space $\|f\| = \|f\|_\infty + \|f\|_{\text{Lip}}$, where

$$\|f\|_\infty = \sup_{\ell \geq 0} e^{-\beta\ell} |f|_{\Delta_\ell}|_\infty$$
$$\|f\|_{\text{Lip}} = \sup_{\ell,j} e^{-\beta\ell} \text{Lip}(f|_{\Delta_{\ell,j}}),$$

and $\text{Lip}(f)$ denotes the Lipschitz constant of $f$ in the metric $d_\beta$. Let $\mathcal{B}_\Delta = \{f \in L^1(\overline{m}) : \|f\| < \infty\}$.

**Problem 5.** *Prove that the unit ball of $\mathcal{B}_\Delta$ is compactly embedded in $L^1(\overline{m})$.*

At this point the following estimates are standard [Y2, D1, BDM].

$$\|\mathring{\mathcal{L}}^n_\Delta f\|_\infty \leq Ce^{-\beta n}\|f\|_{\text{Lip}} + C|f|_{L^1(\overline{m})}, \qquad \|\mathring{\mathcal{L}}^n_\Delta f\|_{\text{Lip}} \leq Ce^{-\beta n}\|f\|_{\text{Lip}} + C|f|_{L^1(\overline{m})}.$$

Using Problem 5, it follows from standard arguments (see for example [B]) as in the proof of Theorem 4.1 that $\mathring{\mathcal{L}}_\Delta$ is quasi-compact as an operator on $\mathcal{B}_\Delta$ with essential spectral radius bounded by $e^{-\beta}$.

However, this is not enough to conclude a spectral gap (even with g.c.d.$\{\tau\} = 1$) since we must first show that the spectral radius of $\mathring{\mathcal{L}}_\Delta$ is strictly larger than $e^{-\beta}$. In this setting, perturbative arguments are not available since shrinking the hole would require a different tower construction for each hole. Thus we would be comparing a sequence of operators on a sequence of Banach spaces, which is somewhat cumbersome.

An easier route is to assume a smallness condition on the size of the hole and then to prove directly a lower bound on the spectral radius of $\mathring{\mathcal{L}}_\Delta$ and a Perron-Frobenius type decomposition of the peripheral spectrum. Such a decomposition, coupled with the mixing assumption g.c.d.$\{\tau\} = 1$, is sufficient to prove a spectral gap for $\mathring{\mathcal{L}}_\Delta$ on $\mathcal{B}_\Delta$. From this, the full set of results presented in Corollary 4.7 hold for the tower with holes (except (d)).

**Remark 5.2.** *We have left open the form of the smallness condition since there are several possible formulations. One possibility is to assume the hole is sufficiently small that the normalization required after pushing a certain class of regular densities forward one step is small compared to* $1 - e^{-\beta}$*. Thus in one step, one sees that the spectral radius must be larger than* $e^{-\beta}$*. This is the approach taken in [D1, D2, BDM].*

*A less restrictive approach is to use an asymptotic requirement: assume* $-\overline{\rho}(\overline{m}) < \beta$ *and show that this implies that the spectral radius is strictly greater than the essential spectral radius. This approach is taken in the recent preprint [DT]. This is a less restrictive requirement, but also less constructive in terms of an explicit condition on the hole.*

At this point, one also defines an invariant measure $\nu$ on $\mathring{\Delta}$ via the analgous limit to (4.4),

$$\nu(\psi) = \lim_{n\to\infty} \lambda^{-n} \int_{\mathring{\Delta}^n} \psi \, g \, d\overline{m}, \tag{5.3}$$

where $\lambda \in (e^{-\beta}, 1)$ is the largest eigenvalue of $\mathring{\mathcal{L}}_\Delta$ and $g \in \mathcal{B}_\Delta$ is the associated (normalized) eigenvector.

**Problem 6.** *Assume that* $g \geq \delta > 0$ *on* $\Delta_0$*. Let* $[i_0, i_1, \ldots, i_{n-1}] \subset \Delta_0$ *denote a cylinder set of length* $n$ *with respect to the first return map,* $S := F^\tau$*. Prove that for* $\nu$ *defined as in* (5.3) *(and* $\mathring{\mathcal{L}}_\Delta$ *having a spectral gap), there exists* $C \geq 1$ *such that for all* $n \in \mathbb{N}$*,*

$$C^{-1} \lambda^{-\tau^n(y_*)} |JS^n(y_*)|^{-1} \leq \nu([i_0, i_1, \ldots, i_{n-1}]) \leq C \lambda^{-\tau^n(y_*)} |JS^n(y_*)|^{-1},$$

*where* $y_*$ *is an arbitrary point in* $[i_0, i_1, \ldots, i_{n-1}]$ *and* $\tau^n$ *is the time of the nth return to* $\Delta_0$*.*

Problem 6 implies that $\nu_0 := \frac{1}{\nu(\Delta_0)} \nu|_{\Delta_0}$ is a Gibbs measure for $S = F^\tau$ with potential $\phi = -\log |\lambda^\tau JS|$.

We define a topology on $\Delta_0 \cap \mathring{\Delta}^\infty$ by using the cylinder sets in $\mathcal{Z} = \{\Lambda_i\}$ as our basis. The fact that $S|_{\Delta_0 \cap \mathring{\Delta}^\infty}$ is topologically mixing then follows from the condition g.c.d.$\{\tau\} = 1$ and the fact that the partition $\mathcal{Z}$ is generating since $F$ is expanding.

Then Problem 6 together with [Sa, Theorems 7 and 8] implies that

$$0 = \sup_{\eta_0 \in \mathcal{M}_S} \left\{ h_{\eta_0}(S) + \int_{\Delta_0} \phi \, d\eta_0 \right\}, \tag{5.4}$$

where $\mathcal{M}_S$ is the set of $S$-invariant Borel probability measures on $\Delta_0 \cap \mathring{\Delta}^\infty$. Moreover, $\nu_0 \in \mathcal{M}_S$ is the unique nonsingular[10] measure which attains the supremum.

We now project this variational principle for $S$ to obtain one for $F$.

**Lemma 5.3.** *Let* $\mathcal{M}_F$ *be the set of* $F$*-invariant Borel probability measures on* $\Delta$*. For any* $\eta \in \mathcal{M}_F$*, let* $\eta_0 = \frac{1}{\eta_0(\Delta_0)} \eta|_{\Delta_0}$*. Then*

$$\int_{\Delta_0} \log JS \, d\eta_0 = \int_\Delta \log JF \, d\eta \int_{\Delta_0} \tau \, d\eta_0.$$

*Proof.* Notice that $\eta_0 \in \mathcal{M}_S$. For $x \in \Delta_0$, $JS(x) = JF^\tau(x) = \Pi_{i=0}^{\tau(x)-1} JF(F^i x)$. However, $JF(F^i x) = 1$ for $i < R(x) - 1$, so that $JS(x) = JF(F^{\tau-1}x)$. In other words, we have

$$\int_{\Delta_0} \log JS \, d\eta_0 = \eta(\Delta_0)^{-1} \int_{F^{-1}\Delta_0} \log JF \, d\eta = \eta(\Delta_0)^{-1} \int_\Delta \log JF \, d\eta. \tag{5.5}$$

[10] Nonsingular in this context means $\eta_0(S(A)) = 0$ if and only if $\eta_0(A) = 0$ for Borel sets $A \subset \Delta_0$.

