

LIMITING DISTRIBUTIONS FOR COUNTABLE STATE TOPOLOGICAL MARKOV CHAINS WITH HOLES

MARK F. DEMERS, CHRIS IANZANO, PHILIP MAYER, PETER MORFE, AND ELIZABETH C. YOO

ABSTRACT. We study the dynamics of countable state topological Markov chains with holes, where the hole is a countable union of 1-cylinders. For a large class of positive recurrent potentials and under natural assumptions on the surviving dynamics, we prove the existence of a limiting conditionally invariant distribution, which is the unique limit of regular densities under the renormalized dynamics conditioned on non-escape. We also prove the existence of a Gibbs measure on the survivor set, the set of points that never enter the hole, which is an equilibrium measure for the punctured potential of the open system. We prove that the Gurevic pressure on the survivor set equals the exponential escape rate from the open system. These results extend to the non-compact setting results previously available for finite state topological Markov chains.

1. INTRODUCTION

The study of dynamical systems with holes is motivated by the study of systems out of equilibrium - systems in which mass or energy is allowed to escape. Since invariant measures cannot be supported on a bulk of the phase space in such systems, the emphasis becomes a search for physically relevant conditionally invariant measures, sometimes called quasi-stationary states, which are invariant under the dynamics conditioned on non-escape.

Given a measure space (X, \mathcal{B}, m) and a nonsingular, measurable transformation T , one identifies a measurable subset H of the phase space X as the hole. Trajectories that are mapped into H disappear forever and one studies the dynamics conditioned on non-escape. Defining $\dot{X} = X \setminus H$ to be the complement of the hole, we study the dynamics of the open system $\dot{T} : \dot{X} \rightarrow X$ on the sequence of noninvariant domains, $\dot{X}^n = \bigcap_{i=0}^n T^{-i} \dot{X}$. A probability measure μ on X is called *conditionally invariant* if

$$\frac{\mu(T^{-1}(A) \cap \dot{X}^1)}{\mu(\dot{X}^1)} = \mu(A), \quad \text{for all } A \in \mathcal{B}.$$

The scaling factor $\lambda := \mu(\dot{X}^1)$ is sometimes referred to as the eigenvalue of the measure μ since the above relation can be iterated to yield $\mu(T^{-n}(A) \cap \dot{X}^n) = \lambda^n \mu(A)$, so that the conditionally invariant measure necessarily predicts an exponential rate of escape of mass from the open system.

Unfortunately, the existence of conditionally invariant measures with any eigenvalue between 0 and 1 is ubiquitous [DY] and so existence alone becomes meaningless. Rather, one focuses on the limit points of the sequence $\frac{\dot{T}_*^n m}{m(\dot{X}^n)}$, where m is a reference measure of interest and the operator \dot{T}_*^n is defined by $\dot{T}_*^n m(A) = m(T^{-n}(A) \cap \dot{X}^n)$, $\forall A \in \mathcal{B}$. Under certain assumptions, the limit of such a sequence is a conditionally invariant measure and is independent of the initial distribution drawn from a reasonable class of measures. It also predicts a unified exponential rate of escape for

Date: October 27, 2016.

2010 Mathematics Subject Classification. Primary: 37B10, 37C30; Secondary: 37D35.

Key words and phrases. Open systems, transfer operator, conditional limiting distribution, pressure, variational principle.

The majority of this paper was completed as part of Fairfield University's REU program, funded by NSF grant DMS 1358454. MD was partially supported by NSF grant DMS 1362420.

this same class of initial distributions. In this case, we call such a limiting distribution a *physically relevant* conditionally invariant measure.

In probabilistic Markov chains, this type of limiting distribution in the presence of holes (or absorbing states) is called the Yaglom limit and has been studied in [V] and more recently in [FKMP]. The study of deterministic systems with holes was initiated by the work of Pianigiani and Yorke [PY], and since extended to a number of hyperbolic systems, beginning with those that admit finite Markov partitions: expanding maps on \mathbb{R}^n [PY, CMS1]; Smale horseshoes [C1]; finite state topological Markov chains [CMS2]; Anosov diffeomorphisms [CM1, CM2, CMT1, CMT2]; and billiards with convex scatterers satisfying a non-eclipsing condition (which makes the open system an Axiom A diffeomorphism) [LM]. These results were then extended to hyperbolic systems without Markov partitions, including piecewise expanding maps of the interval [K, BC, LiM, D1]; certain classes of unimodal maps [D2, BDM]; and more general dispersing billiards [DWY, D4], including those with corner points [D3]. All the systems listed above admit physically relevant conditionally invariant limiting distributions and enjoy a unified exponential rate of escape for a large class of initial distributions.

Recently, there has been interest in open systems exhibiting polynomial rates of escape [DO, APT, DR], and in particular their connection to slowly mixing systems from non-equilibrium statistical mechanics [Y]. Such systems exhibit qualitatively different behavior from systems with exponential escape rates; for example, the limiting distributions obtained by pushing forward and renormalizing, i.e. the limit points of $\frac{\hat{T}_*^n m}{m(\hat{X}^n)}$, are not conditionally invariant measures, but rather singular invariant measures [DF]. Indeed, no physically relevant conditionally invariant measures can exist for such systems due to the subexponential rate of escape.

Typically, such slowly mixing systems are studied via an induced map on the phase space: One chooses a subset $Y \subset X$ and studies the return map $T^R : Y \rightarrow Y$, where R is the first return time to Y . The usual strategy is to prove results for the induced system T^R , which has stronger hyperbolicity than the original map, and then pass those results back to the original system.

In many situations, the return maps can be constructed to admit countable (but not finite) Markov partitions and their dynamics can be studied via conjugacy to symbolic dynamics, i.e., topological Markov chains. Such techniques are very powerful and are by now classical in the study of dynamical systems. Unfortunately, while finite state topological Markov chains with holes have been well-studied ([CMS2], see also the recent book [CMSa]), the corresponding results for countable state chains are so far unavailable. This motivates the present work: To study countable state topological Markov chains with holes and prove, under a natural set of assumptions and for a general class of potentials, results analogous to those proved for many of the hyperbolic systems listed above. Our hope is that this work will provide a standard reference to future studies of open systems, in particular to those seeking to expand the study of systems with subexponential rates of escape.

There are several complications in this setting. First, our space is not necessarily locally compact and so several of the function space arguments previously used in open systems have to be reformulated. Secondly, open systems are not topologically transitive or positive recurrent in the usual sense¹ of the literature (for example [S1]), so one must formulate alternative conditions which generalize the notions of mixing and recurrence sufficiently to prove strong convergence results for the open system.

We introduce our topological Markov chain, formulate our assumptions and state our main theorems in Section 2. Section 3 contains the proofs of some preliminary facts about the spaces of functions we shall use while Section 4 contains the proof of Theorem 2.1 regarding convergence of a large class of initial densities to a conditionally invariant measure and a unified exponential

¹This is due to the fact that $X \setminus H$ can be decomposed (mod 0) into a disjoint countable union of sets, $X \setminus H = \cup_{n=0}^{\infty} \hat{X}^n \setminus \hat{X}^{n+1}$, satisfying $T(\hat{X}^n \setminus \hat{X}^{n+1}) = \hat{X}^{n-1} \setminus \hat{X}^n$ and $T^{n+1}(\hat{X}^n \setminus \hat{X}^{n+1}) \subseteq H$.

rate of escape. In Section 5 we prove a variational principle, along with Theorem 2.2, linking the escape rate of the open system to the pressure of the closed system restricted to the survivor set, the singular set of points that never enters the hole.

2. SETTING AND MAIN RESULTS

Let \mathcal{S} denote the countable state space of the Markov chain, which we take to be a subset of \mathbb{N} , and A the adjacency matrix, i.e. $A_{i,j} = 1$ if the transition from state i to state j is permitted and 0 otherwise. The associated topological Markov chain is defined as the set of all admissible sequences,

$$\Sigma = \{x = (x_0, x_1, x_2, \dots) \in \mathcal{S}^{\mathbb{N}} \mid A_{x_j, x_{j+1}} = 1, \forall j \geq 0\}.$$

We denote by σ the one-sided shift on Σ , i.e. $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$.

We endow Σ with the usual separation time metric: Fix $\theta \in (0, 1)$ and for $x, y \in \Sigma$, define

$$d_\theta(x, y) = \theta^{s(x,y)}, \text{ where } s(x, y) = \min\{i \in \mathbb{N} \mid x_i \neq y_i\}.$$

With this metric, the space (Σ, d_θ) is complete, but may not be compact when \mathcal{S} is infinite. Indeed, it is not even locally compact unless $\#\{k \in \mathcal{S} \mid A_{s,k} = 1\} < \infty$, for each $s \in \mathcal{S}$. In what follows, we will not assume that Σ is locally compact.

We will denote cylinder sets in Σ in the usual way,

$$[i_0, \dots, i_{n-1}] = \{x \in \Sigma \mid x_k = i_k \text{ for } k = 0, 1, \dots, n-1\}.$$

A cylinder of length n is called an n -cylinder. Cylinders are both open and closed as sets and form a basis for the metric topology on Σ .

We will assume that our topological Markov chain is topologically mixing:

$$\forall i, j \in \mathcal{S} \quad \exists N_{ij} \in \mathbb{N} : \forall n \geq N_{ij} \quad \sigma^{-n}([i]) \cap [j] \neq \emptyset.$$

A topological Markov Chain satisfies the *big images and preimages* property (BIP) if there exists a finite set $\Lambda \subset \mathcal{S}$ such that

$$\forall s \in \mathcal{S}, \exists i, j \in \Lambda \text{ such that } A_{s,i} A_{j,s} = 1.$$

2.1. Introduction of the Hole and Mixing for the Open System. We assume that our hole H is a countable union of 1-cylinders in Σ , i.e. $H = \cup_{i \in \mathcal{S}_H} [i]$, for some subset $\mathcal{S}_H \subset \mathcal{S}$. Denote the complement of the hole by $\overset{\circ}{\Sigma} = \Sigma \setminus H$; more generally, $\overset{\circ}{\Sigma}^n = \cap_{k=0}^n \sigma^{-k}(\Sigma \setminus H)$, denotes the set of points which have not entered H by time n . The dynamics of the open system is defined by $\overset{\circ}{\sigma} = \sigma|_{\overset{\circ}{\Sigma}}$ and its iterates, $\overset{\circ}{\sigma}^n = \sigma^n|_{\overset{\circ}{\Sigma}^n}$.

Topological transitivity and mixing on general open sets do not make sense for open systems, yet we are able to formulate a condition on 1-cylinders which ensures that the open system is not split into disjoint components by the introduction of the hole. We assume that our open system satisfies the following condition.

- (H)** There exists a finite set $\Lambda_H \subset \mathcal{S} \setminus \mathcal{S}_H$ such that
- (a) for each $s \in \mathcal{S} \setminus \mathcal{S}_H$, there exists $j, k \in \Lambda_H$ such that $A_{j,s} A_{s,k} = 1$; and
 - (b) for each $k \in \Lambda_H$, there exists $n \in \mathbb{N}$ such that $\overset{\circ}{\sigma}^n([k]) = \overset{\circ}{\Sigma}$.

We remark that when $H = \emptyset$, BIP plus mixing imply **(H)** with $\Lambda_H = \Lambda$ so that **(H)** is a natural adaptation of BIP plus mixing to an open system.

2.2. Transfer Operator. A function $\varphi : \Sigma \rightarrow \mathbb{R}$ is called locally Lipschitz (or locally Hölder continuous with parameter θ) if

$$\text{Lip}(\varphi) := \sup_{i \in \mathcal{S}} \sup_{\substack{x, y \in [i] \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{d_\theta(x, y)} < \infty.$$

Notice that we do not require any regularity between 1-cylinders so that φ may not be bounded on Σ .

Using a locally Lipschitz φ , we define the associated transfer operator \mathcal{L}_φ acting on continuous functions by

$$\mathcal{L}_\varphi f(x) = \sum_{y \in \sigma^{-1}(x)} f(y) e^{\varphi(y)} \quad \text{and its iterates} \quad \mathcal{L}_\varphi^n f(x) = \sum_{y \in \sigma^{-n}(x)} f(y) e^{S_n \varphi(y)},$$

where $S_n \varphi = \sum_{k=0}^{n-1} \varphi \circ \sigma^k$ is the n th ergodic sum. We will assume that $|\mathcal{L}_\varphi 1|_\infty < \infty$. It follows from this plus BIP and mixing that:

- the Gurevic pressure $P_G(\varphi)$ is finite [S1, Theorem 1];
- φ is positive recurrent [S2, Corollary 2] (see also [MU]);
- there exists a finite conformal Borel measure m , positive on cylinders, such that, $\frac{dm}{dm \circ \sigma} = e^{\varphi - P_G(\varphi)}$ [S1, Theorem 4, Proposition 3]

We will define Gurevic pressure in Section 5.3. Since m is finite, we normalize it to be a probability measure. We may also replace φ by $\varphi - P_G(\varphi)$ so without loss of generality, we may assume $P_G(\varphi) = 0$.

When we introduce a hole H comprised of a countable union of 1-cylinders, we define the related ‘punctured’ potential φ_H for the open system by $\varphi_H = \varphi$ on $\mathring{\Sigma}$ and $\varphi_H = -\infty$ on H . Notice that φ_H is still locally Lipschitz where defined. The associated transfer operator for the open system and its iterates are given by

$$\mathring{\mathcal{L}}_\varphi^n f(x) = \mathcal{L}_\varphi^n(\mathbf{1}_{\mathring{\Sigma}^n} f)(x) = \sum_{y \in \mathring{\sigma}^{-n}(x)} f(y) e^{S_n \varphi(y)} \quad \text{for } n \geq 1,$$

where $\mathbf{1}_A$ denotes the indicator function of a set A .

The importance of the operator $\mathring{\mathcal{L}}_\varphi$ from the point of view of the open system stems from the following relation. Recalling that we have normalized the pressure of φ to be 0, we observe,

$$(2.1) \quad \int_\Sigma \mathring{\mathcal{L}}_\varphi^n 1 \, dm = \int_\Sigma \mathcal{L}_\varphi^n(\mathbf{1}_{\mathring{\Sigma}^n}) \, dm = \int_{\mathring{\Sigma}^n} 1 \, dm,$$

due to the conformality of m , so that the iterates of $\mathring{\mathcal{L}}_\varphi$ govern the rate of escape of mass with respect to m .

We will use interchangeably the notation $\eta(f) = \int f \, d\eta$ for a given measure η on Σ .

