EXPONENTIAL DECAY OF CORRELATIONS FOR CONTACT HYPERBOLIC FLOWS

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Abstract. We describe the main ideas and key steps in the proof of exponential decay of correlations for hyperbolic contact flows. The main exposition concerns contact Anosov flows, followed by some comments on the recent extension of the technique to finite horizon Sinai billiard flows.

1. Introduction

These notes are based on a mini-course given as part of the workshop on Statistical Properties of Nonequilibrium Dynamical Systems, held at the South University of Science and Technology of China in Shenzhen, China, July 11-26, 2016. The purpose of these notes is to present the essential features needed to adapt the analysis of the discrete time transfer operator for hyperbolic maps to the semi-group of continuous time transfer operators for hyperbolic flows. There are three main steps needed in the present setting.

(1) Adapt Banach spaces used for hyperbolic maps to the setting of hyperbolic flows: the presence of the neutral flow direction makes this a nontrivial change.
(2) Contrary to the discrete-time case, we do not prove the quasi-compactness of the transfer operator for the time-one map of the flow, but rather for the generator of the semi-group of transfer operators for the flow; this involves the use of the resolvent to 'integrate out' the neutral direction.
(3) The use of the contact form to estimate an oscillatory integral and derive a spectral gap for the generator of the semi-group (the Dolgopyat-type estimate).

It then follows from some general considerations that a spectral gap for the generator of the semi-group implies exponential decay of correlations for the flow.

1.1. **Setting.** For ease of exposition and to more clearly identify the key features of the techniques we shall present, we will limit our setting to that of a 3-dimensional manifold. This will suffice for the purposes of explaining the main ideas of this technique, as well as its eventual application to dispersing planar billiards.

Let Ω be a 3-dimensional compact, smooth Riemannian manifold, and let $\Phi_t : \Omega \to \Omega$ be a C^2 Anosov flow. By this, we mean that $\{\Phi_t\}_{t \in \mathbb{R}}$ is a family of C^2 diffeomorphisms of Ω satisfying the group properties: (a) $\Phi_0 = Id$; (b) $\Phi_t \circ \Phi_s = \Phi_{t+s}$, for all $s, t \in \mathbb{R}$.

Moreover, at each $x \in \Omega$, there is a $D\Phi_t$-invariant splitting of the tangent space, $T_x\Omega = E^s(x) \oplus E^c(x) \oplus E^u(x)$, continuous in x, such that the angles between $E^s(x)$, $E^u(x)$ and $E^c(x)$ are uniformly bounded away from 0 on Ω. $E^c(x)$ is the flow direction at $x \in \Omega$. We assume there exist constants $C, C' > 0$, $\Lambda > 1$, such that for all $x \in \Omega$ and $t \geq 0$,

$$\|D\Phi_t(x)v\| \leq C\Lambda^{-t}\|v\| \quad \forall\, v \in E^s(x), \qquad \|D\Phi_t(x)v\| \geq C'\Lambda^{t}\|v\| \quad \forall\, v \in E^u(x). \tag{1.1}$$

We shall assume throughout that our Anosov flow is *contact*, i.e. it preserves a contact form on Ω. More precisely, we assume there exists a C^2 one-form ω on Ω such that $\omega \wedge d\omega$ is nowhere zero.

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We assume that Φ_t preserves ω:

$$\omega(\Phi_t(x), D\Phi_t(x)v) = \omega(x, v), \qquad \forall x \in \Omega, v \in T_x\Omega. \tag{1.2}$$

It is clear from the invariance described by (1.2) that $\ker(\omega) = E^s(x) \oplus E^u(x)$. It follows that if $v_0 \in E^c(x)$ is a unit vector in the flow direction, then $\omega(v_0) \neq 0$. Thus replacing ω by $\omega/\omega(v_0)$, we may assume without loss of generality that $\omega(v_0) = 1$ and that the contact volume $\omega \wedge d\omega$ coincides with the Riemannian volume on Ω. It follows from these considerations that the Jacobian of the flow is identically equal to 1, i.e. $J\Phi_t = 1$, and that the flow preserves the Riemannian volume on Ω, which we shall denote by m.

1.2. **Decay of Correlations.** The main question we shall address in these notes is that of the rate of decay of correlations of the contact Anosov flow defined in the previous section. For $\alpha > 0$ and $\varphi, \psi \in C^\alpha(\Omega)$, define the *correlation function*,

$$C_t(\varphi, \psi) = \left| \int_\Omega \varphi\, \psi \circ \Phi_t \, dm - \int_\Omega \varphi \, dm \int_\Omega \psi \, dm \right|.$$

If $C_t(\varphi, \psi) \to 0$ as $t \to \infty$ for all Hölder continuous functions φ and ψ, then we say the flow is mixing. The question then becomes, at what rate? The main result that we shall establish in these notes is the following.

Theorem 1.1. *Let Φ_t be a C^2 Anosov flow of a smooth, compact 3-dimensional Riemannian manifold Ω preserving a C^2 contact form ω. Then for each $\alpha > 0$, there exists $\eta = \eta(\alpha)$, and $C > 0$ such that for all $\varphi, \psi \in C^\alpha(\Omega)$ and all $t \geq 0$,*

$$\left| \int_\Omega \varphi\, \psi \circ \Phi_t \, dm - \int_\Omega \varphi \, dm \int_\Omega \psi \, dm \right| \leq C |\varphi|_{C^\alpha(\Omega)} |\psi|_{C^\alpha(\Omega)} e^{-\eta t}.$$

This is a special case of a more general result proved for any odd-dimensional manifold by Liverani [L]. We will limit our exposition to three dimensions in order to maintain the focus on the essential elements of the technique.

From the definition of the correlation function, one can see immediately that, due to the invariance of the measure m, a simple change of variables yields,

$$\int_M \varphi\, \psi \circ \Phi_t \, dm = \int_M \varphi \circ \Phi_{-t}\, \psi \, dm = \int_M \mathcal{L}_t \varphi\, \psi \, dm, \tag{1.3}$$

where for each t, $\mathcal{L}_t \varphi := \varphi \circ \Phi_{-t}$ is the *transfer operator*, or *Ruelle-Perron-Frobenius operator* associated with Φ_t, defined pointwise, for example, on continuous functions. From this change of variables, it follows that the rate of decay of correlations is tied to the spectral properties of the semi-group $\{\mathcal{L}_t\}_{t \geq 0}$. This is the perspective that we will develop in these notes.

1.3. **Some History and Present Approach.** The proof of exponential decay of correlations for some classes of uniformly hyperbolic flows has proved to be much more subtle than the analogous proof for hyperbolic diffeomorphisms. For uniformly hyperbolic diffeomorphisms, there is a type of dichotomy: either the map is exponentially mixing on smooth observables, or it is not mixing at all. This does not hold for uniformly hyperbolic flows. In [R], Ruelle constructed a class of Axiom A suspension flows with piecewise constant roof function that mix at a polynomial rate. Pollicott [P1] then generalized this class of examples to obtain polynomial decay of correlations of any power, indeed even logarithmically slow decay.

Some early success in proving exponential decay for geodesic flows on manifolds of constant negative curvature in 2 and 3 dimensions was achieved by Moore [Mo], Ratner [Ra] and Pollicott [P2], and certain perturbations were considered in [CEG], but the techniques were algebraic and did not generalize to manifolds of variable curvature.

The first dynamical proof of decay of correlations for Anosov flows was given by Chernov [C1], who exploited the 'twist' provided by the contact form in order to estimate a key quantity, the temporal distance function (see (5.13) and Remark 5.1), yet he was only able to obtain a stretched exponential bound using Markov partitions. Next, Dolgopyat [Do] was the first to prove exponential decay of correlations for Anosov flows, using an assumption of C^1 stable and unstable foliations to estimate a crucial oscillatory integral (see Lemma 5.5). This work was further extended by Liverani [L], who proved exponential decay for contact Anosov flows by combining a functional analytic approach with the ideas of Dolgopyat and Chernov. These ideas were then adapted to piecewise cone hyperbolic flows by Baladi and Liverani [BL], and finally[1] to some dispersing billiard flows in [BDL]. It is this line of argument that we shall follow in the present set of notes, and we shall limit our discussion primarily to the smooth, Anosov case, in order to present the key ideas most clearly.[2]

Given this approach, several choices are available with regards to the functional analytic framework in which to view the transfer operator.

(1) The approach via Markov partitions used by Dolgopyat [Do].
(2) The norms originally used in [L], which define norms integrating over the entire phase space of the flow. These were based on the paper [BKL], which introduced a set of Banach spaces for Anosov diffeomorphisms and subsequently inspired a series of papers constructing norms for hyperbolic maps from several points of view (see [B1] for a recent survey of these approaches, and [B2] for a more in-depth treatment).
(3) The Sobolev-type spaces used in [BL] for piecewise cone hyperbolic contact flows. These norms use Fourier transforms and were based on work of Baladi, Tsujii and Gouëzel [BT1, BG] who constructed the analogous norms for diffeomorphisms.
(4) The 'geometric' approach of [GL], which modified the norms of [BKL] to integrate over cone-stable curves only. This modification turned out to be essential for the adaptability of this method to piecewise hyperbolic maps requiring only Hölder continuity in the unstable direction in [DL] and finally to dispersing billiards in [DZ1, DZ3]. Most recently, it was extended to prove exponential decay of correlations for the finite horizon Sinai billiard flow [BDL].

In the present set of notes, we will define a functional analytic setup for contact Anosov flows which follows the technique described in (4) above. As a result, our exposition and some proofs will differ from Liverani's published proof [L]. Yet we choose this method since it combines a relatively simple exposition with a flexible framework. To date, the geometric norms integrating over stable curves has proved to be the most versatile in terms of its applicability to a wide range of hyperbolic systems with discontinuities.

A brief outline of the paper is as follows. In Section 2, we introduce necessary definitions and define the Banach spaces on which our transfer operators and resolvents will act. We also outline some properties of these spaces regarding embeddings and compactness. Unfortunately, Proposition 2.6 does not provide true Lasota-Yorke inequalities for our semi-group $\{\mathcal{L}_t\}_{t\geq 0}$, so in Section 3 we introduce the generator of the semi-group X and the related resolvent $R(z)$, $z \in \mathbb{C}$. As evidenced by Proposition 3.3 and Corollary 3.4, we are able to prove quasi-compactness for $R(z)$, and so obtain useful information about the spectrum of X (Proposition 3.5). In Section 4, we introduce an improved estimate on the spectral radius of $R(z)$ when $|\mathrm{Im}(z)|$ is large, which implies a spectral gap for X, and leads to the proof of Theorem 1.1. This in turn is reduced to a Dolgopyat-type estimate, Lemma 4.3, which is proved in Section 5. In Section 6, we briefly sketch some modifications needed to generalize the present approach to dispersing billiards, as carried out in [BDL].

[1] In the meantime, Chernov [C2] and Melbourne [M] had proved a stretched exponential bound for dispersing billiard flows using the techniques adapted from [C1] and [Do].

[2] A different mechanism for exponential mixing has been proved in the recent work of Tsujii [T], but this lies outside the scope of the present notes.

2. Functional Analytic Framework

In order to define the Banach spaces on which our transfer operator will act, we first extend its definition from acting on continuous functions introduced in Section 1.2 to acting on spaces of distributions.

For $\alpha \in (0,1]$, and W a smooth submanifold of Ω, define the C^α-norm for functions on W by

$$|\varphi|_{C^\alpha(W)} := \sup_{x\in W} |\varphi(x)| + H^\alpha_W(\varphi), \quad H^\alpha_W(\varphi) := \sup_{x\neq y\in W} |\varphi(x)-\varphi(y)| d_W(x,y)^{-\alpha}, \tag{2.1}$$

where $d_W(\cdot,\cdot)$ is the Riemannian metric restricted to W. Notice with this definition that $C^1(W)$ is the set of Lipschitz functions on W.

Since the flow is C^2, if $\psi \in C^1(\Omega)$, then $\psi \circ \Phi_{-t} \in C^1(\Omega)$. Thus we may define $\mathcal{L}_t$ acting on $(C^1(\Omega))^*$, the dual of $C^1(\Omega)$, by

$$\mathcal{L}_t f(\psi) = f(\psi \circ \Phi_t), \quad \text{for all } \psi \in C^1(\Omega),\ f \in (C^1(\Omega))^*.$$

If $f \in L^1(m)$, then we identify f with the measure $f dm \in (C^1(\Omega))^*$. With this identification, $\mathcal{L}_t$ has the pointwise definition stated earlier, $\mathcal{L}_t f = f \circ \Phi_{-t}$, and its action is consistent with (1.3).

2.1. Admissible cone-stable and cone-unstable curves.

Due to the uniform hyperbolicity of Φ_t given by (1.1), we define stable and unstable cones $C^s(x)$, $C^u(x) \subset E^s(x) \oplus E^u(x)$, lying in the kernel of the contact forms. The cones satisfy the strict invariance condition,

$$D\Phi_{-t} C^s(x) \subset C^s(\Phi_{-t}x), \quad D\Phi_t C^u(x) \subset C^u(\Phi_t x), \quad \text{for all } t > 0. \tag{2.2}$$

Note that these cones are 'flat' since they lie in the plane $E^s(x) \oplus E^u(x)$, and have empty interior in $T_x\Omega$. We may choose these cones so that they are continuous and uniformly transverse on Ω. Moreover, the uniform contraction and expansion given by (1.1) extends to all vectors in $C^s(x)$ and $C^u(x)$, respectively, with possibly slightly weaker constants C, C' and Λ.

Let $d_0 > 0$ denote the minimal length of a closed geodesic on Ω.

Definition 2.1. *We define a family of* admissible cone-stable curves*,* $\mathcal{W}^s = \mathcal{W}^s(\delta_0, C_0)$*, in* Ω *satisfying:*

- (W1) *for all* $W \in \mathcal{W}^s$ *and* $x \in W$*, the unit tangent vector to* W *at* x *belongs to* $C^s(x)$*;*
- (W2) *there exists* $\delta_0 \in (0, d_0/2)$ *such that* $|W| \le \delta_0$ *for all* $W \in \mathcal{W}^s$*;*
- (W3) *there exists* $C_0 > 0$ *such that the curvature of* W *is bounded by* C_0*.*

For brevity, we refer to $W \in \mathcal{W}^s$ simply as stable curves. A family of unstable curves $\mathcal{W}^u$ is defined similarly.

Due to the strict invariance of the cones, we have $\Phi_{-t}\mathcal{W}^s \subseteq \mathcal{W}^s$, $t \ge 0$, up to subdivision of curves longer than length δ_0. Similarly, $\Phi_t \mathcal{W}^u \subseteq \mathcal{W}^u$, $t \ge 0$.

In order to compare different curves in $\mathcal{W}^s$, we will introduce a notion of distance between them. To do this, we place finitely many local sections Σ_i in M, which are smooth surfaces with piecewise smooth boundary, such that

- (a) there exists $\tau_0 \in (0, d_0/2)$, such that each $W \in \mathcal{W}^s$ projects as a smooth, connected curve onto at least one Σ_i under $\{\Phi_t\}_{0\le t\le \tau_0}$;
- (b) each Σ_i is uniformly transverse to the flow direction;
- (c) for each i, there exists a common family of stable and unstable cones for all $x \in \Sigma_i$.

On each section, we distinguish a point $\bar{x}_i$ in the approximate center of Σ_i, and define local coordinates $(\bar{x}^s, \bar{x}^u)$ with $\bar{x}_i$ at the origin, and the $\bar{x}^s$ $(\bar{x}^u)$ axis tangent to $E^s(\bar{x}_i)$ $(E^u(\bar{x}_i))$ at $\bar{x}_i$. We may construct the Σ_i so that they are approximately rectangular in these coordinates: $\Sigma_i = \{(\bar{x}^s, \bar{x}^u) : \bar{x}^s \in I^s_i, \bar{x}^u \in I^u_i\}$, where I^s_i and I^u_i are two intervals centered at 0.

On each domain[3] of the form $D_i = \{\Phi_{-t}(\Sigma_i)\}_{0 \le t \le \tau_0}$, let P_i^+ denote the projection onto Σ_i, defined at $x \in D_i$ as the first intersection of $\Phi_t(x)$ with Σ_i, for $t \ge 0$. For $W \in \mathcal{W}^s$, if $P_i^+ W$ is defined, then we may view it as the graph of a function $G_{i,W} : I_{i,W} \to I_i^u$, where $I_{i,W} \subset I_i^s$, in case the curve W is very short.

