# THERMODYNAMIC FORMALISM FOR DISPERSING BILLIARDS

VIVIANE BALADI AND MARK F. DEMERS

ABSTRACT. For any finite horizon Sinai billiard map $T$ on the two-torus, we find $t_* > 1$ such that for each $t \in (0, t_*)$ there exists a unique equilibrium state $\mu_t$ for $-t \log J^u T$, and $\mu_t$ is $T$-adapted. (In particular, the SRB measure is the unique equilibrium state for $-\log J^u T$.) We show that $\mu_t$ is exponentially mixing for Hölder observables, and the pressure function $P(t) = \sup_\mu \{h_\mu - \int t \log J^u T d\mu\}$ is analytic on $(0, t_*)$. In addition, $P(t)$ is strictly convex if and only if $\log J^u T$ is not $\mu_t$-a.e. cohomologous to a constant, while, if there exist $t_a \neq t_b$ with $\mu_{t_a} = \mu_{t_b}$, then $P(t)$ is affine on $(0, t_*)$. An additional sparse recurrence condition gives $\lim_{t \downarrow 0} P(t) = P(0)$.

## CONTENTS



---

*Date*: December 7, 2020.

We thank Yuri Lima for stimulating questions and comments. MD was partly supported by NSF grant DMS 1800321. VB's research is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 787304). VB is grateful to the Knut and Alice Wallenberg Foundation for an invitation to Lund University in 2020.

# 1. Introduction and Statement of Main Results

1.1. **Set-up.** A Sinai or dispersive billiard table $Q$ on the two-torus $\mathbb{T}^2$ is a set $Q = \mathbb{T}^2 \setminus \cup_{i=1}^{\Omega} \mathcal{O}_i$, for some finite number $\Omega \geq 1$ of pairwise disjoint closed domains $\mathcal{O}_i$ (the obstacles, or scatterers) with $C^3$ boundaries having strictly positive curvature $\mathcal{K}$. (In particular, the domains are strictly convex.) The billiard flow, also called a periodic Lorentz gas, is the motion of a point particle traveling in $Q$ at unit speed and undergoing specular reflections at the boundary of the scatterers. (At a tangential — also called grazing — collision, the reflection does not change the direction of the particle.)

We study here the associated billiard map $T : M \to M$ on the compact set $M = \partial Q \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, defined to be the first collision map on the boundary of $Q$. We use the standard coordinates $x = (r, \varphi)$, where $r$ is arclength along $\partial\mathcal{O}_i$ and $\varphi$ is the angle the post-collision trajectory makes with the normal to $\partial\mathcal{O}_i$. Grazing collisions cause discontinuities in the map $T$. We remark, however, that since the flow is continuous, the map $T$ is well-defined and bijective on $M$. There is no need to reduce the domain to a smaller set.

For $x \in M$, let $\tau(x)$ denote the distance from $x$ to $T(x)$ (the free flight time). Set $\mathcal{K}_{\max} = \sup \mathcal{K} < \infty$, $\mathcal{K}_{\min} = \inf \mathcal{K} > 0$, and $\tau_{\min} = \inf \tau > 0$. Then [CM] the cones in $\mathbb{R}^2$ defined by

$$C^u = \{(dr, d\varphi) : \mathcal{K}_{\min} \leq \frac{d\varphi}{dr} \leq \mathcal{K}_{\max} + \frac{1}{\tau_{\min}}\}\,, \; C^s = \{(dr, d\varphi) : -\mathcal{K}_{\min} \geq \frac{d\varphi}{dr} \geq -\mathcal{K}_{\max} - \frac{1}{\tau_{\min}}\}$$

are strictly invariant under $DT$ and $DT^{-1}$, respectively, whenever these derivatives exist.

The map $T$ is uniformly hyperbolic, in the following sense: Let

$$\Lambda := 1 + 2\tau_{\min}\mathcal{K}_{\min} > 1\,. \tag{1.1}$$

Then there exists $C_1 > 0$ such that, for all $x$ for which $DT^n(x)$, respectively $DT^{-n}(x)$, is defined,

$$\|DT^n(x)v\| \geq C_1\Lambda^n\|v\|,\; \forall v \in \mathcal{C}^u, \quad \|DT^{-n}(x)v\| \geq C_1\Lambda^n\|v\|,\; \forall v \in \mathcal{C}^s\,,\; \forall n \geq 0\,. \tag{1.2}$$