Since the $\eta$-measure of a partition element $\Delta_{\ell,j}$ does not change as it moves up the tower, we have

$$1 = \sum_{(\ell,j)} \eta(\Delta_{\ell,j}) = \sum_i \eta(\Lambda_i)\tau(\Lambda_i) = \int_{\Delta_0} \tau \, d\eta.$$

So by definition of $\eta_0$, we have

$$\int_{\Delta_0} \tau \, d\eta_0 = \eta(\Delta_0)^{-1} \int_{\Delta_0} \tau \, d\eta = \eta(\Delta_0)^{-1}.$$

This, together with (5.5), proves the lemma. □

Since $S = F^\tau$ is a first return map to $\Delta_0$, the general formula of Abramov [A] implies that $h_\eta(F) = h_{\eta_0}(S)\eta(\Delta_0)$ so that

$$h_{\eta_0}(S) = \eta(\Delta_0)^{-1} h_\eta(F) = h_\eta(F) \int_{\Delta_0} \tau \, d\eta_0. \tag{5.6}$$

Since $\int_{\Delta_0} \tau \, d\eta_0 = \eta(\Delta_0)^{-1} \neq 0$ and there is a 1-1 correspondence between measures in $\mathcal{M}_S$ and $\mathcal{M}_F$, putting equation (5.6) and Lemma 5.3 together with (5.4), we have

$$\log \lambda = \sup_{\eta \in \mathcal{M}_F} \left\{ h_\eta(F) - \int_\Delta \log JF \, d\eta \right\}. \tag{5.7}$$

Moreover, $\nu$ is the only nonsingular $F$-invariant probability measure which attains the supremum.

## 5.3 Projecting the Results from the Tower

The last step in the use of the Young tower is to project our results down to our original map $T : I \circlearrowleft$.

For any $\eta \in \mathcal{M}_F$, we can define $\tilde{\eta} = \pi_* \eta \in \mathcal{M}_T$. Then given a function $\tilde{f}$ on $X$, we have $\int_X \tilde{f} \, d\tilde{\eta} = \int_\Delta \tilde{f} \circ \pi \, d\eta$. From the relation $\pi \circ F = T \circ \pi$, we have

$$J_{m,\overline{m}}\pi(Fx) J_{\overline{m}}F(x) = J_m T(\pi x) J_{m,\overline{m}}\pi(x)$$

for each $x \in \Delta$, where $J_{m,\overline{m}}\pi = \frac{dm \circ \pi}{d\overline{m}}$. Thus,

$$\int_I \log JT \, d\tilde{\eta} = \int_\Delta (\log JF + \log J\pi \circ F - \log J\pi) \, d\eta = \int_\Delta \log JF \, d\eta$$

since the last two terms cancel by the the $F$-invariance of $\eta$.

The fact that $h_\eta(F) = h_{\tilde{\eta}}(T)$ follows since $\pi$ is at most countable-to-one ([Bu, Proposition 2.8]). Thus

$$h_\eta(F) - \int_\Delta \log JF \, d\eta = h_{\tilde{\eta}}(T) - \int_I \log JT \, d\tilde{\eta}$$

for each $\eta \in \mathcal{M}_F$. And in particular, setting $\tilde{\nu} = \pi_* \nu$, we have

$$\log \lambda = h_{\tilde{\nu}}(T) - \int_I \log JT \, d\tilde{\nu},$$

so that $\tilde{\nu}$ satisfies the escape rate formula.

There is only one issue with the above: Is $-\log \lambda$, where $\lambda$ is the spectral radius of $\mathring{\mathcal{L}}_\Delta$, the same as escape rate from $\mathring{I}$ with respect to Lebesgue measure?

To answer this question, one must consider how the evolution of densities on $\Delta$ parallels the evolution of densities on $I$. Given a measure $\overline{\mu} = f\overline{m}$ on $\Delta$, we may define its projection $\mu = \pi_*\overline{\mu}$ and the density of this projection with respect to $m$ will be given by

$$P_\pi f(x) = \sum_{y \in \pi^{-1}x} \frac{f(y)}{J\pi(y)}.$$

Notice that $|P_\pi f|_{L^1(m)} = |f|_{L^1(\overline{m})}$ and also,

$$P_\pi(\mathring{\mathcal{L}}_F^n f) = \mathring{\mathcal{L}}_T^n(P_\pi f), \qquad \text{for } f \in L^1(\overline{m}).$$

The importance of these relations lies in the fact that if $\mathring{\mathcal{L}}_F g = \lambda g$, then $\mathring{\mathcal{L}}_T(P_\pi g) = \lambda P_\pi g$, and if $\tilde{f} = P_\pi f$, then

$$\frac{\mathring{\mathcal{L}}_F^n f}{|\mathring{\mathcal{L}}_F^n f|_{L^1(\overline{m})}} \to g \qquad \text{in } L^1(\overline{m}) \text{ implies} \qquad \frac{\mathring{\mathcal{L}}_T^n \tilde{f}}{|\mathring{\mathcal{L}}_T^n \tilde{f}|_{L^1(m)}} \to P_\pi g \qquad \text{in } L^1(m). \tag{5.8}$$

Thus $P_\pi g$ seems to define the physically relevant limiting distribution we are looking for.

However, the space of functions $P_\pi \mathcal{B}_\Delta$ is not well understood in general and for some hyperbolic systems, it is even difficult to show that it contains Lebesgue measure. In one dimension, it is possible to identify conditions that guarantee that $C^p(I) \subset P_\pi \mathcal{B}_\Delta$, and so solve the problem of 'liftability' of the types of noninvariant measures that we would like to evolve under the dynamics. The two properties introduced in [BDM] are as follows.

**(A1)** There exist constants $\xi > 1$ and $C_1$, $C_2 > 0$ such that

(a) for any $x \in \Lambda$, $n \geq 1$ and $k < \tau^n(x)$, $|DT^{\tau^n(x)-k}(T^k x)| > C_1 \xi^{\tau^n(x)-k}$.