Now if $W_1, W_2 \in \mathcal{W}^s$, we define a notion of distance between them as follows. If there exists $U \in W^u$ such that $U \cap W_1 \neq \emptyset$ and $U \cap W_2 \neq \emptyset$ and at least one i such that $P_i^+ W_1$ and $P_i^+ W_2$ are both defined, then

$$d_{\mathcal{W}^s}(W_1, W_2) := \min_i \{ |I_{i,W_1} \triangle I_{i,W_2}| + |G_{i,W_1} - G_{i,W_2}|_{C^1(I_{i,W_1} \cap I_{i,W_2})} \}. \tag{2.3}$$

Otherwise,[4] set $d_{\mathcal{W}^s}(W_1, W_2) = \infty$.

The purpose of requiring the existence of $U \in \mathcal{W}^u$ intersecting both curves is to ensure that they are sufficiently close in the flow direction (since the distance in (2.3) only quantifies the distance between projected curves in Σ_i, which quotients out the flow direction).

Remark 2.2. *The choice to compare curves on sections rather than directly on the manifold Ω may seem unnecessarily awkward at this stage. Yet, it simplifies certain norm calculations considerably by introducing a convenient set of local coordinate systems. In addition, it allows for an immediate generalization to billiards since then one can simply take the sections Σ_i to correspond to the smooth parts of the boundary of the billiard table.*

A second point to notice is that the distance defined by (2.3) *does not define a metric, or even a pseudo-metric since it does not satisfy the triangle inequality. This does not affect our analysis at all since the norms we defined will satisfy the triangle inequality, and this is sufficient for our purposes.*

For two curves $W_1, W_2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1, W_2) < \infty$, we can use the same coordinate system to define a notion of distance between test functions supported on these curves. Let $\psi_i \in C^0(W_i)$, $i = 1, 2$. Define

$$d_0(\psi_1, \psi_2) = \min_i \{ |\psi_1 \circ G_{i,W_1} - \psi_2 \circ G_{i,W_2}|_{C^1(I_{i,W_1} \cap I_{i,W_2}} \},$$

where the minimum is taken over all i such that both $P_i^+(W_1)$ and $P_i^+(W_2)$ are both defined.

2.2. **Definition of the norms and Banach spaces.** Given $\alpha \in (0,1)$ and $W \in \mathcal{W}^s$, define $C^\alpha(W)$ to be the closure[5] of $C^1(W)$ in the $C^\alpha(W)$ norm, defined by (2.1). This definition of $C^\alpha(W)$ guarantees that the embedding of our strong space into our weak space is injective (see Lemma 2.4).

Now fix $\alpha \in (0, 1]$. Given $f \in C^1(\Omega)$, define the *weak norm* of f by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \le 1}} \int_W f \, \psi \, dm_W,$$

where m_W is arc length measure along W.

By contrast, our strong norm will have three components. Choose $1 < q < \infty$, $\beta \in (0, \alpha)$ and $0 < \gamma \le \min\{\alpha - \beta, 1/q\}$.

For $f \in C^1(\Omega)$, define the *strong stable norm* of f by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\beta(W) \\ |\psi|_{C^\beta(W)} \le |W|^{-1/q}}} \int_W f \, \psi \, dm_W.$$

[3]Note that these domains may overlap for different i.

[4]That is, if W_1 and W_2 do not project onto a common Σ_i, or if there is no $U \in \mathcal{W}^u$ with the required property.

[5]$C^\alpha(W)$ is strictly smaller than the set of functions with finite $|\cdot|_{C^\alpha(W)}$ norm, yet it contains all functions with finite $|\cdot|_{C^{\alpha'}(W)}$ norm for all $\alpha' > \alpha$.

Define the *neutral norm* of f by

$$\|f\|_0 = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^\alpha(W) \\ |\psi|_{C^\alpha(W)} \le 1}} \int_W \tfrac{d}{dt}(f \circ \Phi_t)|_{t=0} \, \psi \, dm_W .$$

And finally, define the *unstable norm* of f by

$$\|f\|_u = \sup_{\varepsilon > 0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon}} \sup_{\substack{\psi_i \in C^\alpha(W_i) \\ |\psi_i|_{C^\alpha(W_i)} \le 1 \\ d_0(\psi_1, \psi_2) = 0}} \varepsilon^{-\gamma} \left| \int_{W_1} f \, \psi_1, dm_{W_1} - \int_{W_2} f \, \psi_2 \, dm_{W_2} \right| .$$

Define the *strong norm* of f by

$$\|f\|_{\mathcal{B}} = \|f\|_s + \|f\|_0 + c_u \|f_u\|,$$

where $c_u > 0$ is a constant to be chosen later.

Now our weak space $\mathcal{B}_w$ is defined as the completion of $C^2(\Omega)$ in the $|\cdot|_w$ norm, while our strong space $\mathcal{B}$ is defined as the completion of $C^2(\Omega)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

Remark 2.3. *The restrictions on the parameters are placed due to the following considerations. That $\beta < \alpha$ is required for compactness (Lemma 2.5). Then $\gamma \le \alpha - \beta$ is required when adjusting test functions for the unstable norm estimate* (2.9)*, while $\gamma \le 1/q$ allows us to account for short unmatched pieces due to our use of sections in the same estimate. Finally, $q > 1$ is required to obtain contraction in the strong stable norm estimate* (2.8)*. For a C^2 flow, one may take $\alpha = 1$.*

In order to use the Dolgopyat estimate (Lemma 4.3) to prove Proposition 4.2, we shall introduce additional restrictions on the parameters when applying the mollification lemma (Lemma 4.4). For this proof, we shall need β to be sufficiently small and q sufficiently close to 1 so that $(1+\beta-1/q)/\gamma < \gamma_0$, where γ_0 is from Lemma 4.3.

2.3. **Properties of the Banach Spaces.** The spaces $\mathcal{B}$ and $\mathcal{B}_w$ are spaces of distributions, and the following lemma describes some important relations with more familiar spaces.

Lemma 2.4. *The following set of inclusions are continuous, and the first two are injective,*

$$C^1(\Omega) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (C^\alpha(\Omega))^* .$$

Indeed, there exists $C > 0$ such that for all $f \in C^1(\Omega)$, we have

$$|f|_w \le \|f\|_{\mathcal{B}} \le C |f|_{C^1(\Omega)} . \tag{2.4}$$

Moreover,

$$|f(\psi)| \le C |f|_w |\psi|_{C^\alpha(\Omega)} \ \ \forall\, f \in \mathcal{B}_w, \qquad |f(\psi)| \le C \|f\|_s |\psi|_{C^\beta(\Omega)} \ \ \forall\, f \in \mathcal{B}. \tag{2.5}$$

Proof. The bounds in (2.4) are clear from the definitions of the norms, proving the continuity of the first two inclusions. Moreover the injectivity of the first inclusion is obvious, while that of the second follows from the fact that $C^1(W)$ is dense in both $C^\alpha(W)$ and $C^\beta(W)$ because of the way we have defined these spaces of test functions.

It remains to prove the inequalities in (2.5), which in turn imply the continuity of the last inclusion. We prove the first inequality in (2.5), since the proof of the second is similar.

Let $f \in C^2(\Omega)$, $\psi \in C^\alpha(\Omega)$. We subdivide Ω into a finite number of boxes B_i and foliate each box by a smooth foliation of stable curves $\{W_\xi\}_{\xi \in \Xi_i}$. To see that this is possible, we can choose each box B_i to lie inside one of the domains D_i corresponding to surface Σ_i. Choosing a smooth family of stable curves intersecting Σ_i, we can simply flow it to fill B_i.

Now on each B_i, we disintegrate the measure m into conditional measures $\rho_\xi dm_{W_\xi}$ on each W_ξ and a factor measure $\hat{m}_i$ on the index set Ξ_i. Since the foliation is smooth, we have $|\rho_\xi|_{C^1(W_\xi)} \le C_1$

for some $C_1 > 0$ and all $\xi \in \Xi_i$. Then,

$$|f(\psi)| = \left| \int_\Omega f\,\psi\,dm \right| \le \sum_i \int_{\Xi_i} \left| \int_{W_\xi} f\,\psi\,\rho_\xi\,dm_{W_\xi} \right| d\hat{m}_i$$
$$\le \sum_i \int_{\Xi_i} |f|_w |\psi|_{C^\alpha(W_\xi)} |\rho_\xi|_{C^\alpha(W_\xi)} d\hat{m}_i \le C|f|_w|\psi|_{C^\alpha(\Omega)}.$$

Since this bound holds for all $f \in C^2(\Omega)$, by density it holds for all $f \in \mathcal{B}_w$. □

Lemma 2.5. *The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$.*

Proof. The compactness follows from two important points: the compactness of the unit ball of $C^\alpha(W)$ in $C^\beta(W)$ for each $W \in \mathcal{W}^s$; and the compactness in the C^1 norm of the set of graphs $G_{i,W}$ with C^2 norm bounded by C_0 on each section Σ_i. This allows us to prove that for all $\varepsilon > 0$, there exists a finite set of linear functionals $\ell_{i,j}$ on $\mathcal{B}$, with $\ell_{i,j}(f) = \int_{W_i} f\,\psi_j\,dm_{W_i}$, $\psi_j \in C^\alpha(W_i)$, such that

$$\min_{i,j} \left(|f|_w - \ell_{i,j}(f)\right) \le C\varepsilon^\gamma \|f\|_{\mathcal{B}}, \tag{2.6}$$

for a uniform constant $C > 0$. This implies the required compactness. For the details of the approximation needed to carry out the above estimate, see [DZ1, Lemma 3.10] or [BDL, Lemma 3.10]. □

Exercise 1. *Assume that* (2.6) *holds. Show that it implies that the unit ball of $\mathcal{B}$ is compact in $\mathcal{B}_w$.*

2.4. **Lasota-Yorke type inequalities for the semi-group $\mathcal{L}_t$.** The semi-group of transfer operators $\{\mathcal{L}_t\}_{t\ge 0}$ satisfies the following set of dynamical inequalities, often called Lasota-Yorke, of Doeblin-Fortet inequalities, following their seminal role in the development of the spectral theory of transfer operators [DF, LY].

Proposition 2.6. *There exists $C > 0$ such that for all $f \in \mathcal{B}$ and $t \ge 0$,*

$$|\mathcal{L}_t f|_w \le C|f|_w \tag{2.7}$$
$$\|\mathcal{L}_t f\|_s \le C(\Lambda^{-\beta t} + \Lambda^{-(1-1/q)t})\|f\|_s + C|f|_w \tag{2.8}$$
$$\|\mathcal{L}_t f\|_u \le C\Lambda^{-\gamma t}\|f\|_u + C\|f\|_0 + C\|f\|_s \tag{2.9}$$
$$\|\mathcal{L}_t f\|_0 \le C\|f\|_0. \tag{2.10}$$

If $\mathcal{L}_t$ were the transfer operator for a hyperbolic diffeomorphism of a 2-dimensional manifold, the inequalities (2.7) - (2.9) would be the traditional Lasota-Yorke inequalities (there would be no neutral direction), and we would conclude that $\mathcal{L}_t$ was quasi-compact with spectral radius 1, and essential spectral radius strictly smaller than 1. Unfortunately, in the case of a flow, we are left with the inequality (2.10) for the neutral norm, due to the lack of hyperbolicity in the flow direction. Thus the above inequalities do not represent a true set of Lasota-Yorke inequalities since the strong norm does not contract. So we do not prove that $\mathcal{L}_t$ is quasi-compact on $\mathcal{B}$.

Before proceeding to the next step in the argument, which is the introduction of the resolvent and the generator of the semi-group, we prove several items of the proposition, to give a flavor for the estimates required. A full proof of analogous inequalities in a variety of settings can be found in, for example, [GL] for Anosov diffeomorphisms, [DZ1] for dispersing billiard maps, or [BDL] for some dispersing billiard flows.

Proof of Proposition 2.6. Due to the density of $C^2(M)$ in $\mathcal{B}$, it suffices to prove the inequalities for $f \in C^2(M)$. We first prove (2.8).

When we flow a stable curve $W \in \mathcal{W}^s$ backwards, $\Phi_{-t}W$ may grow to have length greater than δ_0. If so, we subdivide it into a finite collection $\mathcal{G}_t(W) = \{W_i\}_i \subset \mathcal{W}^s$ so that each W_i has length between $\delta_0/2$ and δ_0, and $\cup_i W_i = \Phi_{-t}W$.

Let $f \in C^2(M)$, $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \le |W|^{-1/q}$. We must estimate, for $t \ge 0$,

$$\int_W \mathcal{L}_t f\, \psi\, dm_W = \sum_{W_i \in \mathcal{G}_t(W)} \int_{W_i} f\, \psi \circ \Phi_t\, J_{W_i}\Phi_t\, dm_{W_i}, \tag{2.11}$$

where we have changed variables and subdivided the integral on $\Phi_{-t}W$ into a sum of integrals over the $W_i \in \mathcal{G}_t(W)$. The function $J_{W_i}\Phi_t$ denotes the Jacobian of Φ_t along the curve W_i. Due to (1.1), this is a contraction.

Case I. $|\Phi_{-t}W| > \delta_0$.

For each i, define $\overline{\psi}_i$ to be the average value of $\psi \circ \Phi_t$ on W_i. Then subtracting the average on each W_i, we can rewrite (2.11) as,

$$\begin{aligned}\int_W \mathcal{L}_t f\, \psi\, dm_W &= \sum_{W_i \in \mathcal{G}_t(W)} \int_{W_i} f\,(\psi \circ \Phi_t - \overline{\psi}_i)\, J_{W_i}\Phi_t\, dm_{W_i} + \overline{\psi}_i \int_{W_i} f\, J_{W_i}\Phi_t\, dm_{W_i} \\ &\le \sum_i \|f\|_s |\psi \circ \Phi_t - \overline{\psi}_i|_{C^\beta(W_i)} |W_i|^{1/q} |J_{W_i}\Phi_t|_{C^\beta(W_i)} + |f|_w |\psi \circ \Phi_t|_{C^\alpha(W_i)} |J_{W_i}\Phi_t|_{C^\alpha(W_i)},\end{aligned} \tag{2.12}$$

where we have applied the strong stable norm to the first set of terms and the weak norm to the second set.

The C^β norm of $\psi \circ \Phi_t - \overline{\psi}_i$ is easy to estimate using the uniform hyperbolicity of Φ_t given by (1.1), as well as the fact that we have defined stable curves which are transverse to the flow direction, and whose tangent vector lie exactly in the plane where the hyperbolicity of the flow dominates. Thus for $x, y \in W_i$,

$$|\psi \circ \Phi_t(x) - \psi \circ \Phi_t(y)| \le H^\beta_W(\psi) d(\Phi_t(x), \Phi_t(y))^\beta \le C\Lambda^{-\beta t} d(x,y)^\beta. \tag{2.13}$$

This, together with the fact that $\overline{\psi}_i = \psi \circ \Phi_t(y)$ for some $y \in W_i$ yields,

$$|\psi \circ \Phi_t - \overline{\psi}_i|_{C^\beta(W_i)} \le C\Lambda^{-\beta t} |\psi|_{C^\beta(W)} \le C\Lambda^{-\beta t} |W|^{-1/q}. \tag{2.14}$$

By a similar estimate with α in place of β, and using $|\psi \circ \Phi_t|_{C^0(W_i)} \le |\psi|_{C^0(W)}$, yields $|\psi \circ \Phi_t|_{C^\alpha(W_i)} \le C|\psi|_{C^\alpha(W)} \le C|W|^{-1/q}$.

In order to complete the estimate on the strong stable norm, we need the following lemma.

Lemma 2.7. *Let $W \in \mathcal{W}^s$, $t \ge 0$, and suppose $\Phi_{-t}W = \{W_i\}_i \subset \mathcal{W}^s$.*

(a) *There exists $C_d > 0$, independent of W and t, such that for all W_i and $x, y \in W_i$,*

$$\left| \frac{J_{W_i}\Phi_t(x)}{J_{W_i}\Phi_t(y)} - 1 \right| \le C_d d(x,y).$$

(b) $|J_{W_i}\Phi_t|_{C^1(W_i)} \le (1 + C_d)|J_{W_i}\Phi_t|_{C^0(W_i)}$.