2.3. Main Results. We denote by $L^1(m)$ the set of (complex valued) integrable functions on Σ and by $\mathcal{C}^0(\Sigma)$ the set of bounded continuous functions $\Sigma \rightarrow \mathbb{C}$. $\mathcal{C}^0(\Sigma)$ is a Banach space equipped with the norm

$$|f|_\infty = \sup\{|f(x)| : x \in \Sigma\}.$$

Similarly, define a space of bounded, locally Lipschitz functions on Σ by

$$\text{Lip}(\Sigma) = \{f \in \mathcal{C}^0(\Sigma) : \|f\|_{\text{Lip}} < \infty\},$$

where $\|f\|_{\text{Lip}} := |f|_\infty + \text{Lip}(f)$. Notice that with this definition, $\|f \cdot g\|_{\text{Lip}} \leq \|f\|_{\text{Lip}} \|g\|_{\text{Lip}}$. Also, since m is a probability measure, $\mathcal{C}^0(\Sigma) \subset L^1(m)$.

Since m is conformal with respect to φ , \mathcal{L}_φ and $\mathring{\mathcal{L}}_\varphi$ are both bounded linear operators on $L^1(m)$. It follows from the fact that characteristic functions of cylinders are in $\text{Lip}(\Sigma)$ as well as

the assumption that $|\mathcal{L}_\varphi 1|_\infty < \infty$ that \mathcal{L}_φ is a bounded linear operator on both $\mathcal{C}^0(\Sigma)$ and $\text{Lip}(\Sigma)$. Indeed, we will prove that \mathcal{L}_φ has a spectral gap on $\text{Lip}(\Sigma)$.

Theorem 2.1 (Unique Limiting Distribution). *Suppose (Σ, σ) is topologically mixing and satisfies the BIP property. Let φ be a locally Lipschitz potential with $|\mathcal{L}_\varphi 1|_\infty < \infty$ and let H be a hole in Σ satisfying condition **(H)**. Let $\lambda = \rho(\mathcal{L}_\varphi)$ denote the spectral radius of \mathcal{L}_φ on $\text{Lip}(\Sigma)$.*

Then there exists a probability density $g \in \text{Lip}(\Sigma)$, bounded away from 0 on $\dot{\Sigma}$, such that:

- a) $\mathcal{L}_\varphi g = \lambda g$ and $d\mu := gdm$ defines a conditionally invariant probability measure for $\dot{\sigma}$ with eigenvalue λ ;
- b) the rate of escape from the open system is exponential: $\log \lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \log m(\dot{\Sigma}^n)$;
- c) \mathcal{L}_φ has a spectral gap on $\text{Lip}(\Sigma)$: λ is a simple eigenvalue and the remainder of the spectrum of \mathcal{L}_φ is contained in a disk of radius $\rho' < \lambda$;
- d) If $f \in \text{Lip}(\Sigma)$, then

$$\lim_{n \rightarrow \infty} \lambda^{-n} \mathcal{L}_\varphi^n f = c(f)g,$$

for some constant $c(f)$, and convergence is in $\|\cdot\|_{\text{Lip}}$ at an exponential rate. Moreover, $c(f) > 0$ if and only if

$$\left\| \frac{\mathcal{L}_\varphi^n f}{|\mathcal{L}_\varphi^n f|_{L^1(m)}} - g \right\|_{\text{Lip}} \leq C\tau^n \|f\|_{\text{Lip}},$$

for some $C > 0$ independent of f , and $\tau = \rho'/\lambda < 1$.

We remark that for conditionally invariant measures to be physically relevant in the sense of Theorem 2.1(d), one must have convergence (and escape) occurring at an exponential rate. Thus the assumption of big images and preimages and the analogue for the open system, **(H)**, that we have formulated are crucial from this point of view. If one weakens these assumptions to, for example, a positive recurrent potential with finite pressure, then the rate of convergence to equilibrium for the closed system may be subexponential (see [S1]) and thus there will be no physically relevant conditionally invariant measures for the open system. For recent results regarding limiting distributions in open systems with subexponential rates of escape, see [DF].

Next we turn to the survivor set, $\dot{\Sigma}^\infty = \bigcap_{n=0}^\infty \sigma^{-n}(\Sigma \setminus H)$, the zero m -measure set of points that never enter H . While it may seem that λ is an artifact of the function space $\text{Lip}(\Sigma)$ that we have chosen to work with, our next theorem demonstrates that this number (which depends on the potential φ) is intrinsic to the open system in that $\log \lambda$ equals the pressure on the survivor set.

Theorem 2.2 (Gibbs Measure and Variational Principle). *Let \mathcal{M} denote the set of σ -invariant Borel probability measures on Σ . Under the assumptions of Theorem 2.1, the following hold.*

- a) $\log \lambda = \sup \left\{ h_\eta(\sigma) + \int \varphi_H d\eta : \eta \in \mathcal{M}, \eta(-\varphi_H) < \infty \right\},$

where² $h_\eta(\sigma)$ is the measure-theoretic entropy of η .

- b) There exists $\nu \in \mathcal{M}$ which is realized by the following limit,

$$\nu(f) = \lim_{n \rightarrow \infty} \lambda^{-n} \int_{\dot{\Sigma}^n} f g dm \quad \text{for all } f \in \text{Lip}(\Sigma).$$

The measure ν is a Gibbs measure for the potential $\varphi - \log \lambda$, enjoys exponential decay of correlations, and attains the supremum in the variational principle in (a), i.e. ν satisfies

²Since $\varphi_H|_H = -\infty$, the condition $\eta(-\varphi_H) < \infty$ implies that $\eta(H) = 0$, and so $\eta(\varphi_H) = \eta(\varphi)$. Since η is invariant, this in turn implies that the support of η is contained in $\dot{\Sigma}^\infty$.

the escape rate formula,

$$\log \lambda = h_\nu(\sigma) + \int \varphi_H d\nu.$$

- c) An equivalent characterization of ν is $d\nu = g dm_H$, where m_H is a positive Borel measure with support equal to $\mathring{\Sigma}^\infty$ that is conformal for the potential $\varphi_H - \log \lambda$, i.e., $\lambda^{-1} \mathring{\mathcal{L}}_\varphi^* m_H = m_H$.
- d) The following criterion holds for convergence to g : if $f \in \text{Lip}(\Sigma)$ and $f \geq 0$, then

$$\lim_{n \rightarrow \infty} \frac{\mathring{\mathcal{L}}_\varphi^n f}{|\mathring{\mathcal{L}}_\varphi^n f|_{L^1(m)}} = g \iff \int_\Sigma f d\nu > 0,$$

where the convergence is in $\text{Lip}(\Sigma)$ at an exponential rate.

The role played by the invariant Gibbs measure ν in linking the escape rate with the pressure on the survivor set is further justification for the assumption **(H)**: Having big images is a necessary condition for the existence of a Gibbs measure [S1, Theorem 8] (see also [S2, Theorem 1]).

Remark 2.3. Since $\lambda^n = \mu(\mathring{\Sigma}^n)$, the characterization of ν in Theorem 2.2(b) can be restated in terms of the limit of conditional probabilities,

$$\nu([i_0, i_1, \dots, i_{k-1}]) = \lim_{n \rightarrow \infty} \mu([i_0, i_1, \dots, i_{k-1}] \mid \mathring{\Sigma}^n),$$

for any k -cylinder in Σ . Equivalently, for any $f \in \text{Lip}(\Sigma)$, $\nu(f) = \lim_{n \rightarrow \infty} \mathbb{E}_\mu[f \mid \mathring{\Sigma}^n]$.

3. FUNCTION SPACES AND PRELIMINARY ESTIMATES

For the remainder of the paper, we fix a potential φ satisfying $|\mathcal{L}_\varphi 1|_\infty < \infty$ and assume (Σ, σ, H) satisfies the hypotheses of Theorem 2.1. Our main task will be to prove the existence of a spectral gap for $\mathring{\mathcal{L}}_\varphi$, from which most of our other results follow. We begin with a standard distortion estimate, whose proof we record for completeness.

Lemma 3.1. *There exists a constant $C_d > 0$ such that for all n -cylinders $[i_0, \dots, i_{n-1}]$, $n \geq 1$, and all $x, y \in [i_0, \dots, i_{n-1}]$,*

$$(3.1) \quad \log \left(\frac{e^{S_n \varphi(x)}}{e^{S_n \varphi(y)}} \right) \leq C_d d_\theta(\sigma^n(x), \sigma^n(y));$$

$$(3.2) \quad \left| e^{S_n \varphi(x) - S_n \varphi(y)} - 1 \right| \leq C_d d_\theta(\sigma^n(x), \sigma^n(y)).$$

Proof. Fix $x, y \in [i_0, \dots, i_{n-1}]$. Then

$$\begin{aligned} \log \left[\frac{\exp(S_n \varphi(x))}{\exp(S_n \varphi(y))} \right] &= \sum_{i=0}^{n-1} \varphi \circ \sigma^i(x) - \varphi \circ \sigma^i(y) \leq \text{Lip}(\varphi) \sum_{i=0}^{n-1} d_\theta(\sigma^i(x), \sigma^i(y)) \\ &= \text{Lip}(\varphi) \left(\sum_{j=1}^n \theta^j \right) d_\theta(\sigma^n(x), \sigma^n(y)), \end{aligned}$$

and (3.1) follows with constant $C'_d = \text{Lip}(\varphi) \left(\frac{\theta}{1-\theta} \right)$.

To prove (3.2), note that $|e^z - 1| \leq e^{|z|} |z| \forall z \in \mathbb{R}$. Setting $z = S_n \varphi(x) - S_n \varphi(y)$, using the proof of (3.1) and noting that $d_\theta(\cdot, \cdot) \leq 1$, yields (3.2) with constant $e^{C'_d} C'_d$. \square

The distortion bound (3.1) together with the conformality of m implies that for any n -cylinder $[i_0, \dots, i_{n-1}]$ and any $x \in [i_0, \dots, i_{n-1}]$,

$$(3.3) \quad e^{S_n \varphi(x)} \leq e^{C_d} \frac{m([i_0, \dots, i_{n-1}])}{m(\sigma^n([i_0, \dots, i_{n-1}]))} \leq e^{C_d \kappa^{-1}} m([i_0, \dots, i_{n-1}]),$$

where $\kappa = \min\{m([i]) : i \in \Lambda_H\} > 0$ and Λ_H is the finite set in **(H)**.

Since σ^n is injective on each n -cylinder, these estimates immediately imply that for any $f \in \mathcal{C}^0(\Sigma)$, $x \in \Sigma$,

$$|\mathcal{L}_\varphi^n f(x)| = \left| \sum_{y \in \sigma^{-n}(x)} f(y) e^{S_n \varphi(y)} \right| \leq \|f\|_\infty e^{C_d \kappa^{-1}},$$

where we have used the fact that m is a probability measure, so that \mathcal{L}_φ is a bounded, linear operator on $\mathcal{C}^0(\Sigma)$ with spectral radius at most 1. Unfortunately, we need greater regularity for our function spaces, so we will work in $\text{Lip}(\Sigma)$. Since we are not assuming that Σ is compact or even locally compact, it may be that the unit ball of $\text{Lip}(\Sigma)$ is not relatively compact in $\mathcal{C}^0(\Sigma)$. Instead, we will use $L^1(m)$ as our weak norm. It is a standard fact that the closed unit ball of $\text{Lip}(\Sigma)$ is relatively compact in $L^1(m)$. Before characterizing the spectrum of \mathcal{L}_φ on $\text{Lip}(\Sigma)$, we first prove some greater regularity properties of evolved densities $\mathcal{L}_\varphi^n f$.

3.1. A fixed point for the normalized transfer operator. To obtain the spectral decomposition of \mathcal{L}_φ , it will be convenient to first prove the existence of a positive eigenfunction g . To this end, we define a set of log-Lipschitz probability densities: Given $C > 0$, let

$$L_C = \left\{ f \in L^1(m) \mid f \geq 0, \int_\Sigma f dm = 1, \text{Lip}(\log f) \leq C \right\}.$$

By convention, we set $\text{Lip}(\log f|_{[i]}) = 0$ if $f \equiv 0$ on $[i]$. Define the normalized transfer operator by $\mathcal{L}_1 f = \frac{\mathcal{L}_\varphi f}{|\mathcal{L}_\varphi f|_1}$, where $|\cdot|_1$ denotes the $L^1(m)$ norm. Notice that \mathcal{L}_1 is a nonlinear operator. This section is devoted to the proof of the following proposition.

Proposition 3.2. *Let $K = \frac{C_d}{1-\theta}$. Then $\mathcal{L}_1(L_K) \subset L_K$ and L_K contains a function $g \in \text{Lip}(\Sigma)$ with $g \geq 0$ and $\mathcal{L}_\varphi g = \lambda g$ for $\lambda = \int_{\Sigma} g dm > 0$.*

Note that when $\#(\mathcal{S}) = \infty$, L_K is not compact in $L^1(m)$ since the log-Lipschitz constant does not control the L^∞ norm of a function. Thus there are unbounded functions in L_K . As the next several lemmas will show, however, $\mathcal{L}_1(L_K) \subset L_K$ is bounded in $\mathcal{C}^0(\Sigma)$ and so it is possible to define an invariant subset of L_K which is compactly embedded in $L^1(m)$.

Lemma 3.3. *For all $f \in L^1(m)$ such that $\text{Lip}(\log f) < \infty$ and all $n \geq 0$,*

$$\text{Lip}(\log \mathcal{L}_\varphi^n f) \leq \theta^n \text{Lip}(\log f) + C_d.$$

Proof. Recall that \mathcal{S}_H denotes the collection of states in \mathcal{S} corresponding to H . For $i \in \mathcal{S}_H$, $\mathcal{L}_\varphi^n f \equiv 0$ on $[i]$ for each $n \geq 1$.