(b) Let $x, y \in \Lambda_i$ and $\tau_i = \tau(\Lambda_i)$. Then $\left|\frac{DT^\ell(\pi x)}{DT^\ell(\pi y)}\right| \leq C_2$ for $\ell \leq \tau_i$. If $T^\tau(\Lambda_i) \subseteq \Lambda$, then $\left|\frac{DT^{\tau_i}(\pi x)}{DT^{\tau_i}(\pi y)} - 1\right| \leq C_2 d(T^{\tau_i}(\pi x), T^{\tau_i}(\pi y))$.

Property (A1)(a) says that although $T$ may not be expanding everywhere in its phase space, we only count returns to $\Lambda$ during which average expansion has occurred - this permits its application to nonuniformly expanding maps such as unimodal maps. Property (A1)(b) is simply bounded distortion. In fact, (A1) implies the distortion bound (5.1) in the current setting.

**Problem 7.** *Choose $p \geq \beta/\log \xi$ and show using* **(A1)** *that the separation time metric $d_\beta$ on $\Delta$ and the usual Euclidian distance on $I$ are compatible in the following sense: Any $f \in C^p(I)$ has Lip$(f \circ \pi) < \infty$.*

Unfortunately, the result of Problem 7 is not sufficient for our purposes. Since typically $P_\pi(f \circ \pi) \neq f$, we need the following additional property for our tower construction.

**(A2)** There exists an index set $\mathcal{K} \subset \mathbb{N} \times \mathbb{N}$ such that

(a) $m\big(X \setminus \cup_{(\ell,j)\in\mathcal{K}} \pi(\Delta_{\ell,j})\big) = 0$;

(b) $\pi(\Delta_{\ell_1,j_1}) \cap \pi(\Delta_{\ell_2,j_2}) = \emptyset$ for all but finitely many $(\ell_1, j_1)$, $(\ell_2, j_2) \in \mathcal{K}$.

(c) Define $J\pi_{\ell,j} := J\pi|_{\Delta_{\ell,j}}$. Then $\sup_{(\ell,j)\in\mathcal{K}} |J\pi_{\ell,j}|_\infty + \text{Lip}(J\pi_{\ell,j}) = D < \infty$.

This property guarantees that given $f \in C^p(I)$, we may define $\tilde{f} \in \mathcal{B}_\Delta$ such that $\mathcal{P}_\pi \tilde{f} = f$ [BDM, Proposition 4.2]. This guarantees in particular that the evolution of Hölder continuous functions in $I$ follows the same limiting behavior as functions in $\mathcal{B}_\Delta$. In particular, $1 \in \mathcal{P}_\pi \mathcal{B}_\Delta$ and $\log \lambda$ is the desired escape rate with respect to Lebesgue measure. We conclude by remarking that all piecewise expanding maps considered here admit towers satisfying (A1) and (A2), see [BDM, Section 4.2].

### 5.4 Unimodal Maps of the Interval

Once the general machinery for Young towers with holes has been laid out, including the issues involving the projection of results discussed in Section 5.3, it can be used to study many classes of nonuniformly hyperbolic systems. A prime example is unimodal maps of the interval. The big difference from the viewpoint of our current discussion is that such maps have no Banach space for which it is currently known that $\mathring{\mathcal{L}}$ enjoys a spectral gap. So for this class of systems, we must rely exclusively on Markov extensions to provide both the limiting distribution as well as the variational principle characterizing the escape rate.

The classical representative of maps in this class is the logistic family, $T_a = ax(1-x)$, defined on the interval $[0,1]$. For certain parameter values $a \in [0,2]$, $T_a$ admits an absolutely continuous invariant measure and has exponential decay of correlations, so one expects that the standard program for hyperbolic systems with holes should hold: a unified escape rate, a physical conditionally invariant limiting distribution, and the characterization of the escape rate via a variational principle.

To give an idea of the properties required for such an analysis to go through, we recall the restrictions placed in $T$ in [BDM], which in turn uses the framework of [DHL].[11]

The map $T : I \circlearrowleft$ is assumed to be $C^2$ and has a critical point $c$ with critical order $1 < \ell_c < \infty$. Furthermore, $T$ satisfies the following conditions for all $\delta > 0$ (where $B_\delta(c)$ is a $\delta$-neighborhood of $c$):

**(C1)** *Expansion outside $B_\delta(c)$:* There exist $\xi, \kappa > 0$ such that for every $x$ and $n \geq 1$ such that $x_0 = x, \dots, x_{n-1} = T^{n-1}(x) \notin B_\delta(c)$, we have

$$|DT^n(x)| \geq \kappa \delta^{\ell_c - 1} e^{\xi n},$$

Moreover, if $x_0 \in T(B_\delta(c))$ or $x_n \in B_\delta(c)$, then we have

$$|DT^n(x)| \geq \kappa e^{\xi n}.$$

**(C2)** *Slow recurrence and derivative growth along critical orbit:* There exists $\Lambda > 0$ and $\alpha_c \in (0, \Lambda/(5\ell_c))$ such that

$$|DT^k(T(c))| \geq e^{\Lambda k} \quad \text{and} \quad \mathrm{dist}(T^k(c), c) > \delta e^{-\alpha_c k} \qquad \text{for all } k \geq 1.$$

**(C3)** *Density of preimages:* The preimages of $c$ are dense in $I$.

Condition (C1) follows for piecewise $C^2$ maps from Mañé's Theorem, see [MS, Chapter III.5]. The first half of condition (C2) is the Collet-Eckmann condition, and the second half is a slow recurrence condition. Condition (C3) excludes the existence of non-repelling periodic points.

[11] In fact, both [BDM] and [DHL] allow maps with multiple turning points as well as singularities, but we simplify the exposition here.

For this class of maps (and more general ones), with some condition on the size and generic placement of the holes, [BDM] constructs Young towers with exponential tails respecting the holes and satisfying (A1) and (A2). Thus the full set of results for the open system are proved in this setting.

### 5.5 Dispersing Billiards with Small Holes

Young towers can also be constructed for hyperbolic systems with singularities. In this case, the reference set $\Lambda$ is more complicated - typically is has a product structure of local stable and unstable manifolds, and for systems with singularities, $\Lambda$ comprises a positive measure Cantor set in the unstable direction, so it is not connected, complicating the discussion of 'liftability' of measures in Section 5.3.

Perhaps the prime example of hyperbolic systems with singularities in two dimensions are mathematical billiards. In such systems, a point mass moving at constant speed is assumed to reflect elastically off of fixed boundaries in a two-dimensional domain. The hyperbolicity of the system depends entirely on the geometry of these boundaries. Typically, these are either smooth obstacles in $\mathbb{T}^2$, or the piecewise smooth boundary of a bounded domain in $\mathbb{R}^2$ with only finitely many corner points allowed.