(c) *There exists $\bar{C}$, independent of W and $t \ge 0$, such that $\sum_i |J_{W_i}\Phi_t|_{C^0(W_i)} \le \bar{C}$.*

Proof. Item (a) is a standard distortion bound in hyperbolic dynamics. It can be proved, for example, by choosing $\tau_1 > 0$ and subdividing $[0, t]$ into $[t/\tau_1]$ intervals of length τ_1, plus a last one of length

$s \le \tau_1$. Then using again (1.1)

$$\begin{aligned}\log \frac{J_{W_i}\Phi_t(x)}{J_{W_i}\Phi_t(y)} &\le \sum_{j=1}^{[t/\tau_1]} |\log J_{\Phi_{j\tau_1}W_i}\Phi_{\tau_1}(\Phi_{j\tau_1}(x)) - \log J_{\Phi_{j\tau_1}W_i}\Phi_{\tau_1}(\Phi_{j\tau_1}(y))| \\ &\quad + |\log J_{\Phi_{t-s}W_i}\Phi_s(\Phi_{t-s}(x)) - \log J_{\Phi_{t-s}W_i}\Phi_s(\Phi_{t-s}(y))| \\ &\le \sum_{j=1}^{[t/\tau_1]} Cd(\Phi_{j\tau_1}(x), \Phi_{j\tau_1}(y)) + Cd(\Phi_{t-s}(x), \Phi_{t-s}(y)) \\ &\le C' \sum_{j=1}^{[t/\tau_1]} \Lambda^{-j\tau_1} d(x,y) + \Lambda^{-(t-s)} d(x,y) \le C'' d(x,y),\end{aligned}$$

where C'' depends on the maximum C^2 norm of Φ_s, $0 \le s \le \tau_1$.

Item (b) is an immediate consequence of (a).

Item (c) also follows from (a). To see this, note that if $\Phi_{-t}W$ has length less than δ_0, then there is only a single W_i, and the fact that the Jacobian along stable curves is a contraction implies the inequality. If $\Phi_{-t}W$ has length longer than δ_0, then each W_i has length at least $\delta_0/2$. Thus using bounded distortion from (a) yields,

$$\sum_i |J_{W_i}\Phi_t|_{C^0(W_i)} \approx \sum_i \frac{|\Phi_t(W_i)|}{|W_i|} \le 2\delta_0^{-1} \sum_i |\Phi_t(W_i)| \le 2\delta_0^{-1}|W| \le 2. \tag{2.15}$$

□

The items of the lemma allow us to complete the proof of (2.8). Recalling (2.12), and using (2.14) and Lemma 2.7(b) yields,

$$\int_W \mathcal{L}_t f\, \psi\, dm_W \le \sum_i C\Lambda^{-\beta t}\|f\|_s \frac{|W_i|^{1/q}}{|W|^{1/q}} |J_{W_i}\Phi_t|_{C^0(W_i)} + C|f|_w |W|^{-1/q} |J_{W_i}\Phi_t|_{C^0(W_i)}.$$

The first sum is uniformly bounded in t and W by Lemma 2.7(a),(c) and a Hölder inequality,

$$\begin{aligned}\sum_i \frac{|W_i|^{1/q}}{|W|^{1/q}} |J_{W_i}\Phi_t|_{C^0(W_i)} &\le \left(\sum_i (1+C_d)\frac{|\Phi_t(W_i)|}{|W|}\right)^{1/q} \left(\sum_i |J_{W_i}\Phi_t|_{C_0(W_i)}\right)^{1-1/q} \\ &\le (1+C_d)^{1/q}\bar{C}^{1-1/q}.\end{aligned}$$

The second sum is bounded uniformly in t and W since by an estimate similar to (2.15),

$$\sum_i |W|^{-1/q}|J_{W_i}\Phi_t|_{C^0(W_i)} \le 2\delta_0^{-1}|W|^{1-1/q}.$$

Putting these estimates together yields,

$$\int_W \mathcal{L}_t f\, \psi\, dm_W \le C\Lambda^{-\beta t}\|f\|_s + C|f|_w. \tag{2.16}$$

Case II. $|\Phi_{-t}W| \le \delta_0$.

In this case,[6] we do not subtract an average for the test function, and there is simply one term in (2.11), to which we apply the strong stable norm,

$$\int_W \mathcal{L}_t f \, \psi \, dm_W \le \|f\|_s \frac{|\Phi_{-t}(W)|^{1/q}}{|W|^{1/q}} |J_{\Phi_{-t}(W)} \Phi_t|_{C^0},$$

where again, we have used (2.13) and Lemma 2.7 to estimate the norms of the test functions. By bounded distortion, $|J_{\Phi_{-t}(W)} \Phi_t|_{C^0} \approx \frac{|W|}{|\Phi_{-t}(W)|}$, so that

$$\int_W \mathcal{L}_t f \, \psi \, dm_W \le C\|f\|_s \frac{|W|^{1-1/q}}{|\Phi_{-t}(W)|^{1-1/q}} \le C\|f\|_s \Lambda^{-(1-1/q)t}.$$

Putting Cases I and II together and taking the supremum over W and ψ proves (2.8).

The proof of (2.7) follows more simply since the weak norm needs no contraction so we do not subtract the average value of the test function on each curve. Also, there is no weight of the form $|W|^{-1/q}$ since for the weak norm, the test function $\psi \in C^\alpha(W)$ satisfies $|\psi|_{C^\alpha(W)} \le 1$. Thus following (2.11) and applying the weak norm to each term yields,

$$\int_W \mathcal{L}_t f \, \psi \, dm_W \le \sum_i |f|_w |\psi \circ \Phi_t|_{C^\alpha(W_i)} |J_{W_i} \Phi_t|_{C^\alpha(W_i)} \le \sum_i C|f|_w |J_{W_i}\Phi_t|_{C^0(W_i)} \le C'|f|_w,$$

where again we have used Lemma 2.7.

The proof of the neutral norm bound (2.10) is similarly straightforward. Using the group property of Φ_t, we have,

$$\frac{d}{ds}\big((\mathcal{L}_t f) \circ \Phi_s\big)|_{s=0} = \lim_{s \to 0} \frac{(f \circ \Phi_s - f) \circ \Phi_{-t}}{s} = \frac{d}{ds}(f \circ \Phi_s)|_{s=0} \circ \Phi_{-t}. \tag{2.17}$$

Taking $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \le 1$, we use (2.17) and change variables as in (2.11),

$$\begin{aligned}\int_W \frac{d}{ds}\big((\mathcal{L}_t f) \circ \Phi_s\big)|_{s=0} \, \psi \, dm_W &= \sum_i \int_{W_i} \frac{d}{ds}(f \circ \Phi_s)|_{s=0} \, \psi \circ \Phi_t \, J_{W_i}\Phi_t \, dm_{W_i} \\ &\le \sum_i \|f\|_0 |\psi \circ \Phi_t|_{C^\alpha(W_i)} |J_{W_i}\Phi_t|_{C^\alpha(W_i)},\end{aligned}$$

and the sum is uniformly bounded in t and W, again using Lemma 2.7.

The proof of (2.9) is more lengthy, and to avoid cumbersome technicalities, we shall omit the proof in these notes. We refer the interested reader to [DZ1] for the map version or [BDL] for the flow version. □

3. The Generator and The Resolvent: Regaining Quasi-Compactness

The novel idea introduced by Liverani in [L] was to shift attention away from the semi-group of transfer operators, and onto generator of the semi-group, and the associated resolvent. Indeed, the path we shall follow to prove Theorem 1.1 will be to prove a spectral gap for the generator.

For $f \in C^1(\Omega)$, define

$$Xf = \lim_{t \to 0^+} \frac{\mathcal{L}_t f - f}{t}.$$

The operator X is called the *generator* of the semi-group $\{\mathcal{L}_t\}_{t \ge 0}$. Since Φ_t is invertible, in fact $\{\mathcal{L}_t\}_{t \in \mathbb{R}}$ is a group when acting pointwise on functions; however, since we are interested in its action

[6]This case can be eliminated entirely by requiring that curves in $\mathcal{W}^s$ have a minimum length of say, $\delta_0/2$. Then Case I would suffice to estimate all curves, and (2.8) would simplify to $\|\mathcal{L}_t f\|_s \le C\Lambda^{-\beta t}\|f\|_s + C|f|_w$. Since we are interested in presenting norms which can be applied to discontinuous maps and flows, we do not place this additional restriction on curves in $\mathcal{W}^s$.

on the Banach space $\mathcal{B}$, we consider only the semi-group. This is because the dynamical properties of $\mathcal{L}_t$ for $t < 0$ will not preserve the norms: the roles of the stable and unstable directions are exchanged, and so the definition of the anisotropic spaces would also need to be changed in order to study $t < 0$.

Remark that if $f \in C^2(M)$, then $Xf \in C^1(M)$, so $Xf \in \mathcal{B}$ by Lemma 2.4. By definition, this implies that the domain of X is dense in $\mathcal{B}$.

The following lemma provides additional information about the behavior of $\mathcal{L}_t$ for small t.

Lemma 3.1. *There exists $C > 0$ such that for all $f \in \mathcal{B}$,*

(a) $\lim_{t \to 0^+} \|\mathcal{L}_t f - f\|_{\mathcal{B}} = 0$;

(b) $|\mathcal{L}_t f - f|_w \le Ct\|f\|_{\mathcal{B}}$, $t \ge 0$.

Proof. For the proof of (a), see [BDL, Lemma 4.6]. We prove (b).

Let $f \in C^2(\Omega)$, $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \le 1$. Then using (2.17), we estimate

$$\begin{aligned}
\int_W (\mathcal{L}_t f - f)\, \psi \, dm_W &= \int_W \int_0^t \frac{d}{ds}(f \circ \Phi_{-s})\, \psi \, ds\, dm_W \\
&= \int_0^t \int_W \frac{d}{dr}(f \circ \Phi_r)|_{r=0} \circ \Phi_{-s}\, \psi \, dm_W \, ds \\
&= \int_0^t \sum_i \int_{W_i} \frac{d}{dr}(f \circ \Phi_r)|_{r=0}\, \psi \circ \Phi_s \, J_{W_i}\Phi_s \, dm_{W_i}\, ds \\
&\le \int_0^t \|f\|_0 \sum_i |\psi \circ \Phi_s|_{C^\alpha(W_i)} |J_{W_i}\Phi_s|_{C^\alpha(W_i)} \;\le\; Ct\|f\|_0,
\end{aligned}$$

where we have changed variables in the third line, and used Lemma 2.7 in the fourth. Taking the supremum over ψ and W proves (b). □

Remark 3.2. *Item (a) of Lemma 3.1 implies that the semi-group $\{\mathcal{L}_t\}_{t \ge 0}$ acting on $\mathcal{B}$ is strongly continuous. This in turn implies that X is closed as an operator on $\mathcal{B}$, with a dense domain* [Da].

Next, for $z \in \mathbb{C}$, we define the *resolvent* $R(z) : \mathcal{B} \to \mathcal{B}$ by

$$R(z) = (zI - X)^{-1}. \tag{3.1}$$

When $\mathrm{Re}(z) > 0$, $R(z)$ has the following representation,

$$R(z)f = \int_0^\infty e^{-zt} \mathcal{L}_t f \, dt. \tag{3.2}$$

The importance of (3.2) is that the operator $R(z)$ integrates out time, and so eliminates the neutral direction. This will be the key point that enables the subsequent analysis.

Exercise 2. *Use the definition of X to verify that $R(z)$ defined by* (3.2) *satisfies $R(z)Xf = -f + zR(z)f$. This implies that $R(z)$ satisfies* (3.1).

3.1. **Quasi-compactness of $R(z)$.** Define $\lambda = \max\{\Lambda^{-\beta}, \Lambda^{-\gamma}, \Lambda^{-(1-1/q)}\} < 1$.

Proposition 3.3. *There exists $C \ge 1$ such that for all $z \in \mathbb{C}$ with $Re(z) =: a > 0$, and all $f \in \mathcal{B}$ and $n \ge 0$,*

$$|R(z)^n f|_w \le Ca^{-n}|f|_w\,, \tag{3.3}$$

$$\|R(z)^n f\|_s \le C(a - \log\lambda)^{-n}\|f\|_s + Ca^{-n}|f|_w\,, \tag{3.4}$$

$$\|R(z)^n f\|_u \le C(a - \log\lambda)^{-n}\|f\|_u + Ca^{-n}(\|f\|_s + \|f\|_0)\,, \tag{3.5}$$

$$\|R(z)^n f\|_0 \le Ca^{1-n}(1 + |z|/a)|f|_w\,. \tag{3.6}$$

Due to the integration over time provided by (3.2), Proposition 3.3 represents an essential improvement over Proposition 2.6. The key improvement is the weak norm $|f|_w$ appearing on the right hand side of (3.6) in place of the neutral norm $\|f\|_0$ which appeared on the right hand side of (2.10). This permits the following corollary.

Corollary 3.4. *Let $z = a + ib \in \mathbb{C}$ with $a > 0$. The spectral radius of $R(z)$ on $\mathcal{B}$ is at most a^{-1}. For any $\sigma > (1 - a^{-1} \log \lambda)^{-1}$, we may choose $c_u > 0$ such that the essential spectral radius is at most σa^{-1}. Thus the spectrum of $R(z)$ outside the disk of radius σa^{-1} is finite-dimensional, and if it is nonempty, then $R(z)$ is quasi-compact as an operator on $\mathcal{B}$.*

Proof. Using the definition of the strong norm, we estimate,

$$\begin{aligned} a^n \|R(z)^n f\|_{\mathcal{B}} &= a^n \|R(z)^n f\|_s + c_u a^n \|R(z)^n f\|_u + a^n \|R(z) f\|_0 \\ &\le C[(1 - a^{-1} \log \lambda)^{-n} + c_u] \|f\|_s + C c_u (1 - a^{-1} \log \lambda)^{-n} \|f\|_u + C c_u \|f\|_0 \\ &\quad + C(1 + a + |z|)|f|_w. \end{aligned}$$

Now choose $\sigma \in ((1 - a^{-1} \log \lambda)^{-1}, 1)$ and $N > 0$ so large that $C(1 - a^{-1} \log \lambda)^{-N} < \sigma^N/2$. Finally, choose $c_u > 0$ so small that $C c_u < \sigma^N/2$. Then the above estimate yields,

$$a^N \|R(z)^N f\|_{\mathcal{B}} \le \sigma^N \|f\|_{\mathcal{B}} + C(a + |z| + 1)|f|_w, \tag{3.7}$$

which is the traditional Lasota-Yorke inequality. Since this can be iterated, it follows from a classical result of Hennion [H], together with the compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ (Lemma 2.5), that the essential spectral radius of $R(z)$ on $\mathcal{B}$ is at most σa^{-1}. □

The following two facts will be useful for proving Proposition 3.3.

Exercise 3. *Starting from (3.2), prove by induction that $R(z)^n f = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \mathcal{L}_t f \, dt$.*

Exercise 4. *Let $z = a + ib$ with $a > 0$. Show that $\left| \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \, dt \right| \le a^{-n}$, for all $n \ge 1$.*

Proof of Proposition 3.3. As usual, by density, it suffices to prove the inequalities for $f \in C^2(\Omega)$. We begin by proving the weak norm estimate (3.3).

Let $W \in \mathcal{W}^s s$, $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \le 1$. Then for $n \ge 1$,

$$\begin{aligned} \left| \int_W R(z)^n f \, \psi \, dm_W \right| &= \left| \int_0^\infty \int_W \mathcal{L}_t f \, \psi \, dm_W \, \frac{t^{n-1}}{(n-1)!} e^{-zt} \, dt \right| \\ &\le \int_0^\infty |\mathcal{L}_t f|_w \, \frac{t^{n-1}}{(n-1)!} e^{-at} \, dt \le C |f|_w a^{-n}, \end{aligned} \tag{3.8}$$

where in the first line we have used Exercise 3 and reversed the order of integration since the integral of $\mathcal{L}_t f$ on W is uniformly bounded in t; in the second line we have used (2.7) and Exercise 4 to complete the estimate. Taking the appropriate suprema over W and ψ proves (3.3).

The proof of (3.4) is similar, except that we take advantage of the extra contraction provided by (2.8). Taking $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \le |W|^{-1/q}$, we estimate for $n \ge 1$, following (3.8),

$$\begin{aligned} \left| \int_W R(z)^n f \, \psi \, dm_W \right| &\le \int_0^\infty \|\mathcal{L}_t f\|_s \, \frac{t^{n-1}}{(n-1)!} e^{-at} \, dt \\ &\le \int_0^\infty \left[C \|f\|_s \frac{t^{n-1}}{(n-1)!} e^{-(a - \log \lambda)t} + C |f|_w \frac{t^{n-1}}{(n-1)!} e^{-at} \right] dt \\ &\le C(a - \log \lambda)^{-n} \|f\|_s + C a^{-n} |f|_w, \end{aligned}$$

where again we have used Exercises 3 and 4, as well as (2.8).

The estimate for (3.5) is again similar, now using (2.9).