Now fix $i \in \mathcal{S} \setminus \mathcal{S}_H$ and suppose $x, y \in [i]$. For each $n \in \mathbb{N}$, note that $\sigma^{-n}(x)$ and $\sigma^{-n}(y)$ are in 1-1 correspondence since σ^n is a bijection from $\Sigma^n \cap [j_0, j_1, \dots, j_{n-1}, i]$ to its image for each $(n+1)$ -cylinder $[j_0, \dots, j_{n-1}, i]$. So we may enumerate the elements of $\sigma^{-n}(x) = \{u_j\}_{j \in J}$ and $\sigma^{-n}(y) = \{v_j\}_{j \in J}$ so that u_j and v_j lie in the same $(n+1)$ -cylinder for each j , and J is the relevant

index set. Now we estimate,

$$\begin{aligned} \mathcal{L}_\varphi^n f(x) &= \sum_{j \in J} f(u_j) e^{S_n \varphi(u_j)} \leq \sum_{j \in J} f(v_j) e^{\text{Lip}(\log f) d_\theta(u_j, v_j)} e^{S_n \varphi(v_j)} e^{C_d d_\theta(x, y)} \\ &\leq \mathcal{L}_\varphi^n f(y) e^{(\theta^n \text{Lip}(\log f) + C_d) d_\theta(x, y)} \end{aligned}$$

where we have used the fact that $d_\theta(u_j, v_j) = \theta^n d_\theta(x, y)$. The lemma follows by noting that

$$\text{Lip}(\log f) = \sup_{i \in \mathcal{S}} \sup_{x, y \in [i]} \log \left(\frac{f(x)}{f(y)} \right) d_\theta(x, y)^{-1}.$$

□

Let $M_K = \{f \in L^1(m) \mid f \geq 0, \int_{\hat{\Sigma}} f dm \leq 1, \text{Lip}(\log f) \leq K\}$. Thus $L_K \subset M_K$.

Lemma 3.4. *Let $K = \frac{C_d}{1-\theta}$. There exists $C_1 > 0$ such that if $f \in M_K$, then $|\mathcal{L}_\varphi f|_\infty \leq C_1$.*

Proof. Suppose $f \in M_K$ and fix $i \in \Lambda_H$, where Λ_H is the finite set of states from **(H)**. Since $\int_{[i]} f dm \leq 1$, there exists $y \in [i]$ such that $f(y) \leq m([i])^{-1}$. Thus if $x \in [i]$, then due to the regularity of f ,

$$f(x) \leq f(y) e^{K d_\theta(x, y)} \leq m([i])^{-1} e^K.$$

Let $G_{\Lambda_H} = \cup_{i \in \Lambda_H} [i]$. Recalling that $\kappa = \min\{m([i]) : i \in \Lambda_H\}$, we have shown that

$$(3.4) \quad \forall f \in M_K, \sup_{x \in G_{\Lambda_H}} f(x) \leq \kappa^{-1} e^K.$$

Note that $\mathcal{L}_\varphi f \geq 0$ if $f \geq 0$ and $\int_{\hat{\Sigma}^1} \mathcal{L}_\varphi f dm = \int_{\hat{\Sigma}^1} f dm \leq 1$. Also, by Lemma 3.3, if $\text{Lip}(\log f) \leq K$, then $\text{Lip}(\log \mathcal{L}_\varphi f) \leq K$, so that $\mathcal{L}_\varphi(M_K) \subset M_K$. Thus we may apply (3.4) to $\mathcal{L}_\varphi f$.

Next we turn our attention to proving an upper bound on $\mathcal{L}_\varphi f(x)$ for $x \in \Sigma \setminus G_{\Lambda_H}$. For each $y \in \hat{\sigma}^{-1}(x)$, let $[i_y]$ denote the 1-cylinder containing y . Due to **(H)**(a), $\hat{\sigma}([i_y]) \supset [j]$ for some $j \in \Lambda_H$. Thus we may organize $\hat{\sigma}^{-1}(x)$ according to these images. Define for each $j \in \Lambda_H$,

$$P_j(x) = \{y \in \hat{\sigma}^{-1}(x) : \hat{\sigma}([i_y]) \supset [j]\}.$$

For each $j \in \Lambda_H$, choose a point $z_j \in [j]$. Note that each $y \in \hat{\sigma}^{-1}(x)$ may be contained in more than one set $P_j(x)$. But for each $y \in P_j(x)$, there exists $w_y \in [i_y]$ such that $\hat{\sigma}(w_y) = z_j$. Now using the regularity of f and bounded distortion, we estimate

$$\begin{aligned} \mathcal{L}_\varphi f(x) &= \sum_{y \in \hat{\sigma}^{-1}(x)} f(y) e^{\varphi(y)} \leq \sum_{j \in \Lambda_H} \sum_{y \in P_j(x)} f(y) e^{\varphi(y)} \\ &\leq \sum_{j \in \Lambda_H} \sum_{y \in P_j(x)} e^K f(w_y) e^{C_d e^{\varphi(w_y)}} \leq e^K e^{C_d} \sum_{j \in \Lambda_H} \mathcal{L}_\varphi f(z_j) \leq \kappa^{-1} e^{2K+C_d} \#\{j \in \Lambda_H\}, \end{aligned}$$

where for the last inequality, we have applied (3.4) to $\mathcal{L}_\varphi f(z_j)$, since $\mathcal{L}_\varphi f \in M_K$ and $z_j \in G_{\Lambda_H}$. This proves the lemma with $C_1 = \kappa^{-1} e^{2K+C_d} \#\{j \in \Lambda_H\}$. □

Lemma 3.5. *There exists $C_2 \in (0, 1)$ such that if $f \in L_K$, then $\int \mathcal{L}_\varphi f dm \geq C_2$.*

Proof. First notice that for $f \in L_K$,

$$(3.5) \quad \int \mathcal{L}_\varphi f dm = \int_{\hat{\Sigma}^1} f dm \quad \text{and} \quad \int_{\hat{\Sigma} \setminus \hat{\Sigma}^1} f dm + \int_{\hat{\Sigma}^1} f dm = 1.$$

Both $\hat{\Sigma}^1$ and $\hat{\Sigma} \setminus \hat{\Sigma}^1$ are unions of 2-cylinders. For each $i \in \mathcal{S} \setminus \mathcal{S}_H$, define $\Lambda_{H,i} = \{j \in \Lambda_H : A_{i,j} = 1\}$. Also, let $[iH]$ denote the union of 2-cylinders in $[i]$ which map to H . Now

$$\frac{\int_{\hat{\Sigma} \setminus \hat{\Sigma}^1} f dm}{\int_{\hat{\Sigma}^1} f dm} \leq \frac{\sum_{i \in \mathcal{S} \setminus \mathcal{S}_H} \int_{[iH]} f dm}{\sum_{i \in \mathcal{S} \setminus \mathcal{S}_H} \sum_{j \in \Lambda_{H,i}} \int_{[i,j]} f dm} \leq \sup_{i \in (\mathcal{S} \setminus \mathcal{S}_H) \cap Q_f} \frac{\sup_{x \in [i]} f(x) m([iH])}{\inf_{x \in [i]} f(x) \sum_{j \in \Lambda_{H,i}} m([i,j])},$$

where $Q_f = \{i \in S \mid \exists x \in [i] : f(x) > 0\}$ and we have used the fact that for two sequences of positive terms, $a_i, b_i > 0$ with $\sum_i b_i < \infty$, then $\sum_i \frac{a_i}{b_i} \leq \sup_i \frac{a_i}{b_i}$.

By conformality there exist $u_i \in [i]$ and $v_{i,j} \in [i, j]$ such that

$$m([iH]) \leq e^{\varphi(u_i)} m(H) \quad \text{and} \quad m([i, j]) \geq e^{\varphi(v_{i,j})} m([j]).$$

Thus using Lemma 3.1,

$$\frac{\int_{\dot{\Sigma} \setminus \dot{\Sigma}^1} f \, dm}{\int_{\dot{\Sigma}^1} f \, dm} \leq \sup_{i \in S \setminus \mathcal{S}_H} e^K e^{C_d} \frac{m(H)}{\sum_{j \in \Lambda_{H,i}} m([j])} \leq e^{K+C_d} \kappa^{-1} m(H) =: B.$$

On the other hand, using this estimate together with (3.5) yields,

$$B \geq \frac{1 - \int_{\dot{\Sigma}^1} f \, dm}{\int_{\dot{\Sigma}^1} f \, dm} = \frac{1}{\int_{\dot{\Sigma}^1} f \, dm} - 1,$$

which implies that $\int \mathcal{L}_\varphi f \, dm \geq 1/(B+1)$. \square

As mentioned earlier, L_K is not necessarily compact in $L^1(m)$ when \mathcal{S} is infinite. However, motivated by Lemmas 3.4 and 3.5, we set $C_\# = C_1/C_2$ and define a subset of L_K as follows.

$$L_K^\infty = \{f \in L_K : |f|_\infty \leq C_\#\}.$$

Lemma 3.6. $\mathcal{L}_1 : L_K^\infty \rightarrow L_K^\infty$ is a well-defined continuous map, where $K = \frac{C_d}{1-\theta}$.

Proof. As already noted in the proof of Lemma 3.4, $\mathcal{L}_\varphi(M_K) \subset M_K$. Indeed, \mathcal{L}_1 is continuous on L_K since \mathcal{L}_φ is continuous on $\text{Lip}(\Sigma)$ and the normalization factor, $|\mathcal{L}_\varphi f|_1$, is uniformly bounded away from 0 for $f \in L_K$ by Lemma 3.5. Thus $\mathcal{L}_1(L_K) \subset L_K$.

Finally, suppose $f \in L_K^\infty \subset M_K$. Then $|\mathcal{L}_\varphi f|_\infty \leq C_1$ by Lemma 3.4 and $|\mathcal{L}_1 f|_\infty = |\mathcal{L}_\varphi f|_\infty / |\mathcal{L}_\varphi f|_1 \leq C_1/C_2 = C_\#$ by Lemma 3.5. Thus $\mathcal{L}_1 f \in L_K^\infty$ and $\mathcal{L}_1(L_K^\infty) \subset L_K^\infty$. \square

Proposition 3.7. L_K^∞ is a convex, compact subset of $L^1(m)$.

Proof. Recall that the closed balls $B_R = \{f \in \text{Lip}(\Sigma) \mid \|f\|_{\text{Lip}} \leq R\}$ are compact in $L^1(m)$. Thus, to prove that L_K^∞ is compact, it suffices to show that L_K^∞ is bounded in $\text{Lip}(\Sigma)$ and closed in $L^1(m)$.

Claim 1: L_K^∞ is bounded in $\text{Lip}(\Sigma)$.

Suppose $f \in L_K^\infty$. By definition, $|f|_\infty \leq C_\#$. Now fix $i \in \mathcal{S}$, $x, y \in [i]$, $x \neq y$. Then,

$$|f(x) - f(y)| = |f(y)| \left| \frac{f(x)}{f(y)} - 1 \right| \leq |f(y)| e^{Kd_\theta(x,y)} Kd_\theta(x,y),$$

where in the last inequality, we have used the estimate $|e^z - 1| \leq |z|e^{|z|}$ as in the proof of (3.2) in Lemma 3.1. Combining this with the bound on $|f|_\infty$, we conclude that $\|f\|_{\text{Lip}} \leq (Ke^K + 1)C_\#$ for all $f \in L_K^\infty$.

Claim 2: L_K^∞ is closed in $L^1(m)$.

Suppose f is a limit point of L_K^∞ with respect to the topology of $L^1(m)$. Since L_K^∞ is bounded in $\text{Lip}(\Sigma)$, it follows that $f \in \text{Lip}(\Sigma)$ and thus f is continuous. Fix $(f_n)_{n \in \mathbb{N}} \subseteq L_K^\infty$ such that $f_n \rightarrow f$ pointwise a.e. and in $L^1(m)$. Then $\int_\Sigma f \, dm = 1$. Let $G = \{x \in \Sigma \mid f_n(x) \rightarrow f(x)\}$.

Since $m(\Sigma \setminus G) = 0$ and m is positive on cylinders, G is dense in Σ . Since $0 \leq f_n(x) \leq C_\#$, it follows from the continuity of f and the density of G that $0 \leq f(x) \leq C_\#$ for all $x \in \Sigma$.

Now fix $[i]$ and suppose $x, y \in G \cap [i]$. Then $f_n(x) \leq f_n(y)e^{Kd_\theta(x,y)}$ for each $n \in \mathbb{N}$. We conclude by taking limits that $f(x) \leq f(y)e^{Kd_\theta(x,y)}$.

Suppose $x, z \in [i]$. By density of G , we can fix $(y^{(k)})_{k \in \mathbb{N}}, (v^{(k)})_{k \in \mathbb{N}} \subseteq G \cap [i]$ such that $\lim_{k \rightarrow \infty} y^{(k)} = x$ and $\lim_{k \rightarrow \infty} v^{(k)} = z$. The previous paragraph implies $f(v^{(k)}) \leq f(y^{(k)})e^{Kd_\theta(y^{(k)}, v^{(k)})}$

for each $k \in \mathbb{N}$. Since f is continuous, we may take the limit as $k \rightarrow \infty$ to conclude that $f(z) \leq f(x)e^{Kd_\theta(x,z)}$.

We have shown that for $x, z \in [i]$, then

$$|\log f(x) - \log f(z)| \leq Kd_\theta(x, z).$$

Thus $\text{Lip}(\log f) \leq K$ and so $f \in L_K^\infty$. This proves that L_K^∞ is closed in L^1 .

Claim 3: L_K^∞ is convex.

Suppose $f, h \in L_K^\infty$ and $t \in [0, 1]$. Then $tf + (1-t)h \in \text{Lip}(\Sigma)$ and $0 \leq tf + (1-t)h \leq C_\sharp$ by convexity of $[0, C_\sharp]$. Similarly, $\int_\Sigma tf + (1-t)h dm = 1$. It remains to show that $\text{Lip}(\log(tf + (1-t)h)) \leq K$.

Fix $i \in \mathcal{S}$ and $x, y \in [i]$. If either f or h is 0 on $[i]$, then the required inequality is trivial. So assume $f, h \neq 0$ on $[i]$. Then

$$\frac{tf(x) + (1-t)h(x)}{tf(y) + (1-t)h(y)} \leq \max \left\{ \frac{tf(x)}{tf(y)}, \frac{(1-t)h(x)}{(1-t)h(y)} \right\} = \max \left\{ \frac{f(x)}{f(y)}, \frac{h(x)}{h(y)} \right\} \leq e^{Kd_\theta(x,y)},$$

since $f, h \in L_K^\infty$. Taking the appropriate suprema proves $\text{Lip}(\log(tf + (1-t)h)) \leq K$. \square

Collecting these results, we are ready to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemma 3.6, the restriction $\mathcal{L}_1^\circ : L_K^\infty \rightarrow L_K^\infty$ is a well-defined continuous map. Since L_K^∞ is a convex, compact subset of $L^1(m)$, it follows from the Schauder-Tychonoff theorem that \mathcal{L}_1° has a fixed point in L_K^∞ .