The *billiard map* is defined to be the discrete time collision-to-collision map with the boundary, recording position (parametrized by arclength) and the angle an outgoing velocity vector makes with the normal vector to the boundary at each collision. The phase space of the map is a union of cylinders: The periodic coordinate is the position coordinate, while the angular coordinate ranges from $-\pi/2$ to $\pi/2$.

If one asssumes the boundaries are convex with strictly positive curvature, then the billiard is hyperbolic; however, the map has singularities: the derivative blows up at tangential collisions and these complicate the analysis of such systems.

When we place a hole in such a system, we first define a hole in the billiard table (perhaps an open convex set, or an arc in a boundary) and then determine the geometry of the induced holes in the phase space of the map.

Recently, Banach spaces of distributions were constructed where the transfer operator associated to a dispersing billiard map has a spectral gap [DZ1, DZ2]. So from the perturbative point of view, the approach outlined in Section 4.1 is available to this class of billiards with holes. Such an approach was carried out in [D3, D4]. On the other hand, to obtain a variational principle for the open system, one still needs to construct an associated Young tower. This was done first in [DWY1] and more recently in [D3, D4], so that the analogue of full set of results described in Corollary 4.7 has been established for a large class of dispersing billiards with holes. For a discussion of the different types of holes allowed with accompanying pictures, see [D3]. [D4] contains a somewhat technical generalization specific to billiards with infinite horizon.

## 6 Open Systems with Subexponential Rates of Escape

So far we have only considered systems with exponential rates of escape. A natural question is:

*How much of this program involving escape rates, limiting distributions and pressure on the survivor set carries over to systems with subexponential rates of mixing?*

We will address this question in the context of a simple class of intermittent maps of the interval, the Manneville-Pomeau or LSV maps.

### 6.1 Manneville-Pomeau or LSV Maps with Holes

For $\gamma \in (0,1)$, define the following map on $I = [0,1]$.

$$T(x) = \begin{cases} x + 2^\gamma x^{1+\gamma}, & x \in [0, 1/2) \\ 2x - 1, & x \in [1/2, 1] \end{cases} .$$

0 is a *neutral fixed point* since $T(0) = 0$ and $T'(0) = 1$. Elsewhere, $T' > 1$. For $\gamma \in (0,1)$, $T$ admits an invariant probability measure, absolutely continuous with respect to Lebesgue, with density $g_{\text{SRB}}(x) \sim x^{-\gamma}$. With respect to $\mu_{\text{SRB}}$, $T$ has polynomial decay of correlations with rate $n^{-1/\gamma+1}$ [Y3].

A convenient countable Markov partition for $T$ is constructed as follows. Set $a_n = T^{-n}(1/2)$ and $J_0 = [1/2, 1]$, $J_n = [a_n, a_{n-1})$, $n \geq 1$. The partition is $\mathcal{Q} = \{J_n\}_{n \geq 0}$ and follows from the definition of $a_n$ that $T(J_n) = J_{n-1}$ for $n \geq 1$ and $T(J_0) = I$.

It follows from standard estimates [Y3, LSV] that

$$a_n \sim n^{-1/\gamma} \qquad \text{and} \qquad |J_n| \sim n^{-1-1/\gamma}. \tag{6.1}$$

This spacing determines the rate of mixing of the system.

**Introduction of the Hole.** We refine the partition $\mathcal{Q}$ by setting,

$$\mathcal{Q}_n = \mathcal{Q} \vee \left( \bigvee_{i=0}^{n} T^{-i}(\{[0,1/2),[1/2,1]\}) \right).$$

Note that $\{[0,1/2),[1/2,1]\}$ is a finite Markov partition for $T$ and this definition of $\mathcal{Q}_n$ preserves 0 as the only accumulation point of the elements of the partition. We start with $\mathcal{Q}$, however, since this allows us to control distortion (see Lemma 6.1).

Now fix $N_0$ and take $H$ to be a finite union of elements of $\mathcal{Q}_{N_0}$. $H$ is not allowed to contain an element of $\mathcal{Q}$ nor a right neighborhood of $1/2$ since this would trivialize the effect of the neutral fixed point on the dynamics. With this assumption, $\mathcal{Q}_{N_0}$ is a countable Markov partition for $\mathring{T}$ as well as $T$.

In this context, the exponential rate of escape $\rho(m)$ is obviously 0, where $m$ denotes Lebesgue measure. So we instead will ask about a polynomial rate of escape defined by

$$\mathfrak{e}_{\text{poly}}(\mu_f) = \lim_{n \to \infty} -\frac{\log \mu_f(\mathring{I}^n)}{\log n},$$

where $\mathring{I}^n$ denotes as usual the set of points which have not escape by time $n$, and $\mu_f$ is a probability measure having density $f$ with respect to Lebesgue, i.e. $d\mu_f = f dm$.

#### 6.1.1 Initial Estimates

Before proving results about escape rates and limiting distributions, we record the following standard lemma.

**Lemma 6.1.** *Given $k \geq 0$ and $n \geq 1$, let $x, y \in \mathring{I}^n$ lie in the same element of $\mathcal{Q}_{n+1}$ such that $T^n(x), T^n(y) \in J_k$. There exists $C > 0$, independent of $n, k, x$ and $y$ such that*

*(a)* $\displaystyle \frac{1}{DT^n(x)} \leq C \left( \frac{k}{n+k} \right)^{\frac{\gamma+1}{\gamma}}$;

(b) $\left|\log \dfrac{DT^n(x)}{DT^n(y)}\right| \le C|T^n(x) - T^n(y)|^{\frac{\gamma}{\gamma+1}}$.

Statement (a) is standard (see [Y3, LSV]), so we prove (b).

*Proof of (b).* Given $i \in \{0, \cdots, n-1\}$, let $J_{k_i}$ denote the element of $\mathcal{Q}$ containing $T^i(x)$ and $T^i(y)$. Let also $B_{k_i} = 2^\gamma \gamma(\gamma+1)a_{k_i}^{\gamma-1}$ be the maximum value of $|D^2T|$ and $M_{k_i}^{(j)}$ be the minimum value of $|DT^j|$ on $J_{k_i}$, respectively. Setting $p = \frac{\gamma}{\gamma+1}$, we have

$$\begin{aligned}
\left|\log \frac{DT^n(x)}{DT^n(y)}\right| &\le \sum_{i=0}^{n-1} |\log DT \circ T^i(x) - \log DT \circ T^i(y)| \le \sum_{i=0}^{n-1} B_{k_i}|T^i(x) - T^i(y)| \\
&\le \sum_{i=0}^{n-1} B_{k_i}|T^i(x) - T^i(y)|^{1-p} \frac{|T^i(x) - T^i(y)|^p}{|T^n(x) - T^n(y)|^p} |T^n(x) - T^n(y)|^p \\
&\le \sum_{i=0}^{n-1} \frac{B_{k_i}|J_{k_i}|^{1-p}}{(M_{k_i}^{(n-i)})^p} |T^n(x) - T^n(y)|^p.
\end{aligned}$$