Let $\mathcal{S}_0 = \{(r, \varphi) \in M : \varphi = \pm\frac{\pi}{2}\}$ denote the set of tangential collisions on $M$. Then

$$\mathcal{S}_n = \cup_{i=0}^{-n} T^i \mathcal{S}_0\,,\; n \in \mathbb{Z}\,, \tag{1.3}$$

is the singularity set for $T^n$. In other words, there exists $n \in \mathbb{Z}$ such that $DT^n(x)$ is not defined if and only if $x$ belongs to the (invariant and dense, [CM, Lemma 4.55]) set of curves $\cup_{m\in\mathbb{Z}}\mathcal{S}_m$. Let

$$M' = M \setminus \cup_{m\in\mathbb{Z}}\mathcal{S}_m\,. \tag{1.4}$$

The spaces $E^u(x)$ and $E^s(x)$ are defined at any $x \in M'$. Indeed, for each $n \geq 0$, let $x_n = T^n x$, and consider $v_n = DT^{-n}(x_n)v/\|DT^{-n}(x_n)v\|$ for some $v \in C^s$. Since $x \in M'$, we have that $DT^{-n}(x_n)$ is well-defined for each $n \geq 0$. By uniform hyperbolicity, the sequence $v_n$ converges to a vector $v_\infty$. The direction of $v_\infty$ is $E^s(x)$. Similarly, for $y \in M \setminus_{m\leq 0} \mathcal{S}_m$, consider $y_n = T^{-n}y$ and $u_n = DT^n(y_n)v/\|DT^n(y_n)u\|$, for $n \geq 0$ and $u \in C^u$. The limit of $u_n$ is $E^u(y)$.

We have [CM, Theorem 4.66, Theorem 4.75] that Lebesgue$(M' \setminus M) = \mu_{\mathrm{SRB}}(M' \setminus M) = 0$, where $\mu_{\mathrm{SRB}} = (2|\partial Q|)^{-1}\cos\varphi\, dr d\varphi$ is the unique absolutely continuous invariant measure. Also, at each $x \in M'$, the unstable and stable Jacobians $J^uT(x)$ and $J^sT(x)$, with respect to arclength along unstable, respectively stable, manifolds, are well-defined and nonzero. Note also that, if $J_{\mathrm{Leb}}T$ denotes the Jacobian of $T$ with respect to Lebesgue, then,

$$J_{\mathrm{Leb}}T(x) = \frac{\cos(\varphi(x))}{\cos(\varphi(T(x)))} = J^uT(x) \cdot J^sT(x) \cdot \frac{E \circ T(x)}{E(x)}\,,\; \forall x \in M'\,, \tag{1.5}$$

where $E(x) = \sin(\angle(E^s(x), E^u(x)))$. Thus, for any $T$-invariant[1] probability measure $\mu$ on $M$,

$$\text{if } \mu(M \setminus M') = 0 \text{ then } \int_M \log J^uT\, d\mu = -\int_M \log J^sT\, d\mu\,. \tag{1.6}$$

[1] All probability measures in the present work are Borel measures.

Finally, we assume that the billiard table $Q$ has *finite horizon,* i.e., the billiard flow on $Q$ does not have any trajectories making only tangential collisions. This implies (but is not[2] equivalent with) $\tau_{\max} := \sup \tau < \infty$, see [BD, Remark 1.1].

1.2. **Potentials and Pressure. Theorems 1.1–1.2. Corollaries 1.3– 1.5.The Operator $\mathcal{L}_t$.** Since $T$ admits a finite generating partition (see the beginning of Section 2.3), it follows that for any $T$-invariant probability measure $\mu$, the Kolmogorov entropy $h_\mu(T)$ is finite ([W, Theorem 4.10, Theorem 4.17]).

Fix $t \geq 0$. Let $\mu$ be a $T$-invariant probability measure $\mu$. If $\mu(M \setminus M') = 0$, define the *pressure of* $\mu$ for the (so-called *geometric) potential* $-t \log J^u T$ by

$$P_\mu(-t \log J^u T) = h_\mu(T) - t \int_M \log J^u T \, d\mu \, .$$

If $\mu(M \setminus M') \neq 0$, we set $\int_M \log J^u T \, d\mu = \infty$, so that $P_\mu(-t \log J^u T) = -\infty$ if $t > 0$. Due to the invariance of $\mu$, the bound (1.2) implies that $\int_M \log J^u T \, d\mu = \lim_{n \to \infty} \frac{1}{n} \int_M \log J^u T^n \, d\mu \geq \log \Lambda$, thus the integral is either well-defined and nonnegative or infinite. It is known that

$$\chi^u := \int_M \log J^u T \, d\mu_{\mathrm{SRB}} = h_{\mu_{\mathrm{SRB}}}(T) \in (\Lambda, \infty) \, , \tag{1.7}$$

so that $P_{\mu_{\mathrm{SRB}}}(-\log J^u T) = 0$ (this is the Pesin entropy formula, see e.g. [CM, Theorem 3.42]).