Let $g \in L_K^\infty$ be such that $\mathcal{L}_1^\circ g = g$. It follows that $\mathcal{L}_\varphi^\circ g = \lambda g$, where

$$\lambda = \int \mathcal{L}_\varphi^\circ g dm = \int_{\Sigma^1} g dm,$$

and $\lambda \geq C_2$ by Lemma 3.5. Since $g \in L_K^\infty$, we have $|g|_\infty \leq C_\sharp$ and by Claim 1 in the proof of Proposition 3.7, we have $\text{Lip}(g) \leq Ke^K C_\sharp$. Thus $g \in \text{Lip}(\Sigma)$. \square

4. PROOF OF THEOREM 2.1

The proof of Theorem 2.1 rests on the fact that as an operator on $\text{Lip}(\Sigma)$, $\mathcal{L}_\varphi^\circ$ has a spectral gap. Much of this section is dedicated to the proof of this fact.

We begin by proving a version of the standard dynamical Lasota-Yorke or Döblin-Fortet inequality, which provides a bound on the essential spectral radius of $\mathcal{L}_\varphi^\circ$. The novelty of the inequality in this setting is the presence of the integral factors appearing in both terms of the inequality. This will enable us to link the essential spectral radius (not just the spectral radius) to the escape rate of mass from the open system. Note that the presence of these L^1 terms is distinct from the analogous inequalities for topological Markov chains derived from finite state spaces; such inequalities exploit a C^0 bound due to the compactness of Σ that is not available in the present setting.

Proposition 4.1. *There exists $C_0 > 0$ such that for all $f \in \text{Lip}(\Sigma)$, and all $n \geq 0$,*

$$\|\mathcal{L}_\varphi^\circ^n f\|_{\text{Lip}} \leq C_0 \theta^n \|f\|_{\text{Lip}} \int_{\Sigma^{n-1}} 1 dm + C_0 \int_{\Sigma^{n-1}} |f| dm.$$

Proof. For each $x \in \Sigma$ and $y \in \sigma^{-n}(\{x\})$, we can find $z_{y,n} \in [y_0, \dots, y_{n-1}]$ such that

$$(4.1) \quad |f(z_{y,n})| \leq (m([y_0, \dots, y_{n-1}]))^{-1} \int_{[y_0, \dots, y_{n-1}]} |f(x)| dm(x).$$

We first estimate the C^0 -norm of $\mathcal{L}_\varphi^n f$.

$$\begin{aligned} |\mathcal{L}_\varphi^n f(x)| &\leq \sum_{y \in \dot{\sigma}^{-n}(x)} |f(y) - f(z_{y,n})| e^{S_n \varphi(y)} + \sum_{y \in \dot{\sigma}^{-n}(x)} |f(z_{y,n})| e^{S_n \varphi(y)} \\ &\leq \theta^n \text{Lip}(f) \sum_{y \in \dot{\sigma}^{-n}(x)} e^{S_n \varphi(y)} + \sum_{y \in \dot{\sigma}^{-n}(x)} |f(z_{y,n})| e^{S_n \varphi(y)}. \end{aligned}$$

Using (3.3) and recalling that $\kappa = \min\{m([i]) : i \in \Lambda_H\}$, we estimate

$$\begin{aligned} (4.2) \quad |\mathcal{L}_\varphi^n f(x)| &\leq \theta^n \text{Lip}(f) \sum_{y \in \dot{\sigma}^{-n}(x)} e^{C_d \kappa^{-1} m([y_0, \dots, y_{n-1}])} + \sum_{y \in \dot{\sigma}^{-n}(x)} e^{C_d \kappa^{-1}} \int_{[y_0, \dots, y_{n-1}]} |f| dm \\ &\leq e^{C_d \kappa^{-1}} \left(\theta^n \text{Lip}(f) m(\dot{\Sigma}^{n-1}) + \int_{\dot{\Sigma}^{n-1}} |f| dm \right). \end{aligned}$$

Here we made use of large images and the fact that if $y \in \dot{\Sigma}^n$, then $[y_0, \dots, y_{n-1}] \subseteq \dot{\Sigma}^{n-1}$. We conclude that

$$(4.3) \quad \|\mathcal{L}_\varphi^n f\|_\infty \leq C \left(\theta^n m(\dot{\Sigma}^{n-1}) \text{Lip}(f) + \int_{\dot{\Sigma}^{n-1}} |f| dm \right).$$

Next, suppose $x, v \in \Sigma$ with $x_0 = v_0$. For each $y \in \dot{\sigma}^{-n}(x)$, let $z_{y,n}$ be as in (4.1). Since $x_0 = v_0$, there is a one-to-one correspondence between $y \in \dot{\sigma}^{-n}(x)$ and $w \in \dot{\sigma}^{-n}(v)$ that preserves n -cylinders. Now we estimate,

$$\begin{aligned} |\mathcal{L}_\varphi^n f(x) - \mathcal{L}_\varphi^n f(v)| &\leq \sum_{y,w} |f(y) - f(w)| e^{S_n \varphi(y)} + \sum_{y,w} |f(w)| e^{S_n \varphi(w)} \left| \frac{e^{S_n \varphi(y)}}{e^{S_n \varphi(w)}} - 1 \right| \\ &\leq \sum_{y \in \dot{\sigma}^{-n}(x)} \theta^n d_\theta(x, v) \text{Lip}(f) e^{S_n \varphi(y)} + \sum_{w \in \dot{\sigma}^{-n}(v)} |f(w)| e^{S_n \varphi(w)} e^{C_d} d_\theta(x, v). \end{aligned}$$

Summing the series as in (4.2), dividing by $d_\theta(x, y)$ and taking the appropriate suprema, we obtain

$$\text{Lip}(\mathcal{L}_\varphi^n f) \leq 2e^{2C_d \kappa^{-1}} \theta^n \text{Lip}(f) m(\dot{\Sigma}^{n-1}) + e^{2C_d \kappa^{-1}} \int_{\dot{\Sigma}^{n-1}} |f| dm.$$

Adding this to (4.3) concludes the proof of the lemma with $C_0 = 3e^{2C_d \kappa^{-1}}$. \square

The previous proposition and the relative compactness of the closed unit ball of $\text{Lip}(\Sigma)$ in $L^1(m)$ are nearly enough to prove that \mathcal{L}_φ is quasi-compact. Due to the loss of mass caused by the hole, it is still necessary to show that the spectral radius of \mathcal{L}_φ is strictly larger than the contraction provided by Proposition 4.1. This will be done in Section 4.2. In the next section, we prepare some groundwork by investigating further properties of L_K^∞ .

4.1. Regularity of log-Lipschitz functions. In order to make a Perron-Frobenius argument and show that \mathcal{L}_φ has a spectral gap, we first show that functions in L_K^∞ become positive under the action of \mathcal{L}_φ^n (Proposition 4.3). This will imply that eigenfunctions of \mathcal{L}_φ in L_K^∞ are bounded away from 0 (Corollary 4.4). We begin by proving a combinatorial result about admissible sequences in the open system.

Lemma 4.2. *Suppose $(\dot{\Sigma}, \dot{\sigma}, H)$ satisfies **(H)**. Then there exists a finite set $\Lambda'_H \subseteq \mathcal{S}$ and $N_* \in \mathbb{N}$ such that $\Lambda_H \subseteq \Lambda'_H$ and $\forall j, k \in \Lambda'_H, \forall n \geq N_* \exists a_1, \dots, a_n \in \Lambda'_H$ such that*

$$A_{j,a_1} A_{a_1,a_2} \cdots A_{a_{n-1},a_n} A_{a_n,k} = 1.$$

Proof. We construct Λ'_H by adding elements to Λ_H . Suppose $j, k \in \Lambda_H$. By **(H)**(b), there exists $\{a_1, \dots, a_n\} \subseteq \mathcal{S}$ such that

$$A_{j,a_1} A_{a_1,a_2} \cdots A_{a_{n-1},a_n} A_{a_n,k} = 1.$$

Define $\Lambda'_{j,k} = \{a_1, \dots, a_n\}$ when $j \neq k$. In the case when $k = j$, we can choose a set $\{b_1, \dots, b_m\} \subseteq \mathcal{S}$ such that

$$A_{j,b_1} A_{b_1,b_2} \cdots A_{b_{m-1},b_m} A_{b_m,j} = 1$$

and $\gcd(n+1, m+1) = 1$, which also follows as a consequence of **(H)**(b). Define $\Lambda'_{j,j} = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ in this case. Let $\Lambda'_H = \Lambda_H \cup \left(\bigcup_{j,k \in \Lambda_H} \Lambda'_{j,k} \right)$.

To complete the proof of the proposition, we show that the subshift $\Sigma_{\Lambda'_H} = \{x \in \Sigma \mid x_i \in \Lambda'_H\}$ is topologically mixing. Recall that a topologically transitive TMC is topologically mixing if and only if there exist states j, k and relatively prime integers p, q such that $\sigma^{-p}([j]) \cap [j] \neq \emptyset$ and $\sigma^{-q}([k]) \cap [k] \neq \emptyset$ (see [A, Section 4.2]).

We claim that $\Sigma_{\Lambda'_H}$ is topologically transitive. If $j, k \in \Lambda_H$, then $[j, a_1, \dots, a_n, k] \cap \Sigma_{\Lambda'_H} \neq \emptyset$, where $\{a_1, \dots, a_n\} = \Lambda'_{j,k}$. Suppose $j \in \Lambda'_H \setminus \Lambda_H$ and $k \in \Lambda_H$. By **(H)**(a), we can choose $\ell \in \Lambda_H$ such that $A_{j,\ell} = 1$. By construction, we can choose $a_1, \dots, a_n \in \Lambda'_{\ell,k}$ such that

$$A_{\ell,a_1} A_{a_1,a_2} \cdots A_{a_{n-1},a_n} A_{a_n,k} = 1.$$

Thus $A_{j,\ell} A_{\ell,a_1} A_{a_1,a_2} \cdots A_{a_{n-1},a_n} A_{a_n,k} = 1$, with ℓ, a_1, \dots, a_n all belonging to Λ'_H and so $\Sigma_{\Lambda'_H} \cap [j, \ell, a_1, \dots, a_n, k] \neq \emptyset$.

For the case $j \in \Lambda_H$ and $k \in \Lambda'_H \setminus \Lambda_H$, we argue differently due to the asymmetry of **(H)**(a). Since $k \in \Lambda'_H \setminus \Lambda_H$, k must belong to $\Lambda'_{i,\ell}$ for some $i, \ell \in \Lambda_H$. Thus there exists $a_1, \dots, a_n \in \Lambda'_{i,\ell}$ such that $A_{i,a_1} A_{a_1,a_2} \cdots A_{a_n,k} = 1$. Then appending this sequence to $\Lambda'_{j,i}$, since $i, j \in \Lambda_H$, yields the required sequence in Λ'_H connecting j to k . The case when $j, k \in \Lambda'_H \setminus \Lambda_H$ follows by combining the other two cases and using again **(H)**(a). Therefore, $\Sigma_{\Lambda'_H}$ is topologically transitive.

Finally, note that given $j \in \Lambda_H$, the periodic sequences

$$x = (j, a_1, \dots, a_n, j, a_1, \dots), \quad y = (j, b_1, \dots, b_m, j, b_1, \dots),$$

where $\Lambda'_{j,j} = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ and $\gcd(n+1, m+1) = 1$, satisfy $\sigma^{n+1}(x) = x, \sigma^{m+1}(y) = y$. Therefore, $\Sigma_{\Lambda'_H}$ is topologically mixing. Since Λ'_H is finite, the proposition follows. \square

Note that the set Λ'_H given by Lemma 4.2 is not unique: we simply choose one finite set $\Lambda'_{j,k}$ for each pair $j, k \in \Lambda_H$. We consider the set Λ'_H so constructed as fixed for the remainder of the paper.

Proposition 4.3. *There exists $N \in \mathbb{N}$ and a sequence $(B_n)_{n \geq N}$ with $B_n > 0$ such that for all $f \in L_K^\infty$ and all $n \geq N$,*

$$\mathcal{L}_\varphi^n f(x) \geq B_n \quad \text{for all } x \in \mathring{\Sigma}.$$

Proof. Fix $f \in L_K^\infty$. Choose $j = j(C_\sharp) \in \mathbb{N}$ such that $\sum_{i > j} m([i]) < \frac{1}{2C_\sharp}$. Then since $|f|_\infty \leq C_\sharp$, we have $\int_{\bigcup_{i > j} [i]} f \, dm < \frac{1}{2}$.

Since $\int_{\mathring{\Sigma}} f \, dm = 1$, there exists $i_0 \in \mathcal{S} \setminus \mathcal{S}_H$ with $i_0 \leq j(C_\sharp)$ such that

$$\frac{1}{m([i_0])} \int_{[i_0]} f \, dm \geq \frac{1}{2}.$$

Due to the log-regularity of f , it follows that $f(y) \geq \frac{1}{2} e^{-K}$ for all $y \in [i_0]$.

By **(H)**, there exists N_{i_0} such that $\mathring{\sigma}^n([i_0]) = \mathring{\Sigma}$ for all $n \geq N_{i_0}$. Thus for each $n \geq N_{i_0}$ and $x \in \mathring{\Sigma}$, there exists $w_{x,n} \in [i_0]$ such that $\sigma^n(w_{x,n}) = x$. We may increase N_{i_0} to ensure that $N_{i_0} \geq N_*$, where N_* is from Lemma 4.2. Thus using **(H)**(a) and Lemma 4.2, we may choose $w_{x,n}$ so that

$$\sigma^k(w_{x,n}) \in \bigcup_{i \in \Lambda'_H} [i] \quad \text{for all } k = 1, \dots, n-1.$$

This implies in particular that $\mathcal{L}_\varphi^n f(x) \geq \frac{1}{2}e^{-K}e^{S_n\varphi(w_{x,n})}$. Now define

$$B_{n,i_0} = \inf\{e^{S_n\varphi(y)} : \sigma^k(y) \in [i_0] \cup (\cup_{i \in \Lambda'_H} [i]), \forall k = 0, 1, \dots, n-1\}.$$

It follows from the finiteness of $\{i_0\} \cup \Lambda'_H$ and bounded distortion that $B_{n,i_0} > 0$. Finally, define

$$N = \max\{N_{i_0} : i_0 \leq j(C_\#)\} \quad \text{and} \quad B_n = \frac{1}{2}e^{-K} \min\{B_{n,i_0} : i_0 \leq j(C_\#)\}.$$

Then for all $n \geq N$ and $x \in \mathring{\Sigma}$, $\mathcal{L}_\varphi^n f(x) \geq B_n$ as required. \square

Proposition 4.3 has an important consequence for the fixed points of \mathcal{L}_1 in L_K . This in turn gives a uniform estimate for the size of $\lambda^{-n}\mathcal{L}_\varphi^n \mathbf{1}$ on $\mathring{\Sigma}$.