Finally, we prove (3.6). This differs from the previous estimates since we will not simply apply (2.10), which would result in no improvement over Proposition 2.6, but rather we first integrate by parts in order to use the weak norm. Now taking $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \le 1$, we estimate,

$$\begin{aligned}
\int_W \frac{d}{ds}((R(z)^n f)\circ\Phi_s)|_{s=0}\,\psi\,dm_W &= \int_W \int_0^\infty \frac{t^{n-1}}{(n-1)!}e^{-zt}\,\frac{d}{ds}\big((\mathcal{L}_t f)\circ\Phi_s\big)|_{s=0}\,dt\,\psi\,dm_W \\
&= \int_W \int_0^\infty \frac{t^{n-1}}{(n-1)!}e^{-zt}\,\frac{d}{dt}(\mathcal{L}_t f)\,dt\,\psi\,dm_W \\
&= \int_W \int_0^\infty -\left(\frac{t^{n-2}}{(n-2)!}e^{-zt} - ze^{-zt}\frac{t^{n-1}}{(n-1)!}\right)\mathcal{L}_t f\,dt\,\psi\,dm_W \\
&= -\int_0^\infty \left(\frac{t^{n-2}}{(n-2)!} - \frac{zt^{n-1}}{(n-1)!}\right)e^{-zt}\int_W \mathcal{L}_t f\,\psi\,dm_W\,dt.
\end{aligned}$$

Now we use the triangle inequality, apply the weak norm estimate (2.7) to the integral over W, and Exercise 4 to both terms integrated over t to obtain,

$$\|R(z)^n f\|_0 \le C|f|_w\big(a^{1-n} + |z|a^{-n}\big),$$

which proves (3.6). □

3.2. **Initial results on the spectrum of X.** Proposition 3.3 and Corollary 3.4 provide useful information about the spectrum of X, which we denote by $\mathrm{sp}(X)$. First notice that since $\|\mathcal{L}_t\|_{\mathcal{B}}$ is uniformly bounded in t by Proposition 2.6, the spectrum of X on $\mathcal{B}$ is entirely contained in the left half-plane, $Re(z) \le 0$. Moreover, the invariant measure m, identified with the constant function 1 according to our convention, is an eigenvector with eigenvalue 0 for X.

Proposition 3.5. *The spectrum of X on $\mathcal{B}$ is contained in $Re(z) \le 0$. The intersection $sp(X) \cap \{z \in \mathbb{C} : -\log\lambda < Re(z) \le 0\}$ consists of at most countably many isolated eigenvalues of finite multiplicity. The spectrum of X on the imaginary axis contains only an eigenvalue at 0 of multiplicity 1.*

We will not present a formal proof of Proposition 3.5, which by now is standard. We refer the interested reader to [BL, Lemma 3.6, Corollary 3.7] or [BDL, Corollary 5.4]. However, we discuss the main ideas, which are essential for what comes next.

The proof of the proposition relies on the observation that for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, we have,

$$\bar\rho \in \mathrm{sp}(R(z)) \quad \text{if and only if} \quad \bar\rho = (z-\rho)^{-1}, \quad \text{where } \rho \in \mathrm{sp}(X). \tag{3.9}$$

Here $R(z)$ and X are understood as operators on $\mathcal{B}$. The proof of this is classical, see for example [Da, Lemma 8.1.9]. Furthermore, the following fact holds.

Exercise 5. *Suppose $\rho \in sp(X)$ and $\bar\rho = (z-\rho)^{-1} \in sp(R(z))$. Show that for any $k \ge 1$ and $f \in \mathcal{B}$, we have $(R(z)-\bar\rho)^k f = 0$ if and only if $(X-\rho)^k f = 0$. This implies that $\bar\rho$ is an eigenvalue of $R(z)$ of multiplicity k if and only if ρ is an eigenvalue of X of multiplicity k.*

Figure 1 summarizes this relationship. By fixing $a > 0$ and considering the family of parameters $\{z = a + ib : b \in \mathbb{R}\}$, we see that the essential spectrum of X is contained in the half plane $\{Re(w) \le \log\lambda\}$, and so is bounded away from the imaginary axis.

Since the spectrum of $R(z)$ in the annulus $\{(a-\log\lambda)^{-1} < |w| \le a^{-1}\}$ contains only finitely many eigenvalues of finite multiplicity by Corollary 3.4, it follows that for each $b_0 > 0$ the intersection of $\mathrm{sp}(X)$ with the rectangle $\{\mathrm{Re}(w) \in (\log\lambda, 0], |\mathrm{Im}(w)| \le b_0\}$ contains only finitely many eigenvalues of finite multiplicity. Once this identification is made, the fact that the imaginary axis contains only the simple eigenvalue at 0 follows from the fact that contact Anosov flows are mixing, see [BK, Theorem 3.6] together with the classical Hopf argument [LW].

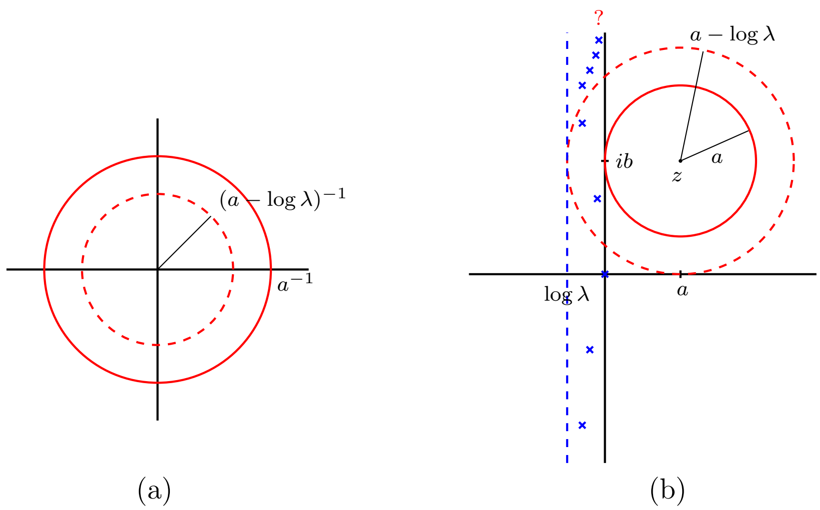

FIGURE 1. (a) The spectrum of $R(z)$ is contained in a disk of radius a^{-1} (solid red circle), and its essential spectrum is contained in a disk of radius $(a - \log\lambda)^{-1}$ (dashed red circle).
(b) The red circles are the images of the corresponding circles in (a) under the transformation $w \mapsto z - w^{-1}$. Due to (3.9), the spectrum of X lies outside the solid red circle, and its essential spectrum must lie outside the dashed red circle. This forces the strip between the dashed blue line ($Im(w) = \log\lambda$) and the imaginary axis to contain only isolated eigenvalues of finite multiplicity. The blue x's are possible eigenvalues of X, which may accumulate on the imaginary axis as $|\mathrm{Im}(w)| \to \infty$.

4. A SPECTRAL GAP FOR X

Unfortunately, Proposition 3.5 is not sufficient to prove the desired result on decay of correlations that is the goal of these notes. The problem is that although the spectrum of X in each rectangle $\{w \in \mathbb{C} : \mathrm{Re}(w) \in (\log\lambda, 0], |\mathrm{Im}(w)| \le b_0\}$ is finite dimensional, and so the minimum distance from an eigenvalue $\rho \neq 0$ in this rectangle to the imaginary axis is positive, it may happen that a sequence of eigenvalues $\rho = u + iv$ approaches the imaginary axis as $|v| \to \infty$.

In order to conclude exponential mixing, we will show that in fact, X has a spectral gap.

Theorem 4.1. *There exists $\nu > 0$ such that $sp(X) \cap \{w \in \mathbb{C} : -\nu < Re(w) \le 0\} = 0$.*

Theorem 4.1 in turn will follow from the following proposition.

Proposition 4.2. *There exist $\bar{\nu} > 0$, $\bar{C} > 0$ and $b_0 > 0$ such that for all $z = a + ib$ with $1 \le a \le 2$ and $|b| \ge b_0$, $\|R(z)^n\|_{\mathcal{B}} \le (a + \bar{\nu})^{-n}$ for all $\bar{C}\log|b| \le n \le 2\bar{C}\log|b|$. Thus the spectral radius of $R(z)$ on $\mathcal{B}$ is at most $(a + \bar{\nu})^{-1}$ for all $1 \le a \le 2$, $|b| \ge b_0$.*

Proof of Theorem 4.1, assuming Proposition 4.2. Due to Proposition 4.2 and (3.9), the set $\{\mathrm{Re}(w) \in (\bar{\nu}, 0], |\mathrm{Im}(w)| \ge b_0\}$ is disjoint from $\mathrm{sp}(X)$. On the other hand, the set $\{\mathrm{Re}(w) \in (\bar{\nu}, 0], |\mathrm{Im}(w)| \le b_0\}$ contains only finitely many eigenvalues by Proposition 3.5, and 0 is the only eigenvalue on the imaginary axis. The finiteness of this set guarantees a positive minimum distance between the imaginary axis and the closest nonzero eigenvalue. □

4.1. **Reduction of Proposition 4.2 to a Dolgopyat estimate.**

Turning our attention to Proposition 4.2, we note that the strength of the claim can be reduced by a couple of straightforward reductions.

The first point to notice is that the constant $|z|$ appearing in (3.6) and (3.7) ruins the uniformity of our estimates when $|b|$ is large. To compensate for this, we introduce the following modified norm, which depends[7] on $|z|$,

$$\|f\|_{\mathcal{B}}^* = \|f\|_s + \frac{c_u}{|z|}\|f\|_u + \frac{1}{|z|}\|f\|_0. \tag{4.1}$$

It suffices to prove Proposition 4.2 for the norm $\|\cdot\|_{\mathcal{B}}^*$, as long as $\bar{C}$ and $\bar{\nu}$ remain independent of $|z|$. For this would imply that the spectral radius of $R(z)$ acting on the space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}}^*)$ is at most $(a+\bar{\nu})^{-1}$. And since

$$\|\cdot\|_{\mathcal{B}}^* \le \|\cdot\|_{\mathcal{B}} \le |z|\|\cdot\|_{\mathcal{B}}^*,$$

the two norms are equivalent for each $|z|$, and so the spectral radius of $R(z)$ on $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is at most $(a+\bar{\nu})^{-1}$ as well.

Exercise 6. *Show that the same choice of N and c_u as in (3.7) yield the inequality,*

$$\|R(z)^n f\|_{\mathcal{B}}^* \le \sigma^n a^{-n}\|f\|_{\mathcal{B}}^* + Ca^{-n}|f|_w, \quad \forall\, f \in \mathcal{B},$$

for all $n \ge N$ and some $\sigma < 1$ and $C > 0$ independent[8] of z.

Next, using Exercise 6 we have the inequality,

$$\|R(z)^{2n} f\|_{\mathcal{B}}^* \le \sigma^n a^{-n}\|R(z)^n f\|_{\mathcal{B}}^* + Ca^{-n}|R(z)^n f|_w, \quad \forall\, f \in \mathcal{B}.$$

For the first term on the right hand side, we estimate $\|R(z)^n f\|_{\mathcal{B}}^* \le (1+C)a^{-n}\|f\|_{\mathcal{B}}^*$, again using Exercise 6 and the bound $|\cdot|_w \le \|\cdot\|_s \le \|\cdot\|_{\mathcal{B}}^*$. Interpolating between σa^{-1} and a^{-1}, and possibly increasing N to overcome the effect of $(1+C)$, this implies the existence of $\nu > 0$ such that the first term contracts at a rate $(a+\nu)^{-2n}\|f\|_{\mathcal{B}}^*$. Thus to prove Proposition 4.2, it suffices to show that the weak norm decays exponentially at a rate faster than a^{-n}, i.e.

$$|R(z)^n f|_w \le (a+\nu)^{-n}\|f\|_{\mathcal{B}}^*, \tag{4.2}$$

for some $\nu > 0$, and z and n as in the statement of the proposition. In fact, we will prove the following key lemma.

Lemma 4.3 (Dolgopyat inequality)**.** *There exists $C_\# > 0$ and for all $0 < \alpha \le 1$, there exists $C_D, \gamma_0, b_0 > 0$ such that for all $f \in C^1(\Omega)$,*

$$|R(z)^{2n} f|_w \le \frac{C_\#}{a^{2n}|b|^{\gamma_0}}\big(|f|_\infty + (1 + a^{-1}\log\Lambda)^{-n}|\nabla f|_\infty\big), \tag{4.3}$$

for all $1 < a < 2$, $|b| \ge b_0$ and $n \ge C_D \ln b$.

Here, $|\cdot|_\infty$ denotes the L^∞ norm of a function.

Equation (4.3) is the Dolgopyat-type estimate that will prove the existence of a spectral gap for X. Given (4.2), one might expect $\|f\|_{\mathcal{B}}^*$ on the right hand side of (4.3) rather than the C^1 norm of f. In fact, the C^1 norm of f can be replaced by the strong norm of f due to the following mollification lemma.

Let $\eta : \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative C^∞ function supported on the unit ball in $\mathbb{R}^3$, with $\int \eta\, dm = 1$ and a unique global maximum at the origin. For $\varepsilon > 0$, define $\eta_\varepsilon(x) = \varepsilon^{-3}\eta(x/\varepsilon)$.

For $f \in C^0(\Omega)$ and $\varepsilon > 0$, define the following mollification operator,

$$M_\varepsilon(f)(y) = \int_M \tilde{\eta}_\varepsilon(y-x) f(x)\, dm(x), \tag{4.4}$$

where $\tilde{\eta}_\varepsilon$ is the function η_ε in a local chart containing y.

---

[7] Note that $|z| > 1$ since $a \ge 1$.

[8] Use the fact that $1 \le a \le 2$ to obtain a choice of σ independent of a. Also, note that $\frac{1+a+|z|}{|z|} \le 3$.

Lemma 4.4. *There exists $C > 0$, such that for all $f \in C^0(\Omega)$ and $\varepsilon > 0$,*

$$|M_\varepsilon(f) - f|_w \leq C\varepsilon^\gamma \|f\|_{\mathcal{B}}\,; \tag{4.5}$$

$$|M_\varepsilon(f)|_\infty \leq C\varepsilon^{-1-\beta+1/q}\|f\|_s\,; \tag{4.6}$$

$$|\nabla(M_\varepsilon(f))|_\infty \leq C\varepsilon^{-2-\beta+1/q}\|f\|_s\,. \tag{4.7}$$

The estimates on the mollification operator are fairly standard, and follow the same lines as the proof of Lemma 2.4: the integral in an ε-neighborhood of a point $x \in \Omega$ is disintegrated using a foliation curves in $\mathcal{W}^s$, and the strong stable norm is applied to the integral on each stable curve. The interested reader is referred to [BDL, Lemmas 7.3 and 7.4], or [BL, Lemmas 5.3 and 5.4].

Proof of Proposition 4.2 via Equation 4.2. Fix z as in the statement of Proposition 4.2 and without loss of generality, assume $b \geq 1$. If necessary, increase b_0 from Lemma 4.3 so that $C_D \log b_0 \geq N$. Then for $n \geq C_D \log b$ and $\varepsilon > 0$ to be chosen later, we have,

$$\begin{aligned}
|R(z)^{2n} f|_w &\leq |R(z)^{2n}(f - M_\varepsilon(f))|_w + |R(z)^{2n} M_\varepsilon(f)|_w \\
&\leq Ca^{-2n}\big(|f - M_\varepsilon(f)|_w + b^{-\gamma_0}|M_\varepsilon(f)|_\infty + (1 + a^{-1}\log\Lambda)^{-n}|\nabla(M_\varepsilon(f))|_\infty\big) \\
&\leq Ca^{-2n}\big(\varepsilon^\gamma\|f\|_{\mathcal{B}} + b^{-\gamma_0}\varepsilon^{-1-\beta+1/q}\|f\|_s + b^{-\gamma_0}\varepsilon^{-2-\beta+1/q}(1 + a^{-1}\log\Lambda)^{-n}\|f\|_s\big) \\
&\leq Ca^{-2n}\|f\|^*_{\mathcal{B}}\big(\varepsilon^\gamma b + b^{-\gamma_0}\varepsilon^{-1-\beta+1/q} + b^{-\gamma_0}\varepsilon^{-2-\beta+1/q}(1 + a^{-1}\log\Lambda)^{-n}\big)\,,
\end{aligned}$$

where in the second line we have used (3.3) for the first term and Lemma 4.3 for the second, while in the third line we have used Lemma 4.4, and in the fourth line $\|f\|_{\mathcal{B}} \leq |z|\|f\|^*_{\mathcal{B}}$.