Now, $T^n(x)$ has no preimage in $\bigcup_{i>n+k} J_i$, so the weakest expansion (and largest $B_{k_i}$) occurs when $x \in J_{k+n}$, i.e. when $k_i = k + n - i$. This is the worst case scenario in computing the upper bound, so using (6.1), there exists $C > 0$ such that

$$\left|\log \frac{DT^n(x)}{DT^n(y)}\right| \le Ck \sum_{i=0}^{n-1} (n+k-i)^{-2} |T^n(x) - T^n(y)|^p \le C'|T^n(x) - T^n(y)|^p$$

proving statement (b). $\square$

For $\alpha \in [0,1)$, define a class of densities by

$$\mathcal{F}_\alpha = \left\{ f \in L^1(m) : f \ge 0, \exists x_0 \in (0,1) \text{ such that } 0 < \inf_{x \in (0,x_0)} x^\alpha f(x) \le \sup_{x \in (0,1)} x^\alpha f(x) < \infty \right\}.$$

**Theorem 6.2** (Escape Rate)**.** *For any $\alpha \in [0,1)$, $f \in \mathcal{F}^\alpha$, there exists $C > 0$ such that for all $n \ge 0$,*

$$C^{-1} n^{-\frac{1}{\gamma}(1-\alpha)} \le \int_{\mathring{I}^n} f \, dm \le C n^{-\frac{1}{\gamma}(1-\alpha)}.$$

Thus is $\mu_f = fm$, then $\mathfrak{e}_{\text{poly}}(\mu_f) = \frac{1}{\gamma}(1-\alpha)$ and in particular, $\mathfrak{e}_{\text{poly}}(m) = \frac{1}{\gamma}$ while $\mathfrak{e}_{\text{poly}}(\mu_{\text{SRB}}) = \frac{1}{\gamma} - 1$. Here we see already a difference with exponential escape: the polynomial rate of escape depends strongly on the initial density.

The proof of Theorem 6.2 relies on estimating repeated passes through a neighborhood of the origin coupled with the fact that outside of this neighborhood, the map is uniformly expanding and mixes exponentially fast [DF].

**Problem 8.** *Prove the following simpler estimate. Assume $H = J_h$ for some $h \ge 1$. Prove that $m(\mathring{I}^{n-1} \setminus \mathring{I}^n) \le Cn^{-\frac{1}{\gamma}-1}$ for all $n \ge 0$. $\mathring{I}^{n-1} \setminus \mathring{I}^n$ is the set of points that enters $H$ for the first time at precisely time $n$. Deduce that $m(\mathring{I}^n) \le Cn^{-\frac{1}{\gamma}}$.*

### 6.1.2 Limiting Distribution

Notice that in this setting, we cannot hope to prove the existence of a spectral gap for $\mathcal{L}$ since that would imply exponential decay of correlations. We must rely instead on softer arguments. To control the evolution of measures, define the following class of densities which are Hölder continuous on elements of the partition $\mathcal{Q}_{N_0}$. Let $C^0(\mathcal{Q}_{N_0})$ denote the set of functions which are continuous on each element of $\mathcal{Q}_{N_0}$. For $f \in C^0(\mathcal{Q}_{N_0})$, $f \geq 0$, $p \in \mathbb{R}^+$ and $J \in \mathcal{Q}_{N_0}$, define

$$H_J^p(f) = \begin{cases} 0 & \text{if } f \equiv 0 \text{ on } J \\ \sup_{x \neq y \in J} \dfrac{\log f(x) - \log f(y)}{|x-y|^p} & \text{otherwise} \end{cases}$$

Let $\|f\|_p = \sup_{J \in \mathcal{Q}_{N_0}} H_J^p(f)$, then define

$$\mathcal{F}^p = \left\{ f \in C^0(\mathcal{Q}_{N_0}) : |f|_{L^1(m)} = 1, \text{ and } \|f\|_p < \infty \right\},$$

and $\mathcal{F}_\alpha^p = \mathcal{F}^p \cap \mathcal{F}_\alpha$.

**Theorem 6.3** ([DF]). *Let $f \in \mathcal{F}_\alpha^p$ for some $p > 0$, $\alpha \in [0,1)$. Then*

$$\lim_{n\to\infty} \frac{\mathring{T}_*^n \mu_f}{\mu_f(\mathring{I}^n)} = \delta_0,$$

*where $\delta_0$ denotes the point mass at 0 and the convergence is in the weak sense. Moreover, we have*

$$\lim_{n\to\infty} \frac{\mu_f(\mathring{I}^{n+1})}{\mu_f(\mathring{I}^n)} = 1.$$

Note that Theorem 6.3 applies to Lebesgue measure and indeed to any measure with strictly positive Holder continuous density on $I$. It also applies to $\mu_{\text{SRB}}$ since its density $g_{\text{SRB}}$ belongs to $\mathcal{F}_\gamma^p$ for any $p \in (0, \frac{\gamma}{\gamma+1}]$.

This theorem implies that arbitrarily small holes in systems with polynomial rates of escape can act as large perturbations from the point of view of the physical limit $\frac{\mathring{T}_*^n \mu_f}{\mu_f(\mathring{I}^n)}$. For $H = \emptyset$, this sequence converges to the absolutely continuous SRB measure for the closed system, while for any positive sized hole, it converges to the point mass at 0. From this point of view, the limiting distribution is unstable with respect to small leaks in the system.

Indeed, this is not a special case, but typical behavior for systems with subexponential escape. Indeed, if $T$ is a Borel measurable map of a compact, separable metric space $X$, then it is enough to assume

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu(\mathring{X}^n) = 0,$$

in order to observe that limit points of $\frac{\mathring{T}_*^n \mu}{\mu(\mathring{X}^n)}$ are typically singular with respect to the initial distribution $\mu$ and supported on the survivor set. See [DF, Theorem 2.3].

### 6.1.3 Proof of Theorem 6.3

We begin by establishing some simple facts about the class of densities with which we shall work. Set $\mathcal{B}^p = \{f \in \mathcal{F}^p : \|f\|_p \leq 1\}$ and define $\mathring{\mathcal{L}}_1 f = \mathring{\mathcal{L}} f / |\mathring{\mathcal{L}} f|_{L^1(m)}$.