Corollary 4.4. *If $h \in L_K^\infty$ is a fixed point of \mathcal{L}_1 , then $\inf\{h(x) \mid x \in \mathring{\Sigma}\} > 0$.*

Proof. Since $h \in L_K^\infty$, Proposition 4.3 implies $\mathcal{L}_\varphi^N h(x) \geq B_N$ for all $x \in \mathring{\Sigma}$. Since h is invariant under \mathcal{L}_1 , we have $\mathcal{L}_\varphi h = \alpha h$, where $\alpha > 0$ satisfies $\alpha = \int_{\mathring{\Sigma}_1} h \, dm$. It follows that $h(x) \geq B_N \alpha^{-N} > 0$ for all $x \in \mathring{\Sigma}$. \square

Corollary 4.5. *There exists $C_3 > 0$ such that for all $n \in \mathbb{N}$,*

$$C_3^{-1} \leq \lambda^{-n}(\mathcal{L}_\varphi^n \mathbf{1})|_{\mathring{\Sigma}} \leq C_3,$$

where $\lambda \in (0, 1)$ is the eigenvalue corresponding to the fixed point g chosen in Proposition 3.2.

Proof. Fix $n \in \mathbb{N}$. By the previous corollary,

$$\forall x \in \mathring{\Sigma} \quad \inf\{g(x) \mid x \in \mathring{\Sigma}\} \cdot \lambda^{-n} \mathcal{L}_\varphi^n \mathbf{1}(x) \leq \lambda^{-n} \mathcal{L}_\varphi^n g(x) \leq |g|_\infty \lambda^{-n} \mathcal{L}_\varphi^n \mathbf{1}(x).$$

Since $\lambda^{-n} \mathcal{L}_\varphi^n g(x) = g(x)$, we obtain the two inequalities:

$$\inf\{g(x) \mid x \in \mathring{\Sigma}\} \cdot \lambda^{-n} \mathcal{L}_\varphi^n \mathbf{1}(x) \leq \lambda^{-n} \mathcal{L}_\varphi^n g(x) \leq |g|_\infty$$

and

$$\inf\{g(x) \mid x \in \mathring{\Sigma}\} \leq \lambda^{-n} \mathcal{L}_\varphi^n g(x) \leq |g|_\infty \lambda^{-n} \mathcal{L}_\varphi^n \mathbf{1}(x).$$

Letting $C_3 = \frac{|g|_\infty}{\inf\{g(x) \mid x \in \mathring{\Sigma}\}}$ gives the result. \square

Notice that Corollary 4.4 together with the proof of Corollary 4.5 imply that if $g, h \in L_K^\infty$ are two fixed points for \mathcal{L}_1 such that $\mathcal{L}_\varphi g = \lambda g$ and $\mathcal{L}_\varphi h = \alpha h$, then $\alpha = \lambda$. The next corollary is used in the proof that \mathcal{L}_φ has a spectral gap.

Corollary 4.6. *There exists $C_4 > 0$ such that if $f \in L_K^\infty$ and there exists an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that $\lambda^{-n_k} \mathcal{L}_\varphi^{n_k} f \rightarrow f$ pointwise, then $\inf\{f(x) \mid x \in \mathring{\Sigma}\} > C_4$.*

Proof. By Proposition 4.3, there exists a $N \in \mathbb{N}$ and $B_N > 0$ such that $\mathcal{L}_\varphi^N f|_{\mathring{\Sigma}} \geq B_N$. It follows that

$$\lambda^{-n_k} \mathcal{L}_\varphi^{n_k} f|_{\mathring{\Sigma}} \geq B_N \lambda^{-N} \cdot \lambda^{-(n_k-N)} \mathcal{L}_\varphi^{n_k-N} \mathbf{1}|_{\mathring{\Sigma}} \geq B_N \lambda^{-N} C_3^{-1},$$

where in the last inequality we used Corollary 4.5. It follows that $f|_{\mathring{\Sigma}} \geq B_N \lambda^{-N} C_3^{-1}$. \square

4.2. Quasi-Compactness of \mathcal{L}_φ . We can use Corollary 4.4 to compute both the spectral radius and the essential spectral radius of $\mathcal{L}_\varphi : \text{Lip}(\Sigma) \rightarrow \text{Lip}(\Sigma)$. This proves once and for all that the essential spectral radius, $\rho_{\text{ess}}(\mathcal{L}_\varphi)$, is strictly less than $\rho(\mathcal{L}_\varphi)$. It turns out this computation also gives us the escape rate.

Theorem 4.7. $\lambda = \rho(\mathcal{L}_\varphi) = \lim_{n \rightarrow \infty} m(\Sigma^n)^{1/n}$. Moreover, $\rho_{\text{ess}}(\mathcal{L}_\varphi) \leq \theta\lambda$. Thus \mathcal{L}_φ is quasi-compact as an operator on $\text{Lip}(\Sigma)$.

Proof. First, we prove that $\limsup_{n \rightarrow \infty} m(\Sigma^n)^{1/n} \leq \rho(\mathcal{L}_\varphi)$. To see this, it is enough to integrate $\mathcal{L}_\varphi^n \mathbf{1}$:

$$m(\mathring{\Sigma}^n) = \int_{\Sigma} \mathcal{L}_\varphi^n \mathbf{1} \, dm \leq \|\mathcal{L}_\varphi^n \mathbf{1}\|_{\text{Lip}} = \|\mathcal{L}_\varphi^n\|_{\text{Lip}}.$$

Taking n -th roots and letting $n \rightarrow \infty$ shows that $\limsup m(\mathring{\Sigma}^n)^{1/n} \leq \rho(\mathcal{L}_\varphi)$.

Next, from Proposition 4.1, we have the following bound for $f \in \text{Lip}(\Sigma)$,

$$\|\mathcal{L}_\varphi^n f\|_{\text{Lip}} \leq C_0 \left(\theta^n \|f\|_{\text{Lip}} \int_{\mathring{\Sigma}^{n-1}} 1 \, dm + \int_{\mathring{\Sigma}^{n-1}} f \, dm \right) \leq C_0 m(\mathring{\Sigma}^{n-1}) \|f\|_{\text{Lip}} (\theta^n + 1)$$

from which it follows that $\|\mathcal{L}_\varphi^n\|_{\text{Lip}} \leq 2C_0 m(\mathring{\Sigma}^{n-1})$ and thus $\rho(\mathcal{L}_\varphi) \leq \liminf_{n \rightarrow \infty} m(\mathring{\Sigma}^{n-1})^{1/n}$. We conclude that $\lim_{n \rightarrow \infty} m(\mathring{\Sigma}^n)^{1/n}$ exists and equals $\rho(\mathcal{L}_\varphi)$.

Integrating g over $\mathring{\Sigma}^n$ gives:

$$\inf\{g(x) \mid x \in \mathring{\Sigma}\} \cdot m(\mathring{\Sigma}^n) \leq \int \mathcal{L}_\varphi^n g \, dm = \lambda^n.$$

By Corollary 4.4, the left hand side is not zero. Taking n -th roots and letting $n \rightarrow \infty$, we conclude that

$$\rho(\mathcal{L}_\varphi) = \lim_{n \rightarrow \infty} m(\mathring{\Sigma}^n)^{1/n} \leq \lambda.$$

Since λ is an eigenvalue, it is bounded above by $\rho(\mathcal{L}_\varphi)$. Thus, $\lambda = \rho(\mathcal{L}_\varphi) = \lim_{n \rightarrow \infty} m(\mathring{\Sigma}^n)^{1/n}$.

In addition, we see that

$$\theta m(\mathring{\Sigma}^{n-1})^{1/n} < \rho(\mathcal{L}_\varphi)$$

if n is sufficiently large. It follows from Hennion's Theorem [H], Proposition 4.1, and the relative compactness of the unit ball of $\text{Lip}(\mathring{\Sigma})$ in $L^1(m)$ that

$$\rho_{\text{ess}}(\mathcal{L}_\varphi) \leq \theta\lambda < \lambda = \rho(\mathcal{L}_\varphi).$$

Thus \mathcal{L}_φ is quasi-compact. □

The previous theorem implies that mass escapes at an exponential rate. In particular,

$$(4.4) \quad \log \lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \log m(\mathring{\Sigma}^n),$$

which is item (b) of Theorem 2.1.

4.3. A Spectral Gap for \mathcal{L}_φ . Before proving that \mathcal{L}_φ has a spectral gap, we prove the following useful result.

Lemma 4.8. Suppose $f \in \text{Lip}(\Sigma)$, $\inf\{f(x) \mid x \in \mathring{\Sigma}\} > 0$ and $\int_{\Sigma} f \, dm = 1$. If there exists a sequence $(n_k) \subseteq \mathbb{N}$ and $\alpha > 0$ such that $\alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f \rightarrow f$ uniformly, then $\alpha = \lambda$ and $f \in L_K^\infty$.

Proof. Since $f \in \text{Lip}(\Sigma)$ and $\inf\{f(x) \mid x \in \overset{\circ}{\Sigma}\} > 0$, it follows that $\text{Lip}(\log f) < \infty$ (notice also that $f|_H \equiv 0$ since $(\mathcal{L}_\varphi^n f)|_H \equiv 0$ and f is the uniform limit of such functions). By Lemma 3.3,

$$\text{Lip}(\log \alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f) \leq \theta^{n_k} \text{Lip}(\log f) + C_d,$$

and thus $\exists M \in \mathbb{N}$ such that

$$\text{Lip}(\log \alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f) \leq K$$

if $k \geq M$. Since $\inf\{f(x) \mid x \in \overset{\circ}{\Sigma}\} > 0$, uniform convergence of $\alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f$ to f implies uniform convergence of $\log(\alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f)$ to $\log f$. Thus $\text{Lip}(\log f) \leq K$.

Next, note that $f \in M_K$, where M_K is defined before Lemma 3.4. Thus by Lemmas 3.4 and 3.5, we have $|\mathcal{L}_1 f|_\infty \leq C_\sharp$, and thus $\mathcal{L}_1 f \in L_K^\infty$. By Lemma 3.6, $\mathcal{L}_1^n f \in L_K^\infty$ for all $n \in \mathbb{N}$. Notice that by uniform convergence, $\alpha^{-n_k} |\mathcal{L}_\varphi^{n_k} f|_1 \xrightarrow{k \rightarrow \infty} |f|_1 = 1$. Thus,

$$f(x) = \lim_{k \rightarrow \infty} \alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f(x) = \lim_{k \rightarrow \infty} \alpha^{-n_k} |\mathcal{L}_\varphi^{n_k} f|_1 \mathcal{L}_1^{n_k} f(x) \leq C_\sharp, \quad \forall x \in \overset{\circ}{\Sigma}.$$

Thus $f \in L_K^\infty$.

Finally, let $s = \inf\{f(x) \mid x \in \overset{\circ}{\Sigma}\} > 0$ and $S = \sup\{f(x) \mid x \in \overset{\circ}{\Sigma}\} \leq C_\sharp$. For each $k \in \mathbb{N}$ and $x \in \overset{\circ}{\Sigma}$,

$$(4.5) \quad s \left(\frac{\lambda}{\alpha} \right)^{n_k} \lambda^{-n_k} \mathcal{L}_\varphi^{n_k} \mathbf{1}(x) \leq \alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f(x) \leq S \left(\frac{\lambda}{\alpha} \right)^{n_k} \lambda^{-n_k} \mathcal{L}_\varphi^{n_k} \mathbf{1}(x).$$

Using Corollary 4.5, if $\alpha < \lambda$, then the first inequality in (4.5) implies the infimum of $\alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f$ on $\overset{\circ}{\Sigma}$ becomes arbitrarily large, contradicting the uniform convergence to f , which has integral 1. On the other hand, if $\alpha > \lambda$, then the second inequality in (4.5) implies that the limit of $\alpha^{-n_k} \mathcal{L}_\varphi^{n_k} f$ is 0, again contradicting the fact that $\int f dm = 1$. Thus $\alpha = \lambda$. \square

The following proposition proves λ is the only eigenvalue on the circle $\{z \in \mathbb{C} \mid |z| = \rho(\mathcal{L}_\varphi)\}$. Moreover, it follows from this argument that the eigenspace corresponding to λ is one-dimensional. A key strategy in the proof is adapted from [PY].

Proposition 4.9. *Suppose $h \in \text{Lip}(\Sigma, \mathbb{C})$ is an eigenfunction of \mathcal{L}_φ with corresponding eigenvalue $\lambda e^{i2\pi\phi}$ with $\lambda = \rho(\mathcal{L}_\varphi)$. Then $h = zg$ for some $z \in \mathbb{C}$ and $\phi = 0$.*

Proof. Let $h_1 = \Re(h)$ and $h_2 = \Im(h)$ denote the real and imaginary parts of h . Equating the real parts of the equality $\mathcal{L}_\varphi^n h = \lambda^n e^{i2\pi\phi n} h$ yields,

$$\mathcal{L}_\varphi^n h_1 = \lambda^n (h_1 \cos(2\pi\phi n) - h_2 \sin(2\pi\phi n)).$$

By properties of circle rotations, there exists an increasing sequence $(n_k) \subseteq \mathbb{N}$ such that $\lim_{k \rightarrow \infty} e^{i2\pi\phi n_k} = 1$. Since $|h_1|_\infty, |h_2|_\infty < \infty$, it follows that $\lambda^{-n_k} \mathcal{L}_\varphi^{n_k} h_1 \rightarrow h_1$ uniformly.

Fix $\alpha_1 > 0$ such that

$$\inf\{h_1(x) + \alpha_1 g(x) \mid x \in \overset{\circ}{\Sigma}\} > 0$$

and let $\alpha_2 = \int_\Sigma (h_1 + \alpha_1 g) dm$. Let $f_0 = \alpha_2^{-1} (h_1 + \alpha_1 g)$. Notice that $\inf\{f_0(x) \mid x \in \overset{\circ}{\Sigma}\} > 0$ by choice of α_1 .

Let $f_t = (1-t)f_0 + tg$ for each $t \in \mathbb{R}$. Let $J = \{t \in \mathbb{R} \mid \inf\{f_t(x) \mid x \in \overset{\circ}{\Sigma}\} > 0\}$. J is nonempty since $J \supseteq [0, 1]$.