Choose $\rho > 1/\gamma$ and set $\varepsilon = b^{-\rho}$. Next, choose β sufficiently small, and $q > 1$ sufficiently close[9] to 1, so that $\rho(1 + \beta - 1/q) < \gamma_0$. Then,

$$|R(z)^{2n} f|_w \leq Ca^{-2n}\|f\|^*_{\mathcal{B}}\big(b^{-\gamma_1} + b^{-\gamma_2} + b^{-\gamma_2}b^\rho(1 + a^{-1}\log\Lambda)^{-n}\big)\,,$$

where $\gamma_1 = \rho\gamma - 1 > 0$ and $\gamma_2 = \gamma_0 - \rho(1 + \beta - 1/q) > 0$. Finally, choosing $n \geq \frac{\rho \log b}{\log(1+a^{-1}\log\Lambda)}$ implies $b^\rho(1 + a^{-1}\log\Lambda)^{-n} \leq 1$. Putting these estimates together yields,

$$|R(z)^{2n} f|_w \leq Ca^{-2n}\|f\|_{\mathcal{B}^*} b^{-\bar\gamma}\,,$$

for $\bar\gamma = \min\{\gamma_1, \gamma_2\}$, and $n \geq \bar C \log b := \max\{\frac{\rho}{\log(1+a^{-1}\log\Lambda)}, C_D\}\log b$. Next, choosing b_0 sufficiently large so that $Cb_0^{-\bar\gamma/2} \leq 1$ eliminates the constant C from the estimate on $|R(z)^{2n} f|_w$. Finally if also $n \leq 2\bar C \log b$, then $b^{-\bar\gamma/2} \leq e^{-n\bar\gamma/(4\bar C)}$, and (4.2) is proved. □

4.2. **Corollary of the Spectral Gap for X: Proof of Theorem 1.1.** Using Proposition 3.5 and Theorem 4.1, we apply the results of [Bu] to obtain the following decomposition for $\mathcal{L}_t$. Let ν be as in Theorem 4.1 and $\bar\nu$ be as in Proposition 4.2.

There exists a finite set of eigenvalues $\{z_j\}_{j=0}^N = \mathrm{sp}(X) \cap \{w \in \mathbb{C} : \mathrm{Re}(w) \in (-\bar\nu, 0]\}$, with $z_0 = 0$ and $Re(z_j) \leq -\nu$ for $1 \leq j \leq N$, a finite rank projector Π, a bounded linear operator P_t on $\mathcal{B}$ satisfying $P_t\Pi = \Pi P_t = 0$, and a matrix $\hat X : \Pi(\mathcal{B}) \circlearrowleft$ having $\{z_j\}_{j=1}^N$ as eigenvalues such that

$$\mathcal{L}_t = e^{t\hat X}\Pi + P_t, \quad t \geq 0.$$

Moreover, for each $\nu_1 < \bar\nu$, there exists $C_{\nu_1} > 0$ such that for all $f \in \mathrm{Dom}(X)$,

$$|P_t f|_w \leq C_{\nu_1} e^{-\nu_1 t}\|Xf\|_{\mathcal{B}}, \quad \text{for all } t \geq 0.$$

Note that according to the above equation, the weak norm of P_t decays on $\mathrm{Dom}(X)$, but not on all of $\mathcal{B}$. Indeed, if $\|P_t f\|_{\mathcal{B}}$ decayed at a uniform exponential rate for all $f \in \mathcal{B}$, this would imply a

[9]Note that this choice of q does not effect the requirement $\gamma \leq 1/q$ from the definition of the norms, since we may safely take $\gamma \leq 1/2$, and so make it independent of $1/q$ when q is close to 1.

spectral gap for $\mathcal{L}_t$, $t > 0$. The above inequality is significantly weaker, yet sufficient to conclude exponential decay of correlations.

For $f \in \mathcal{B}$, let $\Pi_j f = c_j(f) g_j$ denote the projection onto the eigenvector g_j corresponding to z_j. Note that by conformality of the measure m, for $f \in C^2(\Omega)$, we have $c_0(f) = \int_\Omega f\, dm$.

Now let $\varphi \in C^2(\Omega)$, $\psi \in C^\alpha(\Omega)$. Then

$$\begin{aligned}\int_\Omega \varphi \cdot \psi \circ \Psi_t \, dm &= \int_\Omega \mathcal{L}_t \varphi \cdot \psi \, dm = \int_\Omega P_t \varphi \cdot \psi \, dm + \int_\Omega e^{t\hat{X}} (\Pi f) \cdot \psi \, dm \\ &= \int_\Omega P_t \varphi \cdot \psi \, dm + \int_\Omega \Big(c_0(\varphi) + \sum_{j=1}^N e^{tz_j} c_j(\varphi) g_j \Big) \psi \, dm.\end{aligned}$$

Thus recalling (2.5),

$$\begin{aligned}\left| \int_\Omega \varphi \cdot \psi \circ \Phi_t \, dm - \int_\Omega \varphi \, dm \int_\Omega \psi \, dm \right| &\le C |P_t \varphi|_w |\psi|_{C^\alpha(\Omega)} + \sum_{j=1}^N \bar{c}_j \|\varphi\|_{\mathcal{B}} |\psi|_{C^\alpha(\Omega)} e^{-\nu t} \\ &\le C \Big(e^{-\nu_1 t} \|X\varphi\|_{\mathcal{B}} + e^{-\tau_- t} \|\varphi\|_{\mathcal{B}} \Big) |\psi|_{C^\alpha(\Omega)} \\ &\le C e^{-\nu t} |\varphi|_{C^2(\Omega)} |\psi|_{C^\alpha(\Omega)} \,,\end{aligned}$$

where we have used the fact that $c_j(\varphi) \le \bar{c}_j \|\varphi\|_{\mathcal{B}}$ for some $\bar{c}_j$ independent of φ, and recalling (2.4), that $\|X\varphi\|_{\mathcal{B}} \le C|Xf|_{C^1(\Omega)} \le C|f|_{C^2(\Omega)}$.

To complete the proof of Theorem 1.1, it remains only to approximate $\bar{\varphi} \in C^\alpha(\Omega)$ by $\varphi \in C^2(\Omega)$. This is by now a standard approximation, which we recall here for the convenience of the reader.

Let $\bar{\varphi}, \psi \in C^\alpha(\Omega)$ such that $\int_\Omega \psi \, dm = 0$. Given any $\varepsilon > 0$, define $\varphi \in C^2(\Omega)$ such that $|\bar{\varphi} - \varphi|_{L^1(m)} \le \varepsilon |\bar{\varphi}|_{C^\alpha(\Omega)}$ (for example, by using a mollification as in (4.4)). One has then that $|\varphi|_{C^2(\Omega)} \le C\varepsilon^{\alpha-2} |\bar{\varphi}|_{C^\alpha(\Omega)}$. Now for $t \ge 0$,

$$\begin{aligned}\int \bar{\varphi} \cdot \psi \circ \Phi_t \, dm &= \int (\bar{\varphi} - \varphi) \, \psi \circ \Phi_t \, dm + \int \varphi \cdot \psi \circ \Phi_t \, dm \\ &\le \varepsilon |\bar{\varphi}|_{C^\alpha(\Omega)} |\psi|_{C^0(\Omega)} + C e^{-\nu t} |\varphi|_{C^2(\Omega)} |\psi|_{C^\alpha(\Omega)} \\ &\le \Big(\varepsilon + C e^{-\nu t} \varepsilon^{\alpha - 2} \Big) |\bar{\varphi}|_{C^\alpha(\Omega)} |\psi|_{C^\alpha(\Omega)} \,.\end{aligned}$$

Now choosing $\varepsilon = e^{-\nu t/2}$ completes the proof of Theorem 1.1 with $\eta = \nu\alpha/2$.

5. Dolgopyat Estimate: Proof of Lemma 4.3

Let $f \in C^1(\Omega)$, $W \in \mathcal{W}^s$, and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \le 1$. Let $z = a + ib \in \mathbb{C}$ such that $1 \le a \le 2$ and without loss of generality, take $b \ge 1$. For $n \ge 0$, we must estimate $\int_W R(z)^n f \, \psi \, dm_W$.

Remark 5.1. *Most of the calculations in this section are made simply in order to arrive at the oscillatory integral appearing in* (5.16) *and estimated in Lemma 5.5(c) using the smoothness of the* temporal distance function *established in Lemma 5.5(a) and (b). In order to accomplish this, we will localize in both space and time using partitions of unity in order to exploit the presence of cancellations occurring on small scales according to the oscillation provided by e^{ibt}.*

First, we localize in time. Let $\tau > 0$ be a small time to be chosen later. Let $\tilde{p} : \mathbb{R} \to \mathbb{R}$ be an even function supported on $(-1, 1)$ with a single maximum at 0, satisfying $\sum_{\ell \in \mathbb{Z}} \tilde{p}(t - \ell) = 1$ for any $t \in \mathbb{R}$. Define $p(s) = \tilde{p}(s/\tau)$. Then p and $\tilde{p}$ both define partitions of unity on $\mathbb{R}$. Next, using

Exercise 3,

$$
\begin{aligned}
R(z)^n f &= \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} \mathcal{L}_t f \, dt \\
&= \sum_{\ell \in \mathbb{N}^*} \int_{-\tau}^{\tau} p(s) \frac{(s+\ell\tau)^{n-1}}{(n-1)!} e^{-z(s+\ell\tau)} \mathcal{L}_{\ell\tau}(\mathcal{L}_s f) \, ds + \int_0^\tau p(s) \frac{s^{n-1}}{(n-1)!} e^{-zs} \, \mathcal{L}_s f \, ds,
\end{aligned}
\tag{5.1}
$$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. To abbreviate the notation, we introduce the following notation for the kernels,

$$
p_{n,\ell,z}(s) := p(s) \frac{(s+\ell\tau)^{n-1}}{(n-1)!} e^{-z(s+\ell\tau)}, \quad \text{for } \ell \geq 1, \text{ and } \quad p_{n,0,z}(s) := p(s) \frac{s^{n-1}}{(n-1)!} e^{-zs} \mathbf{1}_{s \geq 0},
$$

where $\mathbf{1}_A$ denotes the indicator of a set A.

Using this notation, we write the integral needed to estimate the weak norm as,

$$
\begin{aligned}
\int_W R(z)^n f \, \psi \, dm_W &= \sum_{\ell \in \mathbb{N}} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_W \psi \, \mathcal{L}_{\ell\tau}(\mathcal{L}_s f) \, dm_W ds \\
&= \sum_{\ell \in \mathbb{N}} \sum_{W_j \in \mathcal{G}_{\ell\tau}(W)} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_j} J_{W_j} \Phi_{\ell\tau} \, \psi \circ \Phi_{\ell\tau} \, \mathcal{L}_s f \, dm_{W_j} ds,
\end{aligned}
\tag{5.2}
$$

where in the first line we have reversed order of integration since the integral in t converges uniformly as x ranges over W, and in the second line we have changed variables for each ℓ, recalling the notation $\mathcal{G}_t(W)$ introduced in the proof of Proposition 2.6.

Next, we introduce partitions of unity in space as well, dividing Ω into 'flow boxes' in which we shall compare integrals on stable curves.

Let $r \in (0, \delta_0)$ and $c > 2$ to be determined below. Set,

$$
\tau = r^{1/3}. \tag{5.3}
$$

At the end of this section, r will be taken sufficiently small with respect to b^{-1}. We choose a finite collection of points x_i so that $\cup_i \mathcal{N}_r(x_i) = M$, where $\mathcal{N}_r(x_i)$ denotes the r-neighborhood of x_i in Ω.

Definition 5.2 (Darboux coordinates). *Using the fact that Ω and ω are smooth, and the splitting of the tangent space is continuous, we may choose cr sufficiently small, so that the following local coordinates exist in a $3cr$ neighborhood of each x_i: $x = (x^s, x^u, x^0)$, where*

- a) $x_i = (0,0,0)$ *is placed at the origin;*
- b) $\{(x^s, 0, 0) : |x^s| \leq 2cr\}$ *is a stable curve;*
- c) *the tangent vector* $(0,1,0)$ *at* x_i *belongs to* $E^u(x_i)$*;*
- d) *in these local coordinates, the contact form* ω *is in standard form,* $\omega = dx^0 - x^s dx^u$.

The last item (d) in the definition above, distinguishes x^0 as the flow direction.

In these local coordinates, define for any $\varepsilon \in (0, cr]$, the *flow box*

$$
B_\varepsilon(x_i) = \{y \in \mathcal{N}_{3cr}(x_i) : \max\{|x_i^s - y^s|, |x_i^u - y^u|, x_i^0 - y^0|\} \leq \varepsilon\}.
$$

Notice that two faces of the box can be obtained by flowing a single stable curve (in our coordinates, this would be the top and bottom faces). We call these the *stable sides* of $B_\varepsilon(x_i)$. Similarly, we define the *unstable sides* and the *flow sides* of each box.

Finally, choose $c > 2$ sufficiently large (depending on the maximum curvature of stable curves in $\mathcal{W}^s$, and maximum width of the stable cone) so that if $W \in \mathcal{W}^s$ intersects $B_r(x_i)$, then $\Phi_s(W)$ does not intersect the stable sides of $B_{cr}(x_i)$ for all $s \in [-cr, cr]$.

Now we return to our required estimate of (5.2). We subdivide each curve $W_j \in \mathcal{G}_{\ell\tau}(W)$ into curves $W_{j,i} = W_j \cap B_r(x_i)$, and define

$$A_{\ell,i} = \{j : W_j \in \mathcal{G}_{\ell\tau}(W) \text{ crosses } B_{cr}(x_i) \text{ completely in the stable direction}\}.$$

If $W_j \in G_{\ell\tau}(W)$ intersects $B_r(x_i)$, but does not cross $B_{cr}(x_i)$ completely, then we place $W_{j,i} := W_j \cap B_{cr}(x_i) \in D_\ell$, the set of discarded pieces, and note that

$$\int_{W_j \cap B_{cr}(x_i)} J_{W_j}\Phi_{\ell\tau}\, \psi \circ \Phi_{\ell\tau}\, \mathcal{L}_s f\, dm_{W_j} \le cr |J_{W_j}\Phi_{\ell\tau}|_{C^0(W_j)} |\psi|_\infty |f|_\infty .$$

Then summing over ℓ, we have that the contribution to the integral from discarded pieces is at most,

$$\sum_{\ell \ge 0} \sum_{j \in D_\ell} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_j \cap B_{cr}(x_i)} J_{W_j}\Phi_{\ell\tau}\, \psi \circ \Phi_{\ell\tau}\, \mathcal{L}_s f\, dm_{W_j} \le Cr|f|_\infty a^{-n}, \tag{5.4}$$

for some $C > 0$.

Exercise 7. *Prove* (5.4)*. Hint: Use the fact that due to the choice of c, there are at most two curves in D_ℓ for each $W_j \in \mathcal{G}_{\ell\tau}(W)$. Then Lemma 2.7(c) and Exercise 4 complete the argument.*

Next, set $\ell_0 = \frac{n}{ae^2\tau}$. We estimate the contribution from the terms with $\ell \le \ell_0$. These are the 'short times' $t \le \frac{n}{ae^2}$ in the integral (5.1).

Exercise 8. *Use Stirling's formula to show that the contribution from terms with $\ell \le \ell_0$ is bounded by*

$$\int_0^{\frac{n}{ae^2}} \frac{t^{n-1}}{(n-1)!} e^{-zt} \int_W \mathcal{L}_t f\, \psi\, dm_W\, dt \le C|f|_\infty a^{-n} e^{-n}, \tag{5.5}$$

for some $C > 0$ independent of n and a.