**Proposition 6.4.** *(a) The set* $\{(1-s)\mu_f + s\delta_0 \ : \ s \in [0,1], f \in \mathcal{B}^p\}$ *is compact.*
*(b) Let* $p \in (0, \frac{\gamma}{\gamma+1}]$ *and* $\alpha \in [0,1)$*. We have* $\mathring{\mathcal{L}}_1(\mathcal{F}^p_\alpha) \subset \mathcal{F}^p_\alpha$*. In addition, there exist two constants* $C_1, C_2 \geq 0$*, such that for every* $f \in \mathcal{F}^p_\alpha$*,*

$$\|\mathring{\mathcal{L}}_1^n f\|_p \leq C_1 \|f\|_p + C_2 \text{ for all } n \geq 1. \tag{6.2}$$

**Problem 9.** *Prove Proposition 6.4(a).*

**Problem 10.** *Prove the inequality* (6.2)*.*

The proof of Theorem 6.3 relies on the above proposition in addition to the following more technical volume estimates,

(a) For $f \in \mathcal{F}_0$, there exists $\bar{C} > 0$ such that[12] for any $n \geq 0$, $\mu_f(\mathring{I}^{n-1} \setminus \mathring{I}^n) \leq \bar{C} n^{-\frac{\gamma+1}{\gamma}} \log n$;

(b) For $f \in \mathcal{F}_\alpha$, there exists $\bar{C} > 0$ such that for any $n \geq 0$, $\mu_f(\mathring{I}^{n-1} \setminus \mathring{I}^n) \leq \bar{C} n^{-\frac{\gamma+1-\alpha}{\gamma}}$.

For $\alpha > 0$, item (b) above together with $\mu_f(\mathring{I}^n) \geq C_f n^{-\frac{1-\alpha}{\gamma}}$ from Theorem 6.2 yields

$$1 \geq \frac{\mu_f(\mathring{I}^{n+1})}{\mu_f(\mathring{I}^n)} = \frac{\mu_f(\mathring{I}^n) - \mu_f(\mathring{I}^n \setminus \mathring{I}^{n+1})}{\mu_f(\mathring{I}^n)} \geq 1 - \frac{\bar{C} n^{-\frac{\gamma+1}{\gamma} + \frac{\alpha}{\gamma}}}{C_f n^{-\frac{1-\alpha}{\gamma}}} = 1 - \frac{\bar{C}}{C_f} n^{-1} \to 1 \text{ as } n \to \infty.$$

A similar conclusion holds for $\alpha = 0$ using item (a) and Theorem 6.2. Consequently, for every $k \geq 1$ (and $\alpha \in [0,1)$),

$$\lim_{n\to\infty} \frac{\mu_f(\mathring{I}^{n+k})}{\mu_f(\mathring{I}^n)} = \prod_{i=0}^{k-1} \left( \lim_{n\to\infty} \frac{\mu_f(\mathring{I}^{n+i+1})}{\mu_f(\mathring{I}^{n+i})} \right) = 1. \tag{6.3}$$

Now, fix $\alpha \in [0,1)$, $p \in (0, \frac{\gamma}{\gamma+1}]$, and $f \in \mathcal{F}^p_\alpha$. We apply Proposition 6.4(b) to conclude that the sequence $\left\{ \frac{\mathring{T}^n_* \mu_f}{\mu_f(\mathring{I}^n)} \right\}_{n\in\mathbb{N}}$ is composed of absolutely continuous probability measures with densities in $\mathcal{F}^p_\alpha$ and the log-Hölder constant $\| \cdot \|_p$ is uniformly bounded along this sequence. By Proposition 6.4(a), any of its limit points must have the form $\mu_\infty = (1 - s_\infty)\mu_{f_\infty} + s_\infty \delta_0$ for some $f_\infty \in \mathcal{F}^p$ and $s_\infty \in [0,1]$. We want to prove that $s_\infty = 1$ for any limit point.

Let $J \in \mathcal{Q}_{N_0}$, let $g_n := \mathring{\mathcal{L}}_1^n f$ and consider a converging subsequence $\{g_{n_j}\}_{j\in\mathbb{N}}$ with limit point $(1 - s_\infty) f_\infty$. (Recall that $\mathring{\mathcal{L}}_1^n$ is the normalized transfer operator.) Since $f_\infty \in \mathcal{F}^p$, the convergence $g_{n_j}|_J \to (1 - s_\infty) f_\infty|_J$ holds in the uniform topology of functions defined on this interval. In particular, its integrals against any bounded measurable function converge as well on each $J \in \mathcal{Q}_{N_0}$.

Fixing $k \geq 1$, note that the set $\bigcup_{i=0}^{k} T^{-i}(H)$ is bounded away from 0 and thus intersects only finitely many elements of $\mathcal{Q}_{N_0}$. Thus the sequence $\{g_{n_j}\}_{j\in\mathbb{N}}$ converges uniformly on this set as well. Now, we have

$$\frac{\mu_f(\mathring{I}^{k+n_j})}{\mu_f(\mathring{I}^{n_j})} = \frac{|\mathring{\mathcal{L}}^{n_j+k} f|_1}{|\mathring{\mathcal{L}}^{n_j} f|_1} = \int_I \mathring{\mathcal{L}}^k g_{n_j} \, dm = \int_{\mathring{I}^k} g_{n_j} \, dm = 1 - \int_{\cup_{i=0}^k T^{-i}(H)} g_{n_j} \, dm,$$

using the fact that $\int_I g_{n_j} \, dm = 1$.

[12] It is likely that the $\log n$ factor can be eliminated from this estimate.

Since the limit of the above expression is 1 by (6.3) and the convergence of $g_{n_j}$ to $(1-s_\infty)f_\infty$ is uniform on each $J$, we must have $f_\infty \equiv 0$ on $\bigcup_{i=0}^{k} T^{-i}(H)$. Since $f_\infty \in \mathcal{F}^p$ is log-Hölder continuous on each $J \in \mathcal{Q}_{N_0}$, we conclude that $f_\infty \equiv 0$ on any $J$ such that $J \cap \left(\bigcup_{i=0}^{k} T^{-i}(H)\right) \neq \emptyset$. Since this holds for all $k$, the transitivity of $T$ implies that we must have $f_\infty \equiv 0$ on all $J \in \mathcal{Q}_{N_0}$, i.e. $s_\infty = 1$. Since the subsequence is arbitrary, it follows that $s_\infty = 1$ for any limit point as desired. This completes the proof of Theorem 6.3.

## 6.2 Geometric Potentials for Manneville-Pomeau Maps

The results of the previous section suggest the following questions for this test case involving systems with polynomial escape.

(1) Can we recover some notion of stability for open systems with slow rates of mixing?

(2) Can we obtain a different perspective by varying the potential?

In this section we describe some preliminary attempts to address these questions which are part of the recent preprint [DT].

Consider the family of potentials $\{t\varphi : t \in \mathbb{R}\}$, $\varphi = -\log|DT|$. Define the related pressure,

$$\mathcal{P}(t\varphi) = \sup\left\{h_\nu(T) + t\int \varphi\, d\nu : \nu \in \mathcal{M}_T\right\}$$

where $\mathcal{M}_T$ denotes the set of $T$-invariant, ergodic, Borel probability measures on $I$. Note that $\mathcal{P}(\varphi) = 0$ and $\mathcal{P}(0) = \log 2 =$ topological entropy of $T$.