To see that J is open, let $t \in J$ and set $\delta = \inf\{f_t(x) \mid x \in \overset{\circ}{\Sigma}\} > 0$. If $|s-t| < \frac{\delta}{3} \min\{\frac{1}{|f_0|_\infty}, \frac{1}{|g|_\infty}\}$, then for $x \in \overset{\circ}{\Sigma}$,

$$f_s(x) = f_t(x) + (t-s)f_0(x) + (s-t)g(x) \geq \delta - \frac{\delta}{3|f_0|_\infty} f_0(x) - \frac{\delta}{3|g|_\infty} g(x) \geq \frac{\delta}{3},$$

and we conclude that $f_s \in J$.

We claim that J is also closed. Suppose $t \in \mathbb{R}$ is a limit point of J . Fix $(t_j) \subseteq J$ such that $t_j \rightarrow t$. Let $f_j = f_{t_j}$. For each $j \in \mathbb{N}$, $\inf\{f_j(x) \mid x \in \mathring{\Sigma}\} > 0$, $\int_{\Sigma} f_j dm = 1$, and $\lambda^{-nk} \mathcal{L}_{\varphi}^{nk} f_j \rightarrow f_j$ uniformly as $k \rightarrow \infty$. By Lemma 4.8, it follows that $(f_j)_{j \in \mathbb{N}} \subseteq L_K^{\infty}$. Moreover, $f_j \rightarrow f_t$ uniformly since

$$|f_{t_j}(x) - f_t(x)| \leq |t_j - t|(|f_0|_{\infty} + |g|_{\infty}), \quad \forall x \in \Sigma.$$

Thus, since L_K^{∞} is closed in $L^1(m)$, we conclude that $f_t \in L_K^{\infty}$.

Since $\lambda^{-nk} \mathcal{L}_{\varphi}^{nk} f_t \rightarrow f_t$ uniformly, it follows by Corollary 4.6 that $\inf\{f_t(x) \mid x \in \mathring{\Sigma}\} > 0$. Therefore $t \in J$ and we conclude that J is closed. It follows that $J = \mathbb{R}$.

Suppose $x \in \mathring{\Sigma}$. We have shown that

$$\forall t \in \mathbb{R} \quad (1-t)f_0(x) + tg(x) > 0,$$

which implies that $f_0(x) = g(x)$. On the other hand, if $x \in H$, then $h(x) = 0$ and thus $f_0(x) = 0$. We conclude that $f_0 = g$.

It follows that $h_1 = (\alpha_2 - \alpha_1)g$. Moreover,

$$(\alpha_2 - \alpha_1)g = \lambda^{-1} \mathcal{L}_{\varphi} h_1 = (\alpha_2 - \alpha_1)g \cos(2\pi\phi) - h_2 \sin(2\pi\phi),$$

which implies that $h_2 = \beta g$ for some $\beta \in \mathbb{R}$. It follows that $h = (\alpha_2 - \alpha_1 + i\beta)g$ and

$$\mathcal{L}_{\varphi} g = \mathcal{L}_{\varphi} ((\alpha_2 - \alpha_1 + i\beta)^{-1} h) = \lambda e^{i2\pi\phi} g$$

so that $\lambda g = \lambda e^{i2\pi\phi} g$. We conclude that $\phi = 0$. \square

Finally, we prove that λ is an eigenvalue of \mathcal{L}_{φ} of algebraic multiplicity 1.

Proposition 4.10. *The algebraic multiplicity of λ is 1.*

Proof. It follows directly from Proposition 4.9 that the dimension of the eigenspace E_{λ} corresponding to λ is one. Thus if λ has a non-trivial Jordan block, there must exist $h \in \text{Lip}(\Sigma)$ such that $(\mathcal{L}_{\varphi} - \lambda)h = g$. Iterating this equation, we obtain

$$\mathcal{L}_{\varphi}^n h = n\lambda^{n-1}g + \lambda^n h.$$

Note also that

$$\forall x \in \mathring{\Sigma}, \forall n \in \mathbb{N} \quad -C_3|h|_{C^0} \leq \lambda^{-n} \mathcal{L}_{\varphi}^n h(x) \leq C_3|h|_{C^0},$$

where C_3 is from Corollary 4.5. Fix $x \in \mathring{\Sigma}$. Then $g(x) > 0$ and combining the two previous inequalities, we must have

$$\forall n \in \mathbb{N} \quad -C|h|_{C^0} \leq n\lambda^{-1}g(x) + h(x) \leq C|h|_{C^0},$$

which is impossible.

Therefore, $(\mathcal{L}_{\varphi} - \lambda)h = 0$ from which we conclude that $h \in E_{\lambda}$, i.e. h is a multiple of g . Thus λ has no Jordan block. \square

Since $\rho_{\text{ess}}(\mathcal{L}_{\varphi}) < \rho(\mathcal{L}_{\varphi}) = \lambda$, Propositions 4.9 and 4.10 together imply that \mathcal{L}_{φ} has a spectral gap. We are now ready to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Let g be the eigenfunction of \mathcal{L}_{φ} with eigenvalue λ given by Proposition 3.2. The fact that $g \in L_K^{\infty}$ together with Corollary 4.4 prove the initial statement of Theorem 2.1.

The fact that $\mathcal{L}_{\varphi} g = \lambda g$ together with the conformality of the measure m as expressed in (2.1) proves item (a) of the theorem. Item (b) follows from (4.4) as already noted in Section 4.2. Item (c) is proved in Propositions 4.9 and 4.10.

It remains to prove item (d) of the theorem. Since \mathcal{L}_{φ} has a spectral gap, we let $\Pi_{\lambda} : \text{Lip}(\Sigma) \rightarrow \text{Lip}(\Sigma)$ be the projection onto E_{λ} , which is simply the span of g , and write

$$\mathcal{L}_{\varphi} = \lambda \Pi_{\lambda} + R,$$

where R is a bounded linear operator on $\text{Lip}(\Sigma)$ with spectral radius strictly less than λ and $R \circ \Pi_\lambda = \Pi_\lambda \circ R = 0$.

Define $\mathbb{W} = \Pi_\lambda^{-1}(\{0\})$. Since Π_λ is a projection, we have the following decomposition

$$\text{Lip}(\Sigma) = E_\lambda \oplus \mathbb{W}.$$

It follows that every Lipschitz function f has a unique decomposition

$$f = c(f)g + w,$$

where $c(f) \in \mathbb{C}$ and $w \in \mathbb{W}$. We shall see that the linear functional c has special properties. Note that continuity of Π_λ implies \mathbb{W} is a closed subspace of $\text{Lip}(\Sigma)$. Also note that both E_λ and \mathbb{W} are invariant under \mathcal{L}_φ .

Lemma 4.11. $\lambda^{-n} \mathcal{L}_\varphi^n \rightarrow \Pi_\lambda$ in the norm topology of $\text{Lip}(\Sigma)$.

Proof. Fix $\epsilon > 0$ such that $\rho(R) + \epsilon < \lambda$. Choose $M \in \mathbb{N}$ such that $\|R^n\|_{\text{Lip}}^{1/n} < \rho(R) + \epsilon$ if $n \geq M$. If $f \in \text{Lip}(\Sigma)$ and $n \geq M$, then

$$\|\lambda^{-n} \mathcal{L}_\varphi^n f - \Pi_\lambda(f)\|_{\text{Lip}} \leq \lambda^{-n} \|R^n\|_{\text{Lip}} \cdot \|f\|_{\text{Lip}} < \left(\frac{\rho(R) + \epsilon}{\lambda}\right)^n \|f\|_{\text{Lip}}.$$

□

Lemma 4.12. $c : \text{Lip}(\Sigma) \rightarrow \mathbb{C}$ is a bounded linear functional.

Proof. For $f \in \text{Lip}(\Sigma)$, write $f = c(f)g + w$. For each $n \in \mathbb{N}$,

$$\lambda^{-n} \mathcal{L}_\varphi^n f = \Pi_\lambda f + \lambda^{-n} R^n f \quad \text{and} \quad \lambda^{-n} \mathcal{L}_\varphi^n f = c(f)g + \lambda^{-n} R^n w.$$

Then Lemma 4.11 implies

$$c(f) = \int_\Sigma \Pi_\lambda(f) dm.$$

This together with the fact that Π_λ is a bounded linear operator proves the lemma. □

Lemmas 4.11 and 4.12 imply the first statement of item (d). The proof of Lemma 4.12 also implies that

$$c(f) = \lim_{n \rightarrow \infty} \lambda^{-n} \int_{\Sigma^n} f dm,$$

since $\int_{\Sigma^n} f dm = \int \mathcal{L}_\varphi^n f dm$.

Lemma 4.13. For $f \in \text{Lip}(\Sigma)$, $c(f) > 0$ if and only if $\frac{\mathcal{L}_\varphi^n f}{|\mathcal{L}_\varphi^n f|_1} \rightarrow g$ in $\text{Lip}(\Sigma)$.

Proof. First, consider the case when $c(f) \neq 0$. Then

$$\lim_{n \rightarrow \infty} \frac{\mathcal{L}_\varphi^n f}{|\mathcal{L}_\varphi^n f|_1} = \lim_{n \rightarrow \infty} \left(\frac{\mathcal{L}_\varphi^n f}{\lambda^n} \right) \left(\frac{|\mathcal{L}_\varphi^n f|_1}{\lambda^n} \right)^{-1} = \frac{c(f)g}{|c(f)|}.$$

Thus, $\lim_{n \rightarrow \infty} \frac{\mathcal{L}_\varphi^n f}{|\mathcal{L}_\varphi^n f|_1} = g$ iff. $c(f) > 0$ in this case.

Now suppose $c(f) = 0$. Then $f \in \mathbb{W}$. Since \mathbb{W} is a \mathcal{L}_φ -invariant subspace, it follows that $\frac{\mathcal{L}_\varphi^n f}{|\mathcal{L}_\varphi^n f|_1} \in \mathbb{W}$ for all $n \in \mathbb{N}$. Since \mathbb{W} is closed, any limit point of $\left(\frac{\mathcal{L}_\varphi^n f}{|\mathcal{L}_\varphi^n f|_1} \right)_{n \in \mathbb{N}}$ is in \mathbb{W} . We conclude that g is not a limit point of the sequence. □

Lemma 4.13 completes the proof of item (d) of Theorem 2.1. □

5. GIBBS MEASURE AND VARIATIONAL PRINCIPLE

We next turn our attention to the survivor set, $\hat{\Sigma}^\infty = \bigcap_{n=0}^\infty \sigma^{-n}(\hat{\Sigma})$, and the proof of Theorem 2.2.

We will begin by showing that the functional $c(\cdot)$ induces a measure m_H , supported on $\hat{\Sigma}^\infty$, for which $\int_\Sigma f dm_H = c(f)$ whenever $f \in \text{Lip}(\Sigma)$. We will then show that m_H is conformal with respect to the potential $\varphi_H - \log \lambda$. Recall that $\mathbf{1}_A$ denotes the indicator function of the set A .

Proposition 5.1. *The linear functional c induces a finite, positive Borel measure m_H , which satisfies*

$$m_H([i_0, \dots, i_{n-1}]) = c(\mathbf{1}_{[i_0, \dots, i_{n-1}]})$$

for all cylinder sets $[i_0, \dots, i_{n-1}]$.

Proof. Let \mathcal{C}' denote the collection of cylinders in Σ . Since \mathcal{S} is infinite, \mathcal{C}' need not be a semi-algebra in general. We define a semi-algebra by first introducing the notion of a generalized n -cylinder. Given $i_0, i_1, \dots, i_{n-2} \in \mathcal{S}$ and $k \in \mathbb{N} \cup \{0\}$, we define a generalized n -cylinder by,

$$[i_0, i_1, \dots, i_{n-2}; k] = \bigcup_{j \geq k} [i_0, i_1, \dots, i_{n-2}, j].$$

Note that if $k = 0$, then $[i_0, i_1, \dots, i_{n-2}; k] = [i_0, i_1, \dots, i_{n-2}]$ so that every $(n-1)$ -cylinder is a generalized n -cylinder. Let $\mathcal{C} \supset \mathcal{C}'$ denote the set of all generalized n -cylinders in Σ .

Now define a set function $\tau : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$ by $\tau(E) = c(\mathbf{1}_E)$ for all $E \in \mathcal{C}$. Note that τ is well-defined since $\mathbf{1}_E \in \text{Lip}(\Sigma)$ for each $E \in \mathcal{C}$. Also, τ is non-negative since $c(f)$ is real and non-negative whenever f is.

Suppose $\{E_n\}_{n \in \mathbb{N}}$ is a disjoint collection of sets in \mathcal{C} with $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{C}$. We claim that $c(\mathbf{1}_{\bigcup E_n}) = \sum_{n \in \mathbb{N}} c(\mathbf{1}_{E_n})$. To see this, note that

$$c(\mathbf{1}_{\bigcup E_n}) = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\hat{\Sigma}^k} \sum_{n \in \mathbb{N}} \mathbf{1}_{E_n} dm = \lim_{k \rightarrow \infty} \sum_{n \in \mathbb{N}} \lambda^{-k} \int_{\Sigma} \mathcal{L}_\varphi^k \mathbf{1}_{E_n} dm,$$

where the interchange of the sum and integral is justified by the monotone convergence theorem. Note that the sequence $(a_{k,n})_{k \in \mathbb{N}}$, where $a_{k,n} = \lambda^{-k} \int_{\Sigma} \mathcal{L}_\varphi^k \mathbf{1}_{E_n} dm$, is uniformly (in both n and k) bounded above by C_3 , where C_3 is from Corollary 4.5. Thus, by the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \sum_{n \in \mathbb{N}} a_{k,n} = \sum_{n \in \mathbb{N}} \lim_{k \rightarrow \infty} a_{k,n}.$$

In other words,

$$c(\mathbf{1}_{\bigcup E_n}) = \sum_{n \in \mathbb{N}} \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\Sigma} \mathcal{L}_\varphi^k \mathbf{1}_{E_n} dm = \sum_{n \in \mathbb{N}} c(\mathbf{1}_{E_n}).$$

It follows that $\tau(\bigcup E_n) = \sum_{n \in \mathbb{N}} \tau(E_n)$ and so τ is countably additive. It follows as a consequence of the Caratheodory extension theorem that τ extends uniquely to a positive Borel measure m_H on Σ (see for example [P, Chapter 2]). \square

Proposition 5.2. $\forall f \in \text{Lip}(\Sigma) \quad \int_\Sigma f dm_H = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\hat{\Sigma}^k} f dm = c(f)$.