Now choose n sufficiently large that

$$\max\{e^{-n}, \Lambda^{-\frac{n}{ae^2}}\} \le r\,. \tag{5.6}$$

It remains to estimate terms in the sum (5.2) for large times $\ell \ge \ell_0$ and components $W_{j,i} \subset W_j \in \mathcal{G}_{\ell\tau}(W)$ that completely cross the box $B_{cr}(x_i)$. Define a partition of unity $\{\phi_{r,i}\}_i$ comprised of C^∞ functions $\phi_{r,i}$ centered at each x_i and supported in $B_r(x_i)$. We may choose this partition such that,

$$\|\nabla \phi_{r,i}\|_{L^\infty} \le Cr^{-1} \quad \text{and} \quad \#\{\phi_{r,i}\}_i \le Cr^{-3}, \tag{5.7}$$

for some $C > 0$. Then recalling the definition of $A_{\ell,i}$ together with (5.4) and (5.5), the sum from (5.2) that we must estimate is,

$$\begin{aligned}\int_W R(z)^n f\, \psi\, dm_W = \sum_{\ell \ge \ell_0} \sum_i \sum_{j \in A_{\ell,i}} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_{j,i}} J_{W_j}\Phi_{\ell\tau}\, \psi \circ \Phi_{\ell\tau}\, \phi_{r,i}\, \mathcal{L}_s f\, dm_{W_j} ds,\\ + \mathcal{O}(a^{-n} r |f|_\infty)\,.\end{aligned}$$

We would like to use the oscillation in the kernel $p_{n,\ell,z}$ to create cancellation in the integrals against Lipschitz functions. Unfortunately, our integrands are not Lipschitz, but only Hölder continuous. To correct for this, define

$$\overline{\psi}_{j,i} = |W_{j,i}|^{-1} \int_{W_{j,i}} \psi \circ \Phi_{\ell\tau}\, dm_{W_{j,i}} \quad \text{and} \quad J_{\ell,j,i} = |W_{j,i}|^{-1} \int_{W_{j,i}} J_{W_j}\Phi_{\ell\tau}\, dm_{W_{j,i}}.$$

Due to the regularity of ψ and $J_{W_j}\Phi_{\ell\tau}$, in particular (2.14) and Lemma 2.7(a), we have

$$|\overline{\psi}_{j,i} J_{\ell,j,i} - \psi \circ \Phi_{\ell\tau}\, J_{W_j}\Phi_{\ell\tau}|_{C^0(W_{j,i})} \le Cr^\alpha J_{\ell,j,i}\,,$$

for some $C > 0$. Then summing over ℓ and using the fact that $|\overline{\psi}_{j,i}|_\infty \le 1$, we must estimate,

$$\int_W R(z)^n f\,\psi\,dm_W = \sum_{\ell \ge \ell_0} \sum_i \sum_{j \in A_{\ell,i}} J_{\ell,j,i} \int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_{j,i}} \phi_{r,i}\,\mathcal{L}_s f\,dm_{W_j} ds, \\ + \mathcal{O}(a^{-n} r^\alpha |f|_\infty)\,. \tag{5.8}$$

Now for each $W_{j,i}$, define $W_{j,i}^0 = \{\Phi_s W_{j,i}\}_{s \in (-cr,cr)} \cap B_r(x_i)$ to be the weak stable surface[10] containing $W_{j,i}$. In the local coordinates in $B_r(x_i)$, we view $W_{j,i}^0$ as the graph of the function

$$\mathbb{W}_j^0(x^s, x^0) = \mathbb{W}_j(x^s) + (0, 0, x^0),$$

where

$$\mathbb{W}_j(x^s) = (x^s, E_j(x^s), F_j(x^s)), \quad |x^s|, |x^0| \le r, \tag{5.9}$$

and E_j, F_j are uniformly C^2 functions. Due to the contact form $\omega = dx^0 - x^s dx^u$ in the local coordinates, it follows that $F_j'(x^s) = x^s E_j'(x^s)$.

On each $B_r(x_i)$, we use these functions to change variables in each integral on the domain $S_r = \{(x^s, x^0) : |x^s| \le r, |x^0| \le r\}$. Thus,

$$\int_{-\tau}^{\tau} p_{n,\ell,z}(s) \int_{W_{j,i}} \phi_{r,i}\,\mathcal{L}_s f\,dm_{W_j} ds = \int_{S_r} p_j\,\phi_{r,j}\,f_j\,dx^s\,dx^0, \tag{5.10}$$

where

$$p_j(x^s, x^0) = p_{n,\ell,z}(-x^0), \quad \phi_{r,j}(x^s, x^0) = \phi_{r,i} \circ \mathbb{W}_j^0(x^s, x^0) \cdot \|\mathbb{W}_j'(x^s)\|, \quad f_j(x^s, x^0) = f \circ \mathbb{W}_j^0(x^s, x^0).$$

At this point, given two curves, $W_{j,i}, W_{k,i} \in A_{\ell,i}$, we would like to slide these two curves to the same reference weak stable surface in $B_r(x_i)$. Let us define this surface to be

$$W_i^0 = \{(x^s, 0, x^0) : |x^s|, |x^0| \le r\}\,,$$

which, by choice of coordinates, is precisely the surface obtained by flowing the stable curve through x_i given by $\{(x^s, 0, 0) : |x^s| \le r\}$, according to Definition 5.2(b).

In order to carry out this sliding, we will use a local foliation of real strong unstable manifolds[11] in $B_r(x_i)$.

Definition 5.3 (Unstable foliation)**.** *For each i, define a foliation $\mathbb{F}$ on $B_r(x_i)$, such that for all $x^0 \in [-cr/2, cr/2]$,*

$$\mathbb{F}(x^s, x^u) = \{(G(x^s, x^u), x^u, H(x^s, x^u) + x^0) : |x^s|, |x^u| \le cr/2\},$$

and each curve $x^u \mapsto \gamma_{x^s}^u(x^u) = (G(x^s, x^u), x^u, H(x^s, x^u) + x^0)$ is a local unstable manifold through $(x^s, 0, 0)$. Moreover, for all $x^s \in [-cr/2, cr/2]$,

- (i) *$\partial_{x^u} H = G$, so that $\gamma_{x^s}^u$ lies in the kernel of ω;*
- (ii) *$G(x^s, 0) = x^s$, $H(x^s, 0) = 0$;*
- (iii) *$\Phi_{-s}(\gamma_{x^s}^u) \in \mathcal{W}^u$, for all $s \ge 0$;*
- (iv) *there exists $C > 0$, independent of x^s, such that $C^{-1} \le \|\partial_{x^s} G\|_\infty \le C$, (and so by (i), $\|\partial_{x^s}\partial_{x^u} H\|_{L^\infty} \le C$);*
- (v) *$\|\partial_{x^u}\partial_{x^s} G\|_{C^\eta} \le C$, for some $\eta > 0$ and $C > 0$ independent of x^s;*
- (vi) *$\|\partial_{x^s} H\|_{C^0} \le Cr$, $\|\partial_{x^s} H\|_{C^\eta} \le C$.*

[10]If $W_{j,i}$ is a local strong stable manifold, then $W_{j,i}^0$ is the corresponding local weak stable manifold.

[11]For systems with discontinuities such as billiards, the real unstable manifolds do not create a nice foliation of $B_r(x_i)$, so a smooth local foliation of unstable curves lying in the kernel of the contact form must be constructed. This is quite laborious and outside the scope of these notes. The interested reader should refer to [BDL, Section 6] for the details of the construction.

Remark 5.4. *We list properties (i)-(vi) for the convenience of the reader: it is known that the foliation by local strong unstable manifolds enjoys these properties for Anosov flows (see, for example* [L, Appendix B] *for the Anosov case or* [BL, Appendix D] *for the piecewise Anosov case). Indeed, item (i) is immediate since unstable manifolds lie in the kernel of the contact form; (ii) is simply a normalization that we take, choosing our parametrization to be the identity on the stable manifold of x_i; (iii) holds due to the invariance of unstable manifolds.*

To justify the estimates in (iv)-(vi), we present the following suggestive calculation, which while not a complete proof, does give a flavor for the estimates involved. We consider the 2-dimensional case on one of the sections Σ_i defined in Section 2.1. On such a section, we adopt local coordinates $(\bar{x}^s, \bar{x}^u)$.

For $\xi \in [-r, r]$, let $V_\xi = \{(\bar{x}^s, \bar{x}^u) : \bar{x}^u = \xi)\}$ denote a stable curve in Σ_i. We project the foliation $\mathbb{F}$ onto Σ_i and normalize $\bar{G}(\bar{x}^s, \bar{x}^u)$ so that $\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, 0) = 1$. Define

$$h_{\xi,0} : V_\xi \to V_0$$

to be the holonomy map along the projected unstable foliation. It follows that the Jacobian $Jh_{\xi,0}$ satisfies the following relation,

$$Jh_{\xi,0} = \frac{\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, 0)}{\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, \xi)} = \frac{1}{\partial_{\bar{x}^s}\bar{G}(\bar{x}^s, \xi)},$$

so that $\partial_{\bar{x}^s}\bar{G}$ can be expressed in terms of the Jacobian of the holonomy map, which is known to be Hölder continuous. This is the content of (iv).

Moreover, using the invariance (ii),

$$Jh_{\xi,0}(x) = \prod_{\ell=1}^{\infty} \frac{J_{\Phi_{-\ell}V_0}\Phi_1(\Phi_{-\ell}(x))}{J_{\Phi_{-\ell}V_\xi}\Phi_1(\Phi_{-\ell}(h_{\xi,0}(x)))},$$

and taking $\partial_{\bar{x}^u}$ of this product converges since the unstable direction is the contracting direction for $\Phi_{-\ell}$. This is the main idea behind (v).

Lifting these calculations to the flow yields (iv) and (v) for G. Item (vi) follows from the normalization (ii) together with (iv).

Having defined our foliation, for $j \in A_{\ell,i}$, we consider the associated holonomy map $h_{j,i} : W_{j,i} \to W_i^0$. As a function of x^s, we have,

$$h_{j,i} \circ \mathbb{W}_j(x^s) =: (h_j^s(x^s), 0, h_j^0(x^s)). \tag{5.11}$$

On S_r, define

$$K_{\ell,n,i,j}(x^s, x^0) = \frac{p(x^0)(\ell\tau - x^0)^{n-1}}{|W_{j,i}|(m-1)!} e^{-z\ell\tau} e^{ax^0} \phi_{r,j}(x^s, x^0).$$

Then (5.10) yields,

$$\begin{aligned}
\int_{S_r} p_j\, \phi_{r,j}\, f_j\, dx^s\, dx^0 &= |W_{j,i}| \int_{S_r} K_{\ell,n,i,j}(x^s, x^0) f(\mathbb{W}_j(x^s) + (0,0,x^0)) e^{ibx^0}\, dx^s\, dx^0 \\
&= |W_{j,i}| \int_{S_r} K_{\ell,n,i,j}(x^s, x^0) f(h_j^s(x^s), 0, h_j^0(x^s) + x^0) e^{ibx^0}\, dx^s\, dx^0 \\
&\quad + |W_{j,i}| \mathcal{O}(|\partial^u f|_\infty r^2) \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau},
\end{aligned}$$

where $\partial^u f$ denotes the derivative of f in the unstable direction. Changing variables twice, first $x^0 \mapsto x^0 - h_j^0(x^s)$, and then $x^s \mapsto (h_j^s)^{-1}(x^s)$, results in the following,

$$
\begin{aligned}
\int_{S_r} p_j\, \phi_{r,j}\, f_j\, dx^s\, dx^0 &= |W_{j,i}| \int_{S_r} \frac{K^*_{\ell,n,i,j}(x^s,x^0)}{|(h_j^s)' \circ (h_j^s)^{-1}(x^s)|} f(x^s,0,x^0) e^{ib(x^0-\Delta_j(x^s))}\, dx^s\, dx^0 \\
&\quad + |W_{j,i}|\mathcal{O}(|\partial^u f|_\infty r^2) \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} \\
&= |W_{j,i}| \int_{S_r} K^*_{\ell,n,i,j}(x^s,x^0) f(x^s,0,x^0) e^{ib(x^0-\Delta_j(x^s))}\, dx^s\, dx^0 \\
&\quad + |W_{j,i}|\mathcal{O}(|\partial^u f|_\infty + |f|_\infty) r^2 \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau},
\end{aligned}
\tag{5.12}
$$

where

$$
K^*(x^s,x^0) = K((h_j^s)^{-1}(x^s), x^0 - \Delta_j(x^s)) \quad \text{and} \quad \Delta_j(x^s) = h_j^0 \circ (h_j^s)^{-1}(x^s),
\tag{5.13}
$$

and in the second line we have used the fact that $(h_j^s)' \approx 1 + r$ due to items (ii) and (iv) of Definition 5.3. The function Δ_j is the so-called *temporal distance function* alluded to in Remark 5.1.

Next we use (5.12) to sum over ℓ, i and j in (5.8).

$$
\begin{aligned}
\int_W R(z)^n f\, \psi\, dm_W &= \sum_{\ell,i} \sum_{j \in A_{\ell,i}} J_{\ell,j,i} |W_{j,i}| \int_{S_r} K^*_{\ell,n,i,j}(x^s,x^0) f(x^s,0,x^0) e^{ib(x^0-\Delta_j(x^s))}\, dx^s\, dx^0 \\
&\quad + \sum_{\ell \ge \ell_0} \sum_i \sum_{j \in A_{\ell,i}} J_{\ell,j,i} |W_{j,i}| \mathcal{O}(|\partial^u f|_\infty + |f|_\infty) r^2 \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} \\
&\quad + \mathcal{O}(a^{-n} r^\alpha |f|_\infty).
\end{aligned}
\tag{5.14}
$$

Exercise 9. *Reverse order of summation and use bounded distortion to show that* $\sum_i \sum_{j \in A_{\ell,i}} J_{\ell,j,i} |W_{j,i}| \le C$, *for some* $C > 0$ *independent of* W *and* n.

Exercise 10. *Use* (5.3) *to show that* $\displaystyle\sum_{\ell \ge \ell_0} \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} \le C\tau^{-1} a^{-n} \le C' r^{-1/3} a^{-n}$, *for some constant* $C > 0$ *independent of* ℓ_0 *and* τ.

Summing over ℓ and using Exercises 9 and (10) yields,

$$
\sum_{\ell \ge \ell_0} \sum_i \sum_{j \in A_{\ell,i}} J_{\ell,j,i} |W_{j,i}| \mathcal{O}(|\partial^u f|_\infty + |f|_\infty) r^2 \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} = \mathcal{O}(a^{-n} r^{5/3})(|\partial^u f|_\infty + |f|_\infty).
\tag{5.15}
$$

Next, we estimate the sums over the integrals in (5.14). Setting $Z_{\ell,j,i} = J_{\ell,j,i}|W_{j,i}|$, we have

$$
\begin{aligned}
&\sum_{\ell\geq\ell_0}\sum_i \int_{S_r}\sum_{j\in A_{\ell,i}} Z_{\ell,j,i}K^*_{\ell,n,i,j} f e^{ib(x^0-\Delta_j(x^s))}\,dx^s\,dx^0 \\
&\leq \sum_{\ell\geq\ell_0}\sum_i \left(\int_{S_r}\Big|\sum_{j\in A_{\ell,i}} Z_{\ell,j,i}K^*_{\ell,n,i,j}e^{ib(x^0-\Delta_j(x^s))}\Big|^2\right)^{1/2}\Big(\int_{S_r}|f|^2\Big)^{1/2} \\
&\leq \sum_{\ell\geq\ell_0}\sum_i |f|_\infty r\left(\sum_{j,k\in A_{\ell,i}} Z_{\ell,j,i}Z_{\ell,k,i}\int_{S_r}K^*_{\ell,n,i,j}\overline{K}^*_{\ell,n,i,k}e^{ib(\Delta_k-\Delta_j)}\right)^{1/2} \\
&\leq \sum_{\ell\geq\ell_0}|f|_\infty r^{-1/2}\left(\sum_i\sum_{j,k\in A_{\ell,i}} Z_{\ell,j,i}Z_{\ell,k,i}\int_{S_r}K^*_{\ell,n,i,j}\overline{K}^*_{\ell,n,i,k}\,e^{ib(\Delta_k-\Delta_j)}\right)^{1/2},
\end{aligned}
\tag{5.16}
$$

where in the second line we have used the Cauchy-Schwarz inequality, in the third line we have used that $|\sum_j v_j|^2 = (\sum_j v_j)(\sum_k \overline{v}_k)$ for any set of complex numbers $\{v_j\}_j$, and in the fourth line we have used the Hölder inequality together with the fact that the cardinality of the sum over i is at most Cr^{-3} by (5.7).

The last integral remaining in (5.16) is the oscillatory integral which has been the object of the rearrangements and changes of variables of this entire section. It is at the heart of the Dolgopyat estimate. Define the flow surface, $W^0_{j,i} = B_r(x_i)\cap(\cup_{s\in[-cr,cr]}\Phi_s(W_{j,i}))$.

Lemma 5.5. *There exists $C>0$, independent of r, n and W, such that:*

a) $\inf_{x^s}|\partial_{x^s}(\Delta_j-\Delta_k)(x^s)| \geq Cd(W^0_{j,i},W^0_{k,i})$;

b) $|\Delta_j-\Delta_k|_{C^{1+\eta}(S_r)}\leq Cr$;

c) $\left|\int_{S_r}K^*_{\ell,n,i,j}\overline{K}^*_{\ell,n,i,k}\,e^{ib(\Delta_k-\Delta_j)}\right| \leq C\frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2}e^{-2a\ell\tau}\left[\frac{r}{d(W^0_{j,i},W^0_{k,i})^{1+\eta}b^\eta}+\frac{r^{-1}}{d(W^0_{j,i},W^0_{k,i})b}\right]$,

where $\eta>0$ is from Definition 5.3.

Sketch of proof. We give here the main ideas used in the proof of Lemma 5.5, the end result of which is the estimate (c) of the key oscillatory integral.