We summarize some classical facts about this family of potentials for $T$.

- For $t \in [0,1)$, $\mathcal{P}(t\varphi) > 0$.
- There exists a $t\varphi - \mathcal{P}(t\varphi)$ *conformal measure* $m_t$, i.e. $\frac{dm_t}{d(m_t \circ T)} = e^{t\varphi - \mathcal{P}(t\varphi)}$.
- There exists an *equilibrium state* $\mu_t$ which is an invariant measure absolutely continuous w.r.t. $m_t$ such that $\mu_t$ attains the supremum in the expression for $\mathcal{P}(t\varphi)$.
- For $t < 1$, $\mu_t$ is exponentially mixing with rate $e^{-\mathcal{P}(t\varphi)}$.

As before, we assume $H$ is a finite union of elements of $\mathcal{Q}_{N_0}$ that allows repeated passes through a neighborhood of 0 - this allows the effect of the neutral fixed point to be felt by the open system and does not trivialize the dynamics.

Define the punctured potential $\varphi^H = -\infty$ on $H$ and $\varphi^H = \varphi$ elsewhere. Similarly, define the punctured pressure by,

$$P(t\varphi^H) = \sup\left\{h_\nu(T) + t\int \varphi\, d\nu : \nu \in \mathcal{M}_T \text{ and } \nu(H) = 0\right\}.$$

Note that the condition $\nu(H) = 0$ for an invariant measure is equivalent to requiring that $\nu$ be supported on the survivor set $\mathring{I}^\infty$.

We define the transfer operator for the potential $t\varphi - \mathcal{P}(t\varphi)$ by

$$\mathcal{L}_{t\varphi} f(x) = \sum_{y \in T^{-1}x} f(y) e^{-t\varphi(y) - \mathcal{P}(t\varphi)},$$

and similarly for the punctured potential, $\mathring{\mathcal{L}}_{t\varphi^H}$.

Recall the exponential rate of escape $-\rho(\mu; H)$ defined in (1.1).

**Proposition 6.5** ([DT]). *For $t \in [0,1]$, the exponential escape rate $-\rho(m_t; H)$ with respect to $m_t$ exists and*

*(a)* $\rho(m_t; H) = \mathcal{P}(t\varphi^H) - \mathcal{P}(t\varphi)$;

*(b) for* $t < 1$, $\rho(m_t; H) < 0$.

*(c) Define* $t^H = \sup\{t > 0 : \mathcal{P}(t\varphi^H) > 0\}$. *Then* $t^H = \dim_{Haus}(\mathring{I}^\infty)$.

The proposition suggests that $t^H$ is a dividing line between qualitatively different behaviors with respect to the conformal measures $m_t$.

- If $t < t^H$, then $\mathcal{P}(t\varphi^H) > 0$ so $\rho(m_t; H) > -\mathcal{P}(t\varphi)$.
- If $t \geq t^H$, then $\mathcal{P}(t\varphi^H) = 0$, so $\rho(m_t, H) = -\mathcal{P}(t\varphi)$.

The gap between the escape rate and the pressure for $t < t^H$ means that in this range of parameters, the system behaves as a classical uniformly hyperbolic open system.

Define $\mathring{Y} = [1/2, 1] \setminus H$ and let $\Lambda$ be the recurrent part of $\mathring{Y}$. Define $\tau$ be the first return to $\Lambda$ or the first entry into $H$. Following Section 5, we define the associated Young tower for the open system by

$$\Delta = \{(x, \ell) \in \Lambda \times \mathbb{N} : \ell < \tau(x)\},$$

and the tower map $F$ is defined as usual. In this simple case, since $\tau$ is a first return map, the tail bound is immediate using the conformality of the measure $m_t$ and (6.1),

$$m_t(\tau > n) \leq C e^{-n\mathcal{P}(t\varphi)},$$

since $\{\tau > n\} = T^{-1}(\cup_{k \geq n-1} J_k)$.

**Case 1: Uniformly Hyperbolic Behavior:** $t < t^H$

Since $\mathcal{P}(t\varphi^H) > 0$ for $t$ in this range, we have $-\rho(m_t) < \mathcal{P}(t\varphi)$ by Proposition 6.5(a). So we may choose $\beta \in (-\rho(m_t), \mathcal{P}(t\varphi))$ and define the function space $\mathcal{B}_\Delta$ on $\Delta$ as in Section 5.2 with weight $\beta$.

The potential $t\varphi - \mathcal{P}(t\varphi)$ lifts to a potential $\phi_{\Delta,t}$ on $\Delta$ by setting $\phi_{\Delta,t} = 0$ on $\Delta \setminus F^{-1}(\Delta_0)$ and $\phi_{\Delta,t} = \sum_{i=0}^{\tau-1} t\varphi \circ T^i - \tau\mathcal{P}(t\varphi)$ on $F^{-1}(\Delta_0)$. This definition ensures that induced reference measure $\overline{m}_t$ on $\Delta$ (defined as in the beginning of Section 5.2) is a conformal measure for $\phi_{\Delta,t}$.

One obtains similar Lasota-Yorke inequalities for the associated transfer operator $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$.

$$\|\mathring{\mathcal{L}}^n_{\phi_{\Delta,t}} f\|_{\text{Lip}} \leq C e^{-\beta n} \|f\|_{\text{Lip}} + C|f|_{L^1(\overline{m}_t)} \quad \text{and} \quad \|\mathring{\mathcal{L}}^n_{\phi_{\Delta,t}} f\|_{\text{Lip}} \leq C e^{-\beta n} \|f\|_{\text{Lip}} + C|f|_{L^1(\overline{m}_t)}$$

This implies the essential spectral radius of $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$ is bounded by $e^{-\beta}$. But since the spectral radius of $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$ equals $e^{\rho(m_t)}$ and $-\rho(m_t) < \beta$, we conclude that $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$ is quasi-compact. The mixing property of $T$ and the condition on $H$ imply that in fact $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$ has a spectral gap on $\mathcal{B}_\Delta$. Thus the full set of results for hyperbolic systems with escape as summarized by Corollary 4.7 hold for this class of potentials and the results project down to $\mathring{T}$ and $\mathring{\mathcal{L}}_{t\varphi^H}$ as described in Section 5.3.

**Case 2:** $t \in [t^H, 1)$

In this case, $\rho(m_t; H) = -\mathcal{P}(t\varphi)$. No spectral gap for $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$ exists on $\mathcal{B}_\Delta$ since the spectral radius of $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$ on $\mathcal{B}_\Delta$ = essential spectral radius of $\mathring{\mathcal{L}}_{\phi_{\Delta,t}}$ on $\mathcal{B}$.