Proof. We know this formula holds when f is the characteristic function of a generalized cylinder. As a first step, suppose that $f = \sum_{n \in \mathbb{N}} a_n \mathbf{1}_{E_n}$, where $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ is bounded and $\{E_n\}_{n \in \mathbb{N}}$ is a disjoint collection of sets in \mathcal{C} . We claim that the proposition holds in this case.

Indeed, since $\sum_{n \in \mathbb{N}} a_n \mathbf{1}_{E_n}$ is bounded by $\sup_n \{|a_n|\}$, it follows from the dominated convergence theorem that

$$\int_\Sigma f dm_H = \sum_{n \in \mathbb{N}} \int_\Sigma a_n \mathbf{1}_{E_n} dm_H = \sum_{n \in \mathbb{N}} \lim_{k \rightarrow \infty} \lambda^{-k} \int_\Sigma a_n \mathcal{L}_\varphi^k \mathbf{1}_{E_n} dm.$$

Since $\lambda^{-k} \mathcal{L}_\varphi^k \mathbf{1}_{E_n} \leq C_3$, another application of the dominated convergence theorem gives:

$$\sum_{n \in \mathbb{N}} \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\Sigma} a_n \mathcal{L}_\varphi^k \mathbf{1}_{E_n} dm = \lim_{k \rightarrow \infty} \sum_{n \in \mathbb{N}} \lambda^{-k} \int_{\Sigma} a_n \mathcal{L}_\varphi^k \mathbf{1}_{E_n} dm.$$

Thus, $\int_{\Sigma} f dm_H = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k} f dm$.

The proof of the proposition will be complete once we prove the following claim: Any Lipschitz function f on Σ is the uniform limit of a sequence $(g_n)_{n \in \mathbb{N}}$ satisfying $g_n = \sum_{k \in \mathbb{N}} a_k \mathbf{1}_{E_{n,k}}$, where $\{E_{n,k}\}_{k \in \mathbb{N}}$ is the collection of all n -cylinders and $\{a_k\}_{k \in \mathbb{N}} \subseteq \mathbb{C}$ lies in a disk of radius $|f|_\infty$.

Fix $f \in \text{Lip}(\Sigma)$. For each $n, k \in \mathbb{N}$, let $y_{E_{n,k}}$ be an arbitrary point in $E_{n,k}$. Define $g_n : \Sigma \rightarrow \mathbb{R}$ by

$$g_n = \sum_k f(y_{E_{n,k}}) \mathbf{1}_{E_{n,k}}.$$

If $x \in E_{n,k}$, then

$$|f(x) - g_n(x)| = |f(x) - f(y_{E_{n,k}})| \leq \text{Lip}(f) \theta^n.$$

It follows that $g_n \rightarrow f$ uniformly in Σ .

We can now compute $\int_{\Sigma} f dm_H$ as

$$(5.1) \quad \int_{\Sigma} f dm_H = \lim_{n \rightarrow \infty} \int_{\Sigma} g_n dm_H = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k} g_n dm,$$

where $\int_{\Sigma} g_n dm_H = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k} g_n dm$ by the first step of the proof. We claim that we can interchange the limits.

Fix $n \in \mathbb{N}$ and observe that $|g_n|_\infty \leq |f|_\infty$. Then

$$\begin{aligned} \left| \lambda^{-k} \int_{\dot{\Sigma}^k} g_n dm - \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k} g_n dm \right| &= \left| \lambda^{-k} \int_{\Sigma} \mathcal{L}_\varphi^k g_n dm - \int_{\Sigma} \Pi_\lambda(g_n) dm \right| \\ &\leq \lambda^{-k} \left| \int_{\Sigma} R^k(g_n) dm \right| \leq |f|_{C^0} \cdot \lambda^{-k} \|R^k\|. \end{aligned}$$

Thus, $\lambda^{-k} \int_{\dot{\Sigma}^k} g_n dm \rightarrow \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k} g_n dm$ uniformly (with respect to n). It follows that we can interchange the limits in (5.1) to obtain

$$\int_{\Sigma} f dm_H = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k} g_n dm = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k} f dm.$$

□

5.1. Some properties of m_H . First, we determine the support of m_H .

Lemma 5.3. *The support of m_H equals $\dot{\Sigma}^\infty$.*

Proof. Observe that $\Sigma \setminus \dot{\Sigma}^\infty$ is equal to the union of all cylinders that end in a state that is part of the hole. We will show that all such cylinders have m_H -measure zero. Suppose $[i_0, \dots, i_{n-2}, h] \subseteq \Sigma$ is one such cylinder, where $[h] \subseteq H$. If $k \geq n$, then clearly $[i_0, \dots, i_{n-2}, h] \cap \dot{\Sigma}^k = \emptyset$. Thus,

$$m_H([i_0, \dots, i_{n-2}, h]) = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\dot{\Sigma}^k \cap [i_0, \dots, i_{n-2}, h]} dm = 0.$$

It follows that $m_H(\Sigma \setminus \dot{\Sigma}^\infty) = 0$ so the support of m_H is contained in $\dot{\Sigma}^\infty$.

Now suppose U is open and $U \cap \dot{\Sigma}^\infty \neq \emptyset$. Fix $x \in U \cap \dot{\Sigma}^\infty$ and fix $n \in \mathbb{N}$ such that $[x_0, \dots, x_{n-1}] \subseteq U$. We claim that for each k sufficiently large $m_H([x_0, \dots, x_{n-1}] \cap \dot{\Sigma}^k)$ is uniformly bounded away from zero. Note first that $\mathbf{1}_{[x_0, \dots, x_{n-1}] \cap \dot{\Sigma}^k} = \mathbf{1}_{[x_0, \dots, x_{n-1}]} \cdot \mathbf{1}_{\dot{\Sigma}^k} \in \text{Lip}(\Sigma)$ for all $k \in \mathbb{N}$. Also, since

$\mathbf{1}_{\sigma([x_{n-1}])}$ is log-Lipschitz and bounded in C^0 -norm, by the proof of Proposition 4.3 there exist $N \in \mathbb{N}$ and $C_* > 0$ such that $\mathcal{L}_\varphi^N \mathbf{1}_{\sigma([x_{n-1}])} \geq C_*$. Now

$$\begin{aligned} m_H([x_0, \dots, x_{n-1}] \cap \mathring{\Sigma}^k) &= \lim_{j \rightarrow \infty} \lambda^{-j} \int_{\mathring{\Sigma}^j} \mathbf{1}_{[x_0, \dots, x_{n-1}] \cap \mathring{\Sigma}^k} dm = \lim_{j \rightarrow \infty} \lambda^{-j} \int_{\mathring{\Sigma}^j} \mathbf{1}_{[x_0, \dots, x_{n-1}]} dm \\ &= \lim_{j \rightarrow \infty} \lambda^{-j} \int_{\Sigma} \mathcal{L}_\varphi^j \mathbf{1}_{[x_0, \dots, x_{n-1}]} dm = \lim_{j \rightarrow \infty} \lambda^{-j} \int_{\Sigma} \mathcal{L}_\varphi^{j-n} \mathbf{1}_{\sigma([x_{n-1}])} dm \\ &= \lim_{j \rightarrow \infty} \lambda^{-(j-n-N)} \int_{\Sigma} \mathcal{L}_\varphi^{j-n-N} \left(\lambda^{-(n+N)} \mathcal{L}_\varphi^N \mathbf{1}_{\sigma([x_{n-1}])} \right) dm \\ &\geq C_* \lambda^{-(n+N)} m_H(\Sigma) > 0 \end{aligned}$$

Taking the limit as $k \rightarrow \infty$, we see that $m_H([x_0, \dots, x_{n-1}] \cap \mathring{\Sigma}^\infty) > 0$. It follows that $m_H(U \cap \mathring{\Sigma}^\infty) > 0$, and so the support of m_H equals $\mathring{\Sigma}^\infty$. \square

Next, we show that m_H is conformal for the renormalized punctured potential $\varphi_H - \log(\lambda)$.

Lemma 5.4. $\lambda^{-1} \mathcal{L}_\varphi^* m_H = m_H$. In particular,

$$\forall f \in Lip(\Sigma), \forall A \in \mathcal{B}(\Sigma) \quad \int_{\sigma^{-1}(A)} f dm_H = \lambda^{-1} \int_A \mathcal{L}_\varphi f dm_H,$$

where $\mathcal{B}(\Sigma)$ denotes the sigma-algebra of Borel subsets of Σ .

Proof. We show that $\lambda^{-1} \mathcal{L}_\varphi^* m_H = m_H$. Suppose $E \in \mathcal{C}$. Then

$$\lambda^{-1} \mathcal{L}_\varphi^* m_H(E) = \lim_{k \rightarrow \infty} \lambda^{-(k+1)} \int_{\Sigma} \mathcal{L}_\varphi^{k+1} \mathbf{1}_E dm = \int_{\Sigma} \mathbf{1}_E dm_H = m_H(E).$$

Since the measures agree on n -cylinders, they agree as Borel measures on Σ .

If $f : \Sigma \rightarrow \mathbb{R}$ is Borel measurable and A is a Borel set, then

$$\int_{\sigma^{-1}(A)} f dm_H = \int_{\Sigma} f \cdot \mathbf{1}_A \circ \sigma dm_H = \int_{\Sigma} \lambda^{-1} \mathcal{L}_\varphi(f \cdot \mathbf{1}_A \circ \sigma) dm_H = \int_A \lambda^{-1} \mathcal{L}_\varphi f dm_H. \quad \square$$

5.2. Construction of Gibbs Measure. Let ν be the positive Borel measure defined by

$$\forall A \in \mathcal{B}(\Sigma) \quad \nu(A) = \int_A g dm_H.$$

Proposition 5.5. ν is a σ -invariant probability measure on Σ whose support is $\mathring{\Sigma}^\infty$. Moreover, for each $f \in Lip(\Sigma)$,

$$\nu(f) = \lim_{n \rightarrow \infty} \lambda^{-n} \int_{\mathring{\Sigma}^n} fg dm.$$

Proof. ν is a probability measure since $\nu(\Sigma) = c(g) = 1$. Moreover, since the support of m_H is $\mathring{\Sigma}^\infty$ and g is bounded away from 0, the support of ν is also $\mathring{\Sigma}^\infty$.

Now fix a nonempty cylinder $E_n = [i_0, \dots, i_{n-1}]$. Then by Lemma 5.4,

$$\nu(\sigma^{-1}(E_n)) = \int_{\sigma^{-1}(E_n)} g dm_H = \lambda^{-1} \int_{E_n} \mathcal{L}_\varphi g dm_H = \int_{E_n} g dm_H = \nu(E_n).$$

Since ν agrees with $\sigma_* \nu$ on cylinders, it follows that $\nu = \sigma_* \nu$ and so ν is σ -invariant.

Finally, since $fg \in Lip(\Sigma)$ whenever $f \in Lip(\Sigma)$, we have by Proposition 5.2,

$$\nu(f) = \int fg dm_H = \lim_{n \rightarrow \infty} \lambda^{-n} \int_{\mathring{\Sigma}^n} fg dm. \quad \square$$

The following proposition implies that in fact ν is a Gibbs measure with respect to the potential φ .

Proposition 5.6. *There exists a constant $C_G > 0$ such that for any n -cylinder $E_n \subset \mathring{\Sigma}^n$ and any $y \in E_n$,*

$$C_G^{-1} \exp(S_n \varphi(y)) \lambda^{-n} \leq \nu(E_n) \leq C_G \exp(S_n \varphi(y)) \lambda^{-n}.$$

Proof. Fix an n -cylinder $E_n = [i_0, \dots, i_{n-1}] \subset \mathring{\Sigma}^n$ and let $w \in E_n$.

Lower bound. For any $x \in \mathring{\Sigma}$ and $k > n$,

$$\begin{aligned} \mathcal{L}_\varphi^k(\mathbf{1}_{E_n} g)(x) &= \sum_{y \in \mathring{\sigma}^{-k}(x) \cap E_n} g(y) e^{S_k \varphi(y)} \\ &\geq e^{-C_d} e^{S_n \varphi(w)} \sum_{y \in \mathring{\sigma}^{-k}(x) \cap E_n} \frac{g(y)}{g(\sigma^n(y))} \cdot g(\sigma^n(y)) e^{S_{k-n} \varphi \circ \sigma^n(y)} \\ &\geq C_g^{-1} e^{-C_d} e^{S_n \varphi(w)} \sum_{y \in \mathring{\sigma}^{-k}(x) \cap E_n} g \circ \sigma^n(y) e^{S_{k-n} \varphi \circ \sigma^n(y)} \\ &= C_g^{-1} e^{-C_d} e^{S_n \varphi(w)} \sum_{z \in \mathring{\sigma}^{-(k-n)}(x) \cap \sigma([i_{n-1}])} g(z) e^{S_{k-n} \varphi(z)} \\ &= C_g^{-1} e^{-C_d} e^{S_n \varphi(w)} \mathcal{L}_\varphi^{k-n}(\mathbf{1}_{\sigma([i_{n-1}])} g)(x) \end{aligned}$$

where $C_g = \frac{\sup\{g(x) \mid x \in \Sigma\}}{\inf\{g(x) \mid x \in \Sigma\}}$. The second-to-last equality follows from the fact that the restriction

$$\sigma^n : \mathring{\sigma}^{-k}(x) \cap [i_0, \dots, i_{n-1}] \rightarrow \mathring{\sigma}^{-(k-n)}(x) \cap \sigma([i_{n-1}])$$

is a bijection. Thus,

$$\begin{aligned} \lambda^{-k} \int_{\mathring{\Sigma}^k} \mathbf{1}_{E_n} g \, dm &= \lambda^{-k} \int_{\Sigma} \mathcal{L}_\varphi^k(\mathbf{1}_{E_n} g) \, dm \\ &\geq C_g^{-1} e^{-C_d} e^{S_n \varphi(w)} \lambda^{-k} \int_{\Sigma} \mathcal{L}_\varphi^{k-n}(\mathbf{1}_{\sigma([i_{n-1}])} g) \, dm \\ &= C_g^{-1} e^{-C_d} e^{S_n \varphi(w)} \lambda^{-n} \cdot \lambda^{-(k-n)} \int_{\Sigma} \mathcal{L}_\varphi^{k-n}(\mathbf{1}_{\sigma([i_{n-1}])} g) \, dm \end{aligned}$$

Note that since $\sigma([i_{n-1}])$ is a union of 1-cylinders, we have

$$\text{Lip}(\log(\mathbf{1}_{\sigma([i_{n-1}])} g)) \leq K.$$

Thus, it follows as in the proof of Proposition 4.3 that we can find $N \in \mathbb{N}$ and $B_N > 0$ such that $\mathcal{L}_\varphi^N(\mathbf{1}_{\sigma([i_{n-1}])} g) \geq B_N$. Then using Corollary 4.5, we have for $k > N$,

$$\lambda^{-k} \mathcal{L}_\varphi^k(\mathbf{1}_{\sigma([i_{n-1}])} g) \upharpoonright_{\mathring{\Sigma}} \geq B_N \lambda^{-N} \cdot \lambda^{-(k-N)} \mathcal{L}_\varphi^{k-N} \mathbf{1} \upharpoonright_{\mathring{\Sigma}} \leq B_N \lambda^{-N} C_3^{-1}.$$

It follows that $\nu(\sigma([i_{n-1}])) > 0$. Indeed, due to **(H)**, we can remove the dependence of the lower bound on i_{n-1} : Set

$$\bar{\kappa} = \inf\{\nu(\sigma([i])) : i \in \mathcal{S}\} \geq \min\{\nu([j]) : j \in \Lambda_H\} > 0,$$

where the last expression is positive since Λ_H is finite and the support of ν is $\mathring{\Sigma}^\infty$. It follows that

$$\nu(E_n) \geq C_g^{-1} e^{-C_d} e^{S_n \varphi(w)} \lambda^{-n} \bar{\kappa},$$

which is the required lower bound.