We choose a curve $W_{j,i}$ with $j\in A_{\ell,i}$ crossing the box $B_{cr}(x_i)$. Without loss of generality (by flowing it if necessary), we may assume $W_{j,i}$ intersects the x^u axis in the local coordinates. For a fixed $\xi\in(-r,r)$, we consider the closed path starting at $(\xi,0,0)$ on W^0_i (i.e. $x^s=\xi$ on the strong stable manifold of x_i), running to x_i along the stable manifold of x_i, and up the coordinate axis of x^u (which lies in $\mathcal{W}^u$) to $W_{j,i}$. From there, the path runs along $W_{j,i}$ until it reaches the point $\mathbb{W}_j((h^s_j)^{-1}(\xi))$, then follows the strong unstable manifold γ^u_ξ (this is an element of the foliation defined in Definition 5.3) down to W^0_*, and from there follows the flow direction back to $(\xi,0,0)$. We call this path $\Gamma(\xi)$. See Figure 2.

Recalling (5.13), we notice that $\Delta_j(\xi) = h^0_j((h^s_j)^{-1}(\xi))$ is precisely the distance in the flow direction from $(\xi,0,0)$ to the point of intersection of γ^u_ξ with W^0_i. In addition, every other smooth component of $\Gamma(\xi)$ lies in the kernel of ω by construction of $\mathcal{W}^s$ and $\mathcal{W}^u$. Since $\omega(v)=1$ for every unit vector v in the flow direction, and using Stokes' theorem, we have,

$$\Delta_j(\xi) = \int_{\Gamma(\xi)}\omega = \int_{\Sigma_1}d\omega+\int_{\Sigma_2}d\omega\,,$$

where Σ_1 is the 'vertical' surface defined by the part of the foliation $\mathbb{F}$ connecting $W_{j,i}$ to W^0_i, and Σ_2 is the 'horizontal surface' comprised of the part of W^0_i enclosed by $\Gamma(\xi)$ and the curve $h_{j,i}(W_{j,i})$

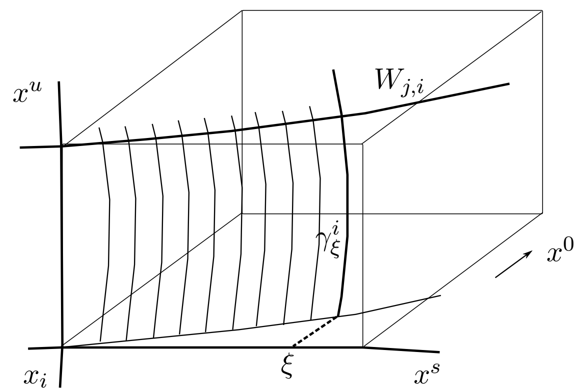

FIGURE 2. Part of a flow box $B_r(x_i)$ with path $\Gamma(\xi)$ and the unstable foliation shown. $\Gamma(\xi)$ starts at ξ, goes along the x^s-axis to x_i, up the x^u-axis to $W_{j,i}$, across $W_{j,i}$ to γ^i_ξ, down γ^i_ξ to the flow surface W^0_i, and then in the flow direction back to ξ. The length of the dotted line is $\Delta_j(\xi)$.

(remembering (5.11). The integral over Σ_2 is 0 since the flow direction lies in the kernel of $d\omega$. Writing the integral over Σ_1 in local coordinates and using (5.9) and Definition 5.3 yields,

$$\Delta_j(\xi) = \int_0^\xi \int_0^{E_j((h^s_j)^{-1}(x^s))} \partial_{x^s} G(x^s, x^u)\, dx^u\, dx^s.$$

And so, assuming that $W_{k,i}$ with $k \in A_{\ell,i}$ is also in standard position intersecting the x^u axis, we obtain

$$\begin{aligned}
\partial_{x^s}\Delta_k(\xi) - \partial_{x^s}\Delta_j(\xi) &= \int_{E_j((h^s_j)^{-1}(\xi))}^{E_k((h^s_k)^{-1}(\xi))} \partial_{x^s} G(x^u, \xi)\, dx^u \\
&= \int_{E_j((h^s_j)^{-1}(\xi))}^{E_k((h^s_k)^{-1}(\xi))} \Big[1 + \int_0^{x^u} \partial_{x^u}\partial_{x^s} G(u,\xi)\, du\Big] dx^u \\
&= \big[E_k((h^s_k)^{-1}(\xi)) - E_j((h^s_j)^{-1}(\xi))\big](1 + \mathcal{O}(r)) \ge d(W_{j,i}, W_{k,i})(1+\mathcal{O}(r))\,.
\end{aligned}$$

This proves item (a) of the lemma, and immediately gives the required bound on the C^0 norm for part (b). The bound on the C^η norm follows from the same integral expression for $\partial_{x^s}(\Delta_k - \Delta_j)$, together with property (v) of the foliation.

For item (c) of the lemma, we follow [BDL, Appendix B]. Define

$$L_{j,k}(x^s, x^0) = K^*_{\ell,n,i,j}(x^s, x^0)\overline{K}^*_{\ell,n,i,k}(x^s, x^0) \quad \text{and} \quad \boldsymbol{\Delta}_{j,k} = \Delta_k - \Delta_j$$

Exercise 11. *Show that* $|L_{j,k}|_\infty \le \frac{(\ell\tau)^{2n-2}}{r^2[(n-1)!]^2} e^{-2a\ell\tau}$ *and* $|\partial_{x^s} L_{j,k}|_\infty \le \frac{(\ell\tau)^{2n-2}}{r^3[(n-1)!]^2} e^{-2a\ell\tau}$.

We define a sequence $\{s_m\}_{m=0}^M \subset \mathbb{R}$ such that $s_0 = -r$, and $\partial_{x^s}\boldsymbol{\Delta}_{j,k}(s_m)\cdot[s_{m+1} - s_m] = 2\pi b^{-1}$, and let $M \in \mathbb{N}$ be such that $s_{M-1} \le r$ and $s_M > r$. Such a finite M exists by part (a) of the lemma. By part (b) of the lemma,

$$|\boldsymbol{\Delta}_{j,k}(x^s) - \boldsymbol{\Delta}_{j,k}(s_m) - \partial_{x^s}\boldsymbol{\Delta}_{j,k}(s_m)[s_{m+1} - s_m] \le Cr|s_m - x^s|^{1+\eta},$$

for all $x^s \in [s_m, s_{m+1}]$. Moreover, using Exercise 11, we have

$$|L_{j,k}(x^s, x^0) - L_{j,k}(s_m, x^0)| \le C\delta_m e_{\ell,n} r^{-3},$$

where $\delta_m = s_{m+1} - s_m$ and $e_{\ell,n} = \frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2} e^{-2a\ell\tau}$. Notice then that by part (a) of the lemma,

$$b\delta_m \le 2\pi d(W^0_{j,i}, W^0_{k,i})^{-1}. \tag{5.17}$$

Now we fix x^0 and estimate for each m,

$$
\begin{aligned}
&\left| \int_{s_m}^{s_{m+1}} e^{-ib\mathbf{\Delta}_{j,k}(x^s)} L_{j,k}(x^s, x^0)\, dx^s \right| \\
&\qquad = \left| \int_{s_m}^{s_{m+1}} e^{-ib[\partial_{x^s}\mathbf{\Delta}_{j,k}(s_m)[x^s - s_m] + \mathcal{O}(r|x^s - s_m|^{1+\eta})]} \big(L_{j,k}(s_m, x^0) + \mathcal{O}(r^{-3}\delta_m e_{\ell,n})\big) dx^s \right| \\
&\qquad \le C\left(b\delta_m^{1+\eta} r^{-1} + r^{-3}\delta_m\right)\delta_m e_{\ell,n} \\
&\qquad \le C\left(\frac{r^{-1}}{d(W^0_{j,i}, W^0_{k,i})^{1+\eta} b^\eta} + \frac{r^{-3}}{d(W^0_{j,i}, W^0_{k,i}) b} \right)\delta_m e_{\ell,n},
\end{aligned}
$$

where again we have used Exercise 11 and in the last line we have used (5.17). The last integral over the interval $[s_{M-1}, r]$ is trivially bounded by $Cr^{-2}\delta_M \le Cr^{-2}(bd(W^0_{j,i}, W^0_{k,i}))^{-1}$, again using (5.17). Then summing over m yields $\sum_{m=0}^{M-1} \delta_m \le 2r$, and integrating over x^0 yields another factor of r, completing the proof of part (c). □

The bound given by Lemma 5.5(c) is nearly what we need to complete the Dolgopyat estimate. We require one more lemma, which allows us to neglect the contribution from curves in $A_{\ell,i}$ that are too close together.

Lemma 5.6. *There exists $C > 0$ such that for each $\ell \ge \ell_0$, $i \in \mathbb{N}$ and $j \in A_{\ell,i}$,*

$$
\sum_{\substack{k \in A_{\ell,i} \\ d(W^0_{j,i}, W^0_{k,i}) \le \rho}} Z_{\ell,k,i} \le C[r(\rho^{1/2} + \Lambda^{-\ell\tau})]\,.
$$

Proof. Let $A(\rho) = \{k \in A_{\ell,i} : d(W^0_{j,i}, W^0_{k,i}) \le \rho\}$. First notice that by bounded distortion,

$$
\sum_{k \in A(\rho)} Z_{\ell,k,i} = \sum_{k \in A(\rho)} |W_{k,i}| |J_{W_{k,i}}\Phi_{\ell\tau}|_{C^0(W_{k,i})} = C^{\pm 1} \sum_{k \in A(\rho)} |\Phi_{\ell\tau}(W_{k,i})|, \tag{5.18}
$$

where the notation $P = C^{-1}Q$ means $C^{-1}Q \le P \le CQ$ for some $C \ge 1$.

Let $W^0_r = \cup_{s \in [-2r, 2r]} \Phi_s(W)$. Fix $\rho^* > 0$, and consider the set of local strong unstable manifolds $\{\gamma^u_x\}_{x \in W^0_r}$ having length ρ^* in both directions, and centered x. Let $G^0_{i,k} = \{x \in W^0_r : x \in \Phi_{\ell\tau}(W^0_{k,i})\}$ and note that the sets $\cup_{x \in G^0_{i,k}} \gamma^u_x$ are disjoint for different k. On the one hand, due to the uniform transversality of E^s, E^u and E^c, we have

$$
\sum_{k \in A(\rho)} m(\cup_{x \in G^0_{i,k}} \gamma^u_x) = \mathcal{O}(r\rho^*) \sum_{k \in A(\rho)} |\Phi_{\ell\tau}(W_{k,i})|\,. \tag{5.19}
$$

On the other hand, for each k, $\Phi_{\ell\tau}(\cup_{x \in G^0_{i,k}} \gamma^u_x)$ is approximately a parallelepiped having length in the flow and stable directions of about r, and having length in the unstable direction at most $2\rho^*\Lambda^{-\ell\tau}$. Moreover, these sets are disjoint for different k and their union lies in a set of length in the unstable direction at most $\rho + 2\rho^*\Lambda^{-\ell\tau}$. Then using the invariance of the measure,

$$
\sum_{k \in A(\rho)} m(\cup_{x \in G^0_{i,k}} \gamma^u_x) = \sum_{k \in A(\rho)} m(\Phi_{\ell\tau}(\cup_{x \in G^0_{i,k}} \gamma^u_x)) \le Cr^2(\rho + \rho^*\Lambda^{-\ell\tau})\,. \tag{5.20}
$$

Using (5.18) in (5.19) and equating this with (5.20) yields,

$$
\sum_{k \in A(\rho)} Z_{\ell,k,i} \le Cr(\rho^*)^{-1}(\rho + \rho^*\Lambda^{-\ell\tau})\,,
$$

and choosing $\rho^* = \rho^{1/2}$ completes the proof of the lemma. □

We will apply Lemma 5.6 with $\rho = r^2$. For each $j \in A_{\ell,i}$ define $A^{\text{close}}_{\ell,i,j} = \{k \in A_{\ell,i} : d(W^0_{k,i}, W^0_{j,i}) \le r^2\}$, and $A^{\text{far}}_{\ell,i,j} = A_{\ell,i} \setminus A^{\text{close}}_{\ell,i,j}$. Then,

$$\sum_i \sum_{j \in A_{\ell,i}} \sum_{k \in A^{\text{close}}_{\ell,i,j}} Z_{\ell,j,i} Z_{\ell,k,i} \le Cr(r + \Lambda^{-\ell\tau}) \le Cr^2, \tag{5.21}$$

remembering (5.6) and using Exercise 9.

Finally, we apply Lemma 5.5(c), summing over $A^{\text{far}}_{\ell,i,j}$,

$$\begin{aligned}
&\Big(\sum_i \sum_{j \in A_{\ell,i}} \sum_{k \in A^{\text{far}}_{\ell,i,j}} Z_{\ell,j,i} Z_{\ell,k,i} \int_{S_r} K^*_{\ell,n,i,j} \overline{K}^*_{\ell,n,i,k} \, e^{ib(\Delta_k - \Delta_j)} \Big)^{1/2} \\
&\le \Big(\sum_i \sum_{j \in A_{\ell,i}} \sum_{k \in A^{\text{far}}_{\ell,i,j}} Z_{\ell,j,i} Z_{\ell,k,i} C \frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2} e^{-2a\ell\tau} \big[r^{-1-2\eta} b^{-\eta} + r^{-3} b^{-1} \big] \Big)^{1/2} \\
&\le C r^{-1/2} [r^{-2\eta} b^{-\eta} + r^{-2} b^{-1}]^{1/2} \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} \,,
\end{aligned} \tag{5.22}$$

where again we have used Exercise 9.

Exercise 12. *Show that for all ℓ, n, i, j, k,* $\left| \int_{S_r} K^*_{\ell,n,i,j} \overline{K}^*_{\ell,n,i,k} \, e^{ib(\Delta_k - \Delta_j)} \right| \le C \frac{(\ell\tau)^{2n-2}}{[(n-1)!]^2} e^{-2a\ell\tau}$.

Now combining Exercises 10 and 12 with with (5.21) and (5.22) in (5.16) yields,

$$\begin{aligned}
&\sum_{\ell \ge \ell_0} \sum_i \int_{S_r} \sum_{j \in A_{\ell,i}} Z_{\ell,j,i} K^*_{\ell,n,i,j} f e^{ib(x^0 - \Delta_j(x^s))} \, dx^s \, dx^0 \\
&\qquad \le \sum_{\ell \ge \ell_0} \frac{(\ell\tau)^{n-1}}{(n-1)!} e^{-a\ell\tau} |f|_\infty \left(r^{1/2} + r^{-1} [r^{-2\eta} b^{-\eta} + r^{-2} b^{-1}]^{1/2} \right) \\
&\qquad \le a^{-n} |f|_\infty \left(r^{1/6} + r^{-4/3} [r^{-2\eta} b^{-\eta} + r^{-2} b^{-1}]^{1/2} \right) .
\end{aligned} \tag{5.23}$$

Now we use (5.15) and (5.23) in (5.14) to estimate,

$$\int_W R(z)^n f \, \psi \, dm_W \le C a^{-n} \left(|f|_\infty \Big(r^\alpha + r^{5/3} + r^{1/6} + r^{-4/3} [r^{-2\eta} b^{-\eta} + r^{-2} b^{-1}]^{1/2} \Big) + r^{5/3} |\partial^u f|_\infty \right) .$$

We can assume without loss of generality that $\eta < 1$ so that the first term in the square root above is the larger of the two. Setting $r = b^{-\frac{\eta}{8+6\eta}}$, bounds the term with the square root by by $b^{-\eta/3}$. Since all other powers of r are positive, we obtain,

$$\int_W R(z)^n f \, \psi \, dm_W \le C a^{-n} b^{-\gamma_0} (|f|_\infty + |\partial^u f|_\infty), \tag{5.24}$$

for some $\gamma_0 > 0$, and all $b \ge b_0$, where b_0 depends only on the maximum size of r determined by Definition 5.2. As a final step, we apply (5.24) to $R(z)^n f$ rather than f.

Exercise 13. *Use* (1.1) *and Exercise 3 to show that* $|\partial^u (R(z)^n f)|_\infty \le C(a + \log \Lambda)^{-n} |\nabla f|_\infty$.

Now Exercise 13 together with (5.24) and the bound $|R(z)^n f|_\infty \le C a^{-n} |f|_\infty$ (from Exercise 4) yield,

$$\begin{aligned}
\int_W R(z)^{2n} f \, \psi \, dm_W &\le C a^{-n} b^{-\gamma_0} \big(|R(z)^n f|_\infty + |\partial^u (R(z)^n f)|_\infty \big) \\
&\le C' a^{-2n} b^{-\gamma_0} \big(|f|_\infty + (1 + a^{-1} \log \Lambda)^{-n} |\nabla f|_\infty \big) \,,
\end{aligned}$$

which completes the proof of Lemma 4.3.