However, we may still consider the evolution of averages,

$$\psi_n^t = \frac{1}{n}\sum_{k=0}^{n-1} \frac{\mathring{\mathcal{L}}^k_{t\varphi^H} 1}{|\mathring{\mathcal{L}}^k_{t\varphi^H} 1|_{L^1(m_t)}}.$$

**Theorem 6.6.** *Fix $t \in [t^H, 1)$. Let $\eta_t$ be a limit point of $\{\psi_n^t m_t\}_{n\in\mathbb{N}}$. Then*

$$d\eta_t = \psi_t dm_t \qquad \text{with} \qquad \mathring{\mathcal{L}}_{t\varphi^H}\psi_t = \lambda\psi_t,$$

*for some $\psi_t \in L^1(m_t)$ and $\lambda \in [e^{-\mathcal{P}(t\varphi)}, 1)$, i.e. every limit point of $\{\psi_n^t m_t\}_{n\in\mathbb{N}}$ is a conditionally invariant measure absolutely continuous w.r.t. $m_t$.*

There are thus three regimes for limiting distributions for this class of potentials:

- $t \in [0, t^H)$: The transfer operator on the tower has a spectral gap. The open system exhibits uniformly hyperbolic behavior: for a large class of initial distributions, a unified escape rate and unique limiting distribution absolutely continuous w.r.t. $m_t$ exist.
- $t \in [t^H, 1)$: No spectral gap for the transfer operator on the tower; however, all limiting distributions are absolutely continuous w.r.t. $m_t$.
- $t = 1$: The only limiting distribution is $\delta_0$.

### 6.3 A Physical Invariant Measure on the Survivor Set

In this section, we drop the Markov requirement on $H$. We consider the induced map $\mathring{S} = \mathring{T}^\tau : Y \circlearrowleft$ and the action of the induced transfer operator acting on $BV(Y)$ with induced potential,

$$\Phi_t = t\sum_{i=0}^{\tau-1} \varphi \circ T^i - \tau\mathcal{P}(t\varphi^H).$$

As before, $\Phi_t^H$ denotes the punctured version of $\Phi_t$. It is a standard result that since $S = T^\tau$ is a full-branched Gibbs-Markov map, that the associated transfer operator $\mathcal{L}_{\Phi_t}$ has a spectral gap on $BV(Y)$.

**Theorem 6.7.** *Fix $t \in [0, 1]$. For sufficiently small holes, the spectral gap for $\mathring{\mathcal{L}}_{\Phi_t^H - \mathcal{P}(\Phi_t)}$ persists. Thus the full set of uniformly hyperbolic results for the induced open system hold.*

The proof of this result, omitted here, relies on the same strategy used in Section 4 by considering $\mathring{\mathcal{L}}_{\Phi_t^H - \mathcal{P}(\Phi_t)}$ acting on functions of bounded variation and applying the alternative perturbative framework of [KL1]. We wind up with Lasota-Yorke inequalities of the form (4.3). Although $\mathring{S}$ has countably many branches (intervals on which $\mathring{T}$ is smooth and monotonic), the number of distinct images of these intervals is finite. Thus we can impose the same condition (H) as in Section 4.1 on the minimum size of the images of intervals of monotonicity for $\mathring{S}$ and this is satisfied for a sequence of holes shrinking to a point. This, plus the analogue of Lemma 4.5, are the essential estimates to proving the persistence of a spectral gap for $\mathring{\mathcal{L}}_{\Phi_t^H - \mathcal{P}(\Phi_t)}$ on $BV(Y)$.

Now let $\Lambda_t$ be the largest eigenvalue of $\mathring{\mathcal{L}}_{\Phi_t^H - \mathcal{P}(\Phi_t)}$, and let $\bar{\mu}_t$ define the associated conditionally invariant measure for $\mathring{S}$. Define

$$\overline{\nu}_t(f) = \lim_{n\to\infty} \Lambda_t^{-n} \int_{\mathring{Y}^n} f \, d\bar{\mu}_t \qquad \forall f \in C^0(Y). \tag{6.4}$$

**Theorem 6.8.** *In the setting of Theorem 6.7, the following hold.*

*(a) $\bar{\nu}_t$ is an equilibrium measure for the potential $\Phi_t^H - \mathcal{P}(\Phi_t)$ on $Y$.*

*(b) $\bar{\nu}_t$ projects to an invariant measure $\nu_t$ for $\mathring{T}$, supported on $\mathring{I}^\infty$.*

*(c) If $H_\varepsilon = (z-\varepsilon, z+\varepsilon)$, then as $\varepsilon \to 0$, $\nu_t$ converges weakly to the equilibrium state $\mu_t$ for the potential $t\varphi - \mathcal{P}(t\varphi)$ for the closed system.*

**Remark 6.9.** *This construction works and Theorem 6.8 holds for $t = 1$: $\nu_1 \to \mu_{SRB}$ as $\varepsilon \to 0$. Thus we regain a notion of stability for the open system even in the case in which the limiting distribution is singular.*

We summarize our findings in this section with regard to the pressure of the invariant measures we have constructed on the survivor set.

- If $t \in [0, t^H)$, then $\nu_t$ is an equilibrium state for the potential $t\varphi^H - \mathcal{P}(t\varphi^H)$, i.e.
$$h_{\nu_t}(T) + \int t\varphi^H d\nu_t = \mathcal{P}(t\varphi^H) = \mathcal{P}(t\varphi) + \rho(m_t; H) > 0.$$
- If $t = t^H$, then $\nu_t$ is an equilibrium state for $t\varphi^H - \mathcal{P}(t\varphi^H) = t\varphi^H$, i.e. $h_{\nu_t}(T) + \int t\varphi^H d\nu_t = 0$.
- If $t \in (t^H, 1]$, then $\nu_t$ is not an equilibrium state for $t\varphi^H - \mathcal{P}(t\varphi^H) = t\varphi^H$ and its pressure is negative,
$$h_{\nu_t}(T) + \int t\varphi^H d\nu_t = \frac{\log \Lambda_t + P(\Phi_t)}{\int \tau \, d\overline{\nu}_t} < 0.$$

We end with some open questions for this class of open systems and the potentials we have considered here.

(1) Is there a dynamical characterization of $\nu_t$ similar to (6.4) for $t \in [t^H, 1]$?

(2) If we graph the pressures $\mathcal{P}_{\nu_t} = h_{\nu_t}(T) + \int t\varphi^H d\nu_t$ as a function of $t$ for $t > t^H$, how smooth is this function?

(3) Is there some dynamical significance to the values of $\mathcal{P}_{\nu_t}$ for $t > t^H$?

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