Upper bound. Let $w \in E_n \subset \hat{\Sigma}^n$ as before. If $x \in \Sigma$, then

$$\begin{aligned} \mathcal{L}_\varphi^k(\mathbf{1}_{E_n}g)(x) &= \sum_{y \in \hat{\sigma}^{-k}(x) \cap E_n} g(y)e^{S_k\varphi(y)} \leq e^{C_d} e^{S_n\varphi(w)} \sum_{y \in \hat{\sigma}^{-k}(x) \cap E_n} g(y)e^{S_{k-n}\varphi \circ \sigma^n(y)} \\ &\leq C_g e^{S_n\varphi(w)+C_d} \sum_{y \in \hat{\sigma}^{-k}(x) \cap E_n} g(\sigma^n(y))e^{S_{k-n}\varphi(\sigma^n(y))}. \end{aligned}$$

By our observation above of the injectivity of σ^n on n -cylinders, we can make the substitution $z = \sigma^n(y)$ and obtain

$$\begin{aligned} \mathcal{L}_\varphi^k(\mathbf{1}_{E_n}g)(x) &\leq C_g e^{S_n\varphi(w)+C_d} \sum_{z \in \hat{\sigma}^{-(k-n)}(x) \cap \sigma([i_{n-1}])} g(z)e^{S_{k-n}\varphi(z)} \\ &= C_g e^{S_n\varphi(w)+C_d} \mathcal{L}_\varphi^{k-n}(\mathbf{1}_{\sigma([i_{n-1}])}g)(x). \end{aligned}$$

This implies that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_\Sigma \lambda^{-k} \mathcal{L}_\varphi^k(\mathbf{1}_{E_n}g) dm &\leq C_g e^{S_n\varphi(w)+C_d} \lambda^{-n} \cdot \lim_{k \rightarrow \infty} \int_\Sigma \lambda^{-(k-n)} \mathcal{L}_\varphi^{k-n}(\mathbf{1}_{\sigma([i_{n-1}])}g) dm \\ &\leq C_g e^{S_n\varphi(w)+C_d} \lambda^{-n} \cdot \lim_{k \rightarrow \infty} \int_\Sigma \lambda^{-(k-n)} \mathcal{L}_\varphi^{k-n} g dm. \end{aligned}$$

Therefore, we conclude that $\nu(E_n) \leq C_g e^{S_n\varphi(w)+C_d} \lambda^{-n}$. \square

5.3. Proof of Theorem 2.2. We begin by defining the Gurevich pressure of a locally Lipschitz continuous potential φ . Define the partition function $Z_n(\varphi, a)$ on states $a \in \mathcal{S}$ by

$$Z_n(\varphi, a) = \sum_{\sigma^n(x)=a} \mathbf{1}_{[a]}(x) e^{S_n\varphi(x)}.$$

The *Gurevich pressure* $P(\varphi)$ of a topologically mixing topological Markov chain (Σ, σ) is then defined by

$$P(\varphi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\varphi, a).$$

The Gurevich pressure always exists (though it may equal ∞) and is independent of the choice of state a . When $|\mathcal{L}_\varphi \mathbf{1}|_\infty < \infty$, the Gurevich pressure is finite [S1, Theorem 1].

We now verify the items of Theorem 2.2. Item (c) follows from Lemmas 5.3 and 5.4 together with Proposition 5.5. Proposition 5.5 also proves the first half of (b).

It follows from Proposition 5.6 together with [S1, Theorem 8], that ν is an equilibrium state for $(\hat{\Sigma}^\infty, \sigma)$ for the potential $\varphi_H - \log \lambda$. Moreover, the Gurevich pressure $P_G(\varphi_H) = \log \lambda$. We thus have,

$$\log \lambda = P_G(\varphi_H) = \sup \left\{ h_\eta(\sigma) + \int \varphi_H d\eta : \eta \in \mathcal{M}, \int -\varphi_H d\eta < \infty \right\},$$

where $h_\eta(\sigma)$ denotes the metric entropy of σ with respect to η . Moreover, ν is the unique nonsingular invariant measure which attains the supremum. (Nonsingular in this context means that $\eta \circ \sigma$ is absolutely continuous with respect to η .) This proves item (a) as well as the escape rate formula in item (b). In order to complete the proof of (b), we must prove that ν enjoys exponential decay of correlations on functions in $\text{Lip}(\Sigma)$.

Let $f_1, f_2 \in \text{Lip}(\Sigma)$. It follows that $f_2 \circ \sigma^n \in \text{Lip}(\Sigma)$ for each $n \in \mathbb{N}$. Thus,

$$(5.2) \quad \int f_1 f_2 \circ \sigma^n d\nu = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\hat{\Sigma}^k} f_1 f_2 \circ \sigma^n g dm = \lim_{k \rightarrow \infty} \lambda^{-k} \int_{\hat{\Sigma}^{k-n}} \mathcal{L}_\varphi^n(f_1g) f_2 dm.$$

Notice also that $\nu(f_1) = c(f_1g)$, while for each $n \in \mathbb{N}$,

$$\nu(f_2) = \lim_{k \rightarrow \infty} \lambda^{-k+n} \int_{\hat{\Sigma}^{k-n}} f_2 g dm.$$

Using these observations together with (5.2), we use Theorem 2.1(d) to estimate,

$$\begin{aligned} \left| \int f_1 f_2 \circ \sigma^n d\nu - \nu(f_1)\nu(f_2) \right| &= \lim_{k \rightarrow \infty} \left| \lambda^{-k+n} \int_{\dot{\Sigma}^{k-n}} [\lambda^{-n} \mathcal{L}_\varphi^n(f_1 g) - c(f_1 g)g] f_2 dm \right| \\ &\leq \lim_{k \rightarrow \infty} C \tau^n \|f_1 g\|_{\text{Lip}} \lambda^{-k+n} \int_{\dot{\Sigma}^{k-n}} |f_2| dm \\ &\leq C' \tau^n \|f_1\|_{\text{Lip}} \nu(|f_2|), \end{aligned}$$

which completes the proof of item (b).

It remains to prove (d) of Theorem 2.2. Let $f \in \text{Lip}(\Sigma)$ with $f \geq 0$. By Lemma 4.13, it suffices to show that $c(f) > 0$ if and only if $\nu(f) > 0$. Now $c(f) = m_H(f)$ while $\nu(f) = m_H(gf)$. Thus the equivalence of $c(f)$ and $\nu(f)$ follows immediately from the fact that g is bounded away from 0 and infinity.

REFERENCES

- [A] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Mathematical Surveys and Monographs, American Mathematical Society (1997), 284 pp.
- [APT] E.G. Altmann, J.S.E. Portela and T. Tél, *Leaking chaotic systems*, Rev. Mod. Phys. **85** (2013), 869-918.
- [C1] N. N. Čencova, *A natural invariant measure on Smale's horseshoe*, Soviet Math. Dokl. **23** (1981), 87-91.
- [CM1] N. Chernov and R. Markarian, *Ergodic properties of Anosov maps with rectangular holes*, Bol. Soc. Bras. Mat. **28** (1997), 271-314.
- [CM2] N. Chernov and R. Markarian, *Anosov maps with rectangular holes. Nonergodic cases*, Bol. Soc. Bras. Mat. **28** (1997), 315-342.
- [CMT1] N. Chernov, R. Markarian and S. Troubetsky, *Conditionally invariant measures for Anosov maps with small holes*, Ergod. Th. and Dynam. Sys. **18** (1998), 1049-1073.
- [CMT2] N. Chernov, R. Markarian and S. Troubetsky, *Invariant measures for Anosov maps with small holes*, Ergod. Th. and Dynam. Sys. **20** (2000), 1007-1044.
- [BC] H. van den Bedem and N. Chernov, *Expanding maps of an interval with holes*, Ergod. Th. and Dynam. Sys. **22** (2002), 637-654.
- [CMSa] P. Collet, S. Martínez and J. San Martín, *Quasi-Stationary Distributions*, Probability and Its Applications, Springer-Verlag: Berlin Heidelberg (2013), 280 pp.
- [CMS1] P. Collet, S. Martínez and B. Schmitt, *The Yorke-Pianigiani measure and the asymptotic law on the limit Cantor set of expanding systems*, Nonlinearity **7** (1994), 1437-1443.
- [CMS2] P. Collet, S. Martínez and B. Schmitt, *The Yorke-Pianigiani measure for topological Markov chains*, Israel J. of Math. **97** (1997), 61-70.
- [D1] M.F. Demers, *Markov Extensions for Dynamical Systems with Holes: An Application to Expanding Maps of the Interval*, Israel J. of Math. **146** (2005), 189-221.
- [D2] M.F. Demers, *Markov Extensions and Conditionally Invariant Measures for Certain Logistic Maps with Small Holes*, Ergod. Th. and Dynam. Sys. **25**:4 (2005), 1139-1171.
- [D3] M.F. Demers, *Dispersing billiards with small holes*, in *Ergodic theory, open dynamics and coherent structures*, Springer Proceedings in Mathematics **70** (2014), 137-170.
- [D4] M.F. Demers, *Escape rates and physical measures for the infinite horizon Lorentz gas with holes*, Dynamical Systems: An International Journal **28**:3 (2013), 393-422.
- [BDM] H. Bruin, M.F. Demers and I. Melbourne, *Existence and convergence properties of physical measures for certain dynamical systems with holes*, Ergod. Th. and Dynam. Sys. **30** (2010), 687-728.
- [DF] M.F. Demers and B. Fernandez, *Escape rates and singular limiting distributions for intermittent maps with holes*, to appear in Trans. Amer. Math. Soc.
- [DWY] M.F. Demers, P. Wright and L.-S. Young, *Escape rates and physically relevant measures for billiards with small holes*, Comm. Math. Phys. **294**:2 (2010), 353-388.
- [DY] M.F. Demers and L.-S. Young, *Escape rates and conditionally invariant measures*, Nonlinearity **19** (2006), 377-397.
- [DO] C.P. Dettmann and O. Georgiou, *Survival probability for the stadium billiard*, Physica D **238** (2009), 2395-2403.
- [DR] C.P. Dettmann and M.R. Rahman, *Survival probability for open spherical billiards*, Chaos **24** (2014).
- [FKMP] P.A. Ferrari, H. Kesten, S. Martínez, and P. Picco, *Existence of quasi-stationary distributions. A renewal dynamical approach*, Annals of Prob. **23**:2 (1995), 501-521.

- [H] H. Hennion, *Sur un théorème spectral et son application aux noyaux lipschitziens*, Proc. Amer. Math. Soc. **118**:2 (1993), 627-634.
- [K] G. Keller, *Markov extensions, zeta functions, and Fredholm theory for piecewise invertible dynamical systems*, Trans. Amer. Math. Soc. **314**:2 (1989), 433-497.
- [LY] A. Lasota and J.A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481-488.
- [LiM] C. Liverani and V. Maume-Deschamps, *Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set*, Annales de l'Institut Henri Poincaré Probability and Statistics, **39** (2003), 385-412.
- [LM] A. Lopes and R. Markarian, *Open Billiards: Cantor sets, invariant and conditionally invariant probabilities*, SIAM J. Appl. Math. **56** (1996), 651-680.
- [MU] R.D. Maudlin and M. Urbanski, *Gibbs states on the symbolic space over an infinite alphabet*, Israel J. Math. **125** (2001), 93-130.
- [P] K.R. Parthasarathy, *Introduction to Probability and Measure*, Springer-Verlag (1977).
- [PY] G. Pianigiani and J.A. Yorke, *Expanding maps on sets which are almost invariant: decay and chaos*, Trans. Amer. Math. Soc. **252** (1979), 351-366.
- [S1] O. Sarig, *Thermodynamical Formalism for Countable Markov Shifts*, Ergodic Theory Dyn. Syst. **19** (1999), 1565-1593.
- [S2] O. Sarig, *Existence of Gibbs measures for countable Markov shifts*, Proc. Amer. Math. Soc. **131**:6 (2003), 1751-1758.
- [V] D. Vere-Jones, *Geometric ergodicity in denumerable Markov chains*, Quart. J. Math. **13** (1962), 7-28.
- [Y] T. Yarmola, *Sub-exponential mixing of random billiards driven by thermostats*, Nonlinearity **26**:7 (2013), 1825-1837.

MARK F. DEMERS, DEPARTMENT OF MATHEMATICS, FAIRFIELD UNIVERSITY, FAIRFIELD, CT 06824, USA
E-mail address: `mdemers@fairfield.edu`

CHRISTOPHER J. IANZANO, DEPARTMENT OF MATHEMATICS, STONY BROOK UNIVERSITY, STONY BROOK, NY 11794, USA
E-mail address: `christopher.ianzano@stonybrook.edu`

PHILIP MAYER, DEPARTMENT OF MATHEMATICS, FAIRFIELD UNIVERSITY, FAIRFIELD, CT 06824, USA
E-mail address: `philip.mayer@student.fairfield.edu`

PETER MORFE, DEPARTMENT OF ELECTRICAL ENGINEERING, THE COOPER UNION, NEW YORK, NY 10003, USA
E-mail address: `morfe@cooper.edu`

ELIZABETH C. YOO, DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027, USA
E-mail address: `ecy@columbia.edu`