6. Extension to Dispersing Billiards

In this section, we briefly describe some of the ideas needed to adapt the technique and framework presented in these notes to the continuous time billiard flow associated with a dispersing billiard table. This is done in full detail in [BDL] for the finite horizon periodic Lorentz gas, and we only recall here in broad terms some of the adjustments that must be made. We remark that although presently a proof of exponential decay of correlations exists only in this context, these results are expected to generalize to dispersing billiard tables with corner points, and cusps (the fact that the discrete time billiard map for tables with cusps has a polynomial rate of decay of correlations will not prevent the associated continuous time flow from having an exponential one), and some billiard tables with focusing boundaries, such as those studied in [BM]. The flow associated with the infinite horizon periodic Lorentz gas, however, is known to have decay of correlations at the polynomial rate of $1/t$ [BBM].

6.1. **The Billiard Table.** Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-torus, and place finitely many open convex sets Γ_i, $i = 1, \dots d$, in $\mathbb{T}^2$ so that their closures are pairwise disjoint and the boundary of each set Γ_i is a C^3 curve with strictly positive curvature. We shall call these sets *scatterers* and the billiard table is $Q = \mathbb{T}^2 \setminus (\cup_{i=1}^d \Gamma_i)$.

The *billiard flow* is defined by the motion of a point particle traveling at unit speed in Q and colliding elastically at the boundaries of the scatterers. The particle's velocity changes only at collisions, which are defined when the particle belongs to $\partial\Gamma_i$ for some i. We assume that the table satisfies a *finite horizon* condition: there is a finite upper bound on the time between consecutive collisions in Q.

Define $\Omega_0 = Q \times \mathbb{S}^1 \subset \mathbb{T}^3$. In Ω_0, we may describe the billiard flow in the coordinates (x, y, θ), where $(x, y) \in Q$ denotes position and $\theta \in \mathbb{S}^1$ denotes velocity. Then,

$$\Phi_t(x, y, \theta) = (x + t\cos\theta, y + t\sin\theta, \theta), \tag{6.1}$$

between collisions, and at collisions the velocity changes from θ^- (pre-collision) to θ^+ (post-collision) according to the usual law of reflection. If we identify $(x, y, \theta^-) \sim (x, y, \theta^+)$, then the flow becomes continuous on the phase space $\Omega := \Omega_0/\sim$. We will find it convenient to work in both the spaces Ω_0 and Ω depending on the context.

Analysis of the flow is often aided by appealing to the associated discrete time *billiard map*. This is defined by introducing coordinates to track each collision (r for position on $\partial\Gamma_i$ parametrized by arc length, and φ for the angle the post-collision velocity vector makes with the normal to $\partial\Gamma_i$). The two-dimensional phase space for the map is then a union of cylinders $M = \cup_{i=1}^d \partial\Gamma_i \times [-\pi/2, \pi/2]$ and the billiard map $T(r, \varphi) = (r_1, \varphi_1)$ maps one collision to the next.

6.2. **Hyperbolicity and Contact Structure.** In the coordinates described above, the flow preserves the one form defined by,

$$\omega = \cos\theta\, dx + \sin\theta\, dy\,.$$

Between collisions, this is obvious from the definition (6.1) since θ is constant except at collisions. That the one form is preserved through collisions is a simple calculation (see [CM, Section 3.3]). Since $(\cos\theta, \sin\theta)$ is the direction of motion of the particle in the table Q, we see that geometrically, the kernel of the one form is the plane perpendicular to the flow direction in Ω, and $\omega(v) = 1$ for any unit vector $v \in \mathbb{R}^3$ pointing in the flow direction.

Exercise 14. *Show that $\omega \wedge d\omega = dx \wedge d\theta \wedge dy$.*

Exercise 14 shows that the contact volume is Lebesgue measure on Ω_0, and this is preserved by the flow. Thus the flow and one form are already normalized according to the requirements of Section 1.1.

Due to the strictly positive curvature of the $\partial\Gamma_i$, both the map and the flow are hyperbolic. Let $\tau_{\min}, \mathcal{K}_{\min} > 0$ denote the minimum time between collisions and the minimum curvature, respectively, and let $\tau_{\max} < \infty$ denote the maximum time between collisions, which exists due to the finite horizon condition. The constant $\Lambda_0 = 1 + 2\tau_{\min}\mathcal{K}_{\min}$ represents the minimum hyperbolicity constant for the map; then setting $\Lambda = \Lambda_0^{1/\tau_{\max}}$ gives a lower bound on the hyperbolicity constant for the flow satisfying (1.1).

The billiard map T preserves the following stable cone on all of M,

$$C^s(r,\varphi) = \{(dr, d\varphi) \in \mathbb{R}^2 : -\mathcal{K}_{\min} \geq d\varphi/dr \geq -\mathcal{K}_{\max} - \tau_{\min}^{-1}\}, \tag{6.2}$$

and an analogous unstable cone C^u is defined by $\mathcal{K}_{\min} \leq d\varphi/dr \leq \mathcal{K}_{\max} + \tau_{\min}^{-1}$. Then flowing C^u forward between consecutive collisions and C^s backwards between collisions defines a family of cones in Ω that is invariant under the flow (satisfying (2.2)) and lies in the kernel of ω. This family of cones is continuous on each component of Ω_0 that does not cross one of the singularity surfaces (defined below). See [BDL, Section 2.1].

6.3. **Singularities.** The singularities for both the map and the flow are created by tangential collisions with the scatterers. For the map, this is the set $\mathcal{S}_0 = \{(r,\varphi) \in M : \varphi = \pm\frac{\pi}{2}\}$. For $n \geq 1$, the sets $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$ and $\mathcal{S}_{-n} = \cup_{i=0}^n T^i\mathcal{S}_0$ are the singularity sets for T^n and T^{-n}, respectively. The map T is discontinuous at $\mathcal{S}_1$. Moreover, its derivative satisfies

$$\|DT(z)\| \approx d(z, \mathcal{S}_1)^{-1/2}, \quad \text{for } z = (r,\varphi) \in M,$$

so that the derivative becomes infinite at tangential collisions.

The local sections Σ_i introduced for Anosov flows in Section 2.1 can be defined naturally for the billiard flow as the boundaries of the scatterers, $\partial\Gamma_i$. The projections P^+ and P^- are defined for $Z \in \Omega$ as the first intersection of $\Phi_t(Z)$ with one of the Γ_i, for $t > 0$ for P^+ and for $t < 0$ for P^-.

While the flow remains continuous on Ω, its derivative also becomes infinite at tangential collisions (with the same order of magnitude as the map). Thus the flow is only Hölder continuous with exponent $1/2$ due to the tangential collisions. Let $\mathcal{S}_0^+$ denote the surface in Ω_0 created by flowing $\mathcal{S}_0$ forward to its next collision (on $\mathcal{S}_{-1}$). Then the family of unstable cones C^u is continuous in Ω_0 away from the surface $\mathcal{S}_0^+$. Similarly, let $\mathcal{S}_0^-$ denote the surface obtained by flowing $\mathcal{S}_0$ under the inverse flow to $\mathcal{S}_1$. The family of stable cones C^s is continuous in Ω_0 away from $\mathcal{S}_0^-$.

In order to regain control of distortion, one introduces *homogeneity strips*, which are artificial subdivisions of the phase space on which the derivative has comparable rates of expansion and contraction. For the map, the standard choice is to choose $k_0 > 0$ and then define the homogeneity strip

$$\mathbb{H}_k = \{(r,\varphi) : k^{-2} \leq \tfrac{\pi}{2} - \varphi \leq (k+1)^{-2}\} \quad \text{for } k \geq k_0,$$

with a similar definition for $\mathbb{H}_{-k}$ for φ near $-\frac{\pi}{2}$. Since expansion factors for the map are proportional to $1/\cos\varphi_1$ when $T(r,\varphi) = (r_1,\varphi_1)$, these subdivisions of the space imply that the Jacobians of the map satisfy distortion bounds as in Lemma 2.7(a), but with[12] Hölder exponent $1/3$.

Exercise 15. *Suppose $z, \tilde{z} \in \mathbb{H}_k$ for some $k \in \mathbb{Z}$. Show that*

$$\left|\frac{\cos\varphi(z)}{\cos\varphi(\tilde{z})} - 1\right| \leq C d(z,\tilde{z})^{1/3}, \quad \textit{for some } C > 0 \textit{ independent of } k.$$

Here, $\varphi(z)$ denotes the second coordinate of $z = (r,\varphi) \in M$.

One extends this distortion control to the Jacobians of the flow by only comparing derivatives at points whose next collisions lie in the same homogeneity strip under the forward flow (for the unstable Jacobian) or the backward flow (for the stable Jacobian).

[12] The exponent $1/3$ is a simple consequence of defining the homogeneity strips to decay like k^{-2}. If, instead, one chooses a decay rate of k^{-p}, $p > 1$, then the Hölder exponent becomes $1/(p+1)$. Thus it is possible to obtain a Hölder exponent arbitrarily close to $1/2$ by choosing p close to 1. However, $p = 1$ is not an acceptable choice since it ruins the summability of the series and the growth lemma needed for the analogue of Lemma 2.7(c) fails.

6.4. **Admissible Curves and Definition of the Norms.** Since our invariant cones C^u and C^s satisfy (2.2), we may define a family of *admissible cone-stable curves* $\mathcal{W}^s$ which is invariant under Φ_{-t}, $t > 0$, and satisfies the requirements of Definition 2.1. In addition, we require stable curves to be disjoint from $\partial\Omega_0$. Thus if a stable curve is in the midst of a collision, we omit the collision points, and consider each of the two or three connected components as separate stable curves.

Due to our definition of C^s, we have that $P^+(W)$ is a stable curve for the map whenever $W \in \mathcal{W}^s$. Due to our discussion of distortion in Section 6.3, we call a stable curve $W \in \mathcal{W}^s$ *homogeneous* if $P^+(W) \subset \mathbb{H}_k$ for some $k \in \mathbb{Z}$. Similarly, we define an invariant family of unstable curves $\mathcal{W}^u$ and call an unstable curve U homogeneous if $P^-(U) \subset \mathbb{H}_k$ for some $k \in \mathbb{Z}$.

Using the (global) coordinates (r, φ) in M and (6.2) allows us to view each map-stable curve $P^+(W)$ as the graph of a function G_W over the r-coordinate. We then use the same definition of distance between stable curves, $d_{\mathcal{W}^s}(W_1, W_2)$, as given in (2.3), with the added requirement that $d_{\mathcal{W}^s}(W_1, W_2) = \infty$ unless $P^+(W_1)$ and $P^+(W_2)$ lie in the same homogeneity strip.

With these conventions in place, we may define the weak and strong norms for $f \in C^1(\Omega_0)$ precisely as in Section 2.2. Due to Exercise 15, we choose $\alpha \le 1/3$ in order that the Jacobian along a stable curve may be a viable test function. The other restrictions on the parameters remain the same.

The definitions of the weak and strong Banach spaces are again the closures with respect to $|\cdot|_w$ and $\|\cdot\|_{\mathcal{B}}$, respectively, but now $C^1(\Omega_0)$ is replaced by slightly different function spaces, see [BDL, Definition 2.12]. However, Lemmas 2.4 (embedding) and 2.5 (compactness) continue to hold as stated.

6.5. **Lasota-Yorke Inequalities and Complexity Bounds.** The Lasota-Yorke inequalities of Proposition 2.6 continue to hold as written as well, except that their proofs change considerably.

As an example, consider the proof of the weak norm inequality, (2.7). Following (2.11), we write,

$$\begin{aligned}
\int_W \mathcal{L}_t f\, \psi\, dm_W &= \sum_{W_i \in \mathcal{G}_t(W)} \int_{W_i} f\, \psi \circ \Phi_t\, J_{W_i}\Phi_t\, dm_{W_i} \\
&\le \sum_{W_i \in \mathcal{G}_t(W)} |f|_w |\psi \circ \Phi_t|_{C^\alpha(W_i)} |J_{W_i}\Phi_t|_{C^\alpha(W_i)} \qquad (6.3)\\
&\le C|f|_w \sum_{W_i \in \mathcal{G}_t(W)} |J_{W_i}\Phi_t|_{C^0(W_i)},
\end{aligned}$$

where we have used bounded distortion and the equivalent of (2.13) to estimate the Hölder norms of $\psi \circ \Phi_t$ and $J_{W_i}\Phi_t$. However, the counterpart of the bound on the sum over the Jacobians, Lemma 2.7(c), is not immediately available due to the cutting caused by the singularities. Indeed, the set $\mathcal{G}_t(W)$ contains a countably infinite number of stable curves since in order to have bounded distortion for $J_{W_i}\Phi_t$, we must subdivide $\Phi_{-t}W$ so that for each $W_i \in \mathcal{G}_t(W)$, $P^+(\Phi_s W_i)$ lies in a single homogeneity strip for all $s \in [0, t]$.

Despite the countable subdivision of $\Phi_{-t}W$ which defines $\mathcal{G}_t(W)$, one can show that the sum over Jacobians in (6.3) remains uniformly bounded in t and $W \in \mathcal{W}^s$. This is an essential property of both the map and the flow: that the hyperbolicity dominates the complexity due to cuts created by singularities, including the countable collection of cuts made by the boundaries of homogeneity strips. The key estimate which encapsulates this property is the *one step expansion* for the map, due to Chernov. Let $\overline{\mathcal{W}}^s$ denote the set of homogeneous stable curves for the map.

Lemma 6.1 (One Step Expansion). *For any $W \in \overline{\mathcal{W}}^s$, let V_i denote the connected homogeneous components of $T^{-1}W$. There exists an adapted metric $\|\cdot\|_*$, equivalent to the Euclidean metric in*

$\mathbb{R}^2$, *such that*

$$\lim_{\delta\downarrow 0} \sup_{\substack{W\in\overline{\mathcal{W}}^s \\ |W|\le\delta}} \sum_i |J_{V_i}T|_* < 1,$$

where $|J_{V_i}T|_*$ *is the minimum contraction factor on* V_i *in the adapted metric* $\|\cdot\|_*$.

This is proved, for example, in [CM, Lemma 5.56]. The main idea is that on homogeneity strips, the contraction factor is $\approx k^{-2}$, so one can choose k_0 sufficiently large to make the sum $\sum_{k\ge k_0} k^{-2}$ as small as one likes. The constant $\Lambda_0 > 1$ defined earlier gives the minimum contraction factor Λ_0^{-1} in the adapted metric, and then choosing δ small enough guarantees that $T^{-1}W$ can contain at most one component in $M \setminus (\cup_{|k|\ge k_0}\mathbb{H}_k)$, and a bounded number of components[13] that must be divided according to homogeneity strips $\mathbb{H}_k$ with $|k|\ge k_0$.

Then choosing δ_0 in the definition of $\mathcal{W}^s$ (Definition 2.1) and the analogous map-stable family $\overline{\mathcal{W}}^s$ according to Lemma 6.1, the one-step expansion can be iterated for the map ([DZ1, Lemmas 3.1 and 3.2]) and then extended to the flow ([BDL, Lemma 3.8]), yielding finally that the sum in (6.3) is bounded uniformly in t and W, proving (2.7) for the billiard flow.

Similar adjustments must be made for the strong norm estimates, with increased complexity due to cutting and distortion control.

6.6. **The Generator and The Resolvent.** The definition of the generator X and the resolvent $R(z)$ proceeds as described in Section 3. Lemma 3.1 and the Lasota-Yorke inequalities of Proposition 3.3 go through with minor changes. Thus the characterization of the spectra of X and $R(z)$ given by Corollary 3.4 and Proposition 3.5 hold for the billiard flow.

To prove that in fact, X has a spectral gap, one can follow again the path outlined in Section 4. The major difference is in the proof of the Dolgopyat estimate, Lemma 4.3. In Section 5, we used a local foliation of strong unstable manifolds to compare the integrals on stable curves in the same flow box in Lemma 5.5. Unfortunately, the foliation of unstable manifolds for the billiard flow is only measurable due to the density of the sets $\{\Phi_t(\mathcal{S}_0)\}_{t\in\mathbb{R}}$ in Ω, so that Definition 5.3 is no longer valid.

Instead, one must construct a foliation of flow-unstable curves, lying in the kernel of the contact form, which approximate the properties enumerated in Definition 5.3. Since the curves are not real unstable manifolds, in item (iii) of the definition, they only remain invariant for a specified amount of time χ, chosen proportional to $\log|b|$. And due to the singularities, there are gaps in the parts of the foliation that can be mapped backwards for time χ. These gaps must be interpolated across in order to obtain the required smoothness for the foliation. Finally, item (v) of the foliation fails, yet a four-point estimate does hold which suffices to prove the items in Lemma 5.5. The construction of this foliation is carried out in detail in [BDL, Section 6], and is one of the most technical parts of that paper.

With the Dolgopyat estimate proved, the proof of Theorem 1.1 can proceed as in Section 4.

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