For a bounded function $g : M \to \mathbb{R}$, we set $P_\mu(-t \log J^u T + g) = P_\mu(-t \log J^u T) + \int g \, d\mu$, and we define the *pressure* $P(t, g)$ *of the potential* $-t \log J^u T + g$ by

$$P(t, g) := \sup\{P_\mu(-t \log J^u T + g) : \mu \text{ a } T\text{-invariant probability measure }\} \, , \; P(t) := P(t, 0) \, . \tag{1.8}$$

We call $\mu$ an *equilibrium state* for the potential $-t \log J^u T + g$ if $P_\mu(-t \log J^u T + g) = P(t, g)$.

The case $t = 0$, $g = 0$, corresponds to the measure of maximal entropy. Under an additional condition of "sparse recurrence to the singularity set" (see Definition 5.4), a measure $\mu_0$ with $P(0) = P_{\mu_0}(0)$ was recently constructed in [BD] ($\mu_0$ was called $\mu_*$ there), shown to be mixing (in fact, Bernoulli), to be the unique measure $\mu$ satisfying $P_\mu(0) = P(0)$, and to satisfy the $T$-adapted condition (1.10) below. The speed of mixing of $\mu_0$ is not known.)

For $t = 1$, we mentioned above that $P_{\mu_{\mathrm{SRB}}}(-\log J^u T) = 0$. In addition, $\mu_{\mathrm{SRB}}$ is $T$-adapted and, for any $T$-invariant probability measure $\mu$ giving small enough weight to neighbourhoods of singularity sets [KS, Part IV, Theorem 1.1], the Ruelle inequality $P_\mu(-\log J^u T) \leq 0$ holds. The measure $\mu_{\mathrm{SRB}}$ is mixing, in fact, correlations for Hölder observables decay exponentially [Y].

For $t$ in a small interval[3] around 1, [CWZ] established the existence of equilibrium states for the potential $-t \log J^u T$ using a Young tower construction with exponential tails, proving that these measures are exponentially mixing on Hölder observables and are unique in the class of measures that lift to the Young tower.

We establish a thermodynamic formalism for Sinai billiards for $t \in (0, t_*)$, with $t_* > 1$ defined by

$$t_* := \sup\{t > 0 : \Lambda^{-t} < e^{P(t)}\} = \sup\left\{t > 0 : t > -\frac{P(t)}{\log \Lambda}\right\} \, . \tag{1.9}$$

(That $t_* > 1$ follows since $\Lambda > 1$ from (1.1), while $P(t) \geq 0$ for $t \leq 1$.) The definition of $t_*$ can be viewed as a pressure gap condition, controlling by $P(t)$ the contribution from pieces that constantly get cut by the singularities. In particular, for any $t < t_*$, we may[4] choose $\theta \in (\Lambda^{-1}, 1)$ in the one-step expansion Lemma 3.1 so that $\theta^t < e^{P(t)}$. This complexity bound permits us to prove the required growth lemmas essential to our analysis. Our first main result is the following theorem:

---

[2] We need the stronger condition e.g. in the proof of Proposition 5.2.

[3] The interval depends on the exponential rate of return (itself close to 1) to the Young tower coupling magnet.

[4] It is in fact enough to require there that $\theta^t < e^{P_*(t)}$.

**Theorem 1.1** (Thermodynamic Formalism for Sinai Billiards)**.** *For each $t \in (0, t_*)$, the potential $-t \log J^u T$ admits a unique equilibrium state $\mu_t$. The measure $\mu_t$ is mixing, gives positive mass to any nonempty open set, and does not have atoms. Moreover, $\mu_t$ is $T$-adapted, that is*[5]

$$\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_t < \infty \,. \tag{1.10}$$

*In addition, $\mu_t$ has exponential decay of correlations for Hölder observables. Finally, if $T$ satisfies the sparse recurrence condition then $\lim_{t \downarrow 0} P(t) = P(0)$.*

We prove Theorem 1.1 for $t \in (0, t_*)$ in three steps:

- First, we introduce in Section 2.2 an equivalent (topological) expression $P_*(t)$ for $P(t)$, generalising what was done in [BD] for $t = 0$, and we show that $P_*(t)$ is convex and strictly decreasing (Proposition 2.5), and that $P(t) \le P_*(t)$ (Proposition 2.3), for all $t > 0$.
- Next, for $t \in (0, t_*)$, we prove the following properties for the transfer operator

$$\mathcal{L}_t f = \frac{f \circ T^{-1}}{|J^s T|^{1-t} \circ T^{-1}} \tag{1.11}$$

  acting on an anisotropic Banach[6] space $\mathcal{B}$ (Theorem 4.1): The operator $\mathcal{L}_t$ has spectral radius $e^{P_*(t)}$, essential spectral radius strictly smaller than $e^{P_*(t)}$, and the maximal eigenvectors of $\mathcal{L}_t$ and its dual give rise to a $T$-invariant probability measure $\mu_t$. In addition, $\mathcal{L}_t$ has a spectral gap on $\mathcal{B}$, so that $\mu_t$ is exponentially mixing on Hölder observables.
- Finally, in Section 5, still for $t \in (0, t_*)$, we show that $P_{\mu_t}(-t \log J^u T) = P_*(t)$ , so that $P(t) = P_*(t)$ Corollary 5.3, as well as the remaining claims about $\mu_t$: in particular that $\mu_t$ is the unique equilibrium state among all $T$-invariant Borel probability measures realising the variational principle $P(t) = P_*(t)$ (Theorem 2.4), and that sparse recurrence implies that $P(t)$ tends to $P(0)$ as $t \downarrow 0$ (Proposition 5.5). Our proof of uniqueness also gives a more general variational principle, $P(t, g) = P_*(t, g)$, Theorem 5.8.

We use the Banach spaces $\mathcal{B}$ introduced in [DZ2], except that we work with (exact) stable manifolds $\mathcal{W}^s$ (as in [BD]) instead of cone stable curves $\widehat{\mathcal{W}}^s$ (see Section 2.1). More importantly, we must tune the parameters used to define $\mathcal{B} = \mathcal{B}(t_0, t_1)$ in Section 4.1 to an interval $[t_0, t_1] \subset (0, t_*)$ containing $t$. In particular, the decay rate $k^{-q}$ defining the homogeneity strips (2.1) in [DZ2] was $q = 2$, while we need to assume $qt > 1$ here (due to (3.2)). Also, we need to let the parameter $p$ used in the definition (4.3) of the strong stable norm tend to infinity when $t \to t_*$ (see Lemma 4.7). It follows that our bound for the essential spectral radius of $\mathcal{L}_t$ on $\mathcal{B}(t_0, t_1)$ deteriorates as $t_0 \to 0$ or $t_1 \to t_*$, and we lose the spectral gap in both limits.

The keys to the proof of the spectral Theorem 4.1 are the delicate growth lemmas given in Sect. 3. To prove these growth lemmas, subtle modifications of the fundamental ideas of Chernov [CM] and of the original techniques introduced in [DZ1, BD] were necessary. In particular, the analysis for $t > 1$ required a new bootstrap argument (see the beginning of Sect. 3 and Sect. 3.4 and 3.6).

In Section 6, a more careful study of the operator $\mathcal{L}_t$ yields our second main result:

**Theorem 1.2** (Strict Convexity)**.** *The function $t \mapsto P(t)$ is analytic on $(0, t_*)$, with*

$$P'(t) = \int \log J^s T \, d\mu_t = -\int \log J^u T \, d\mu_t < 0 \,, \tag{1.12}$$

*and*

$$P''(t) = \sum_{k \ge 0} \left[ \int (\log J^s T \circ T^k) \log J^s T \, d\mu_t - (P'(t))^2 \right] \ge 0 \,. \tag{1.13}$$

[5]The $T$-adapted property appears in particular in the work of Lima–Matheus [LM].

[6]We attract the reader's attention to Lemma 4.3 showing $\mathcal{L}_t(C^1) \subset \mathcal{B}$, which furnishes the proof of [BD, Lemma 4.9], which had been omitted there, see Remark 4.4.

*Moreover,* $P''(t) = 0$ *if and only if* $\log J^sT = f - f \circ T + \int \log J^sT\, d\mu_t$ *($\mu_t$ a.e.) for some* $f \in L^2(\mu_t)$. *Finally, both* $t \mapsto \int \log J^uT\, d\mu_t$ *and* $t \mapsto h_{\mu_t}$ *are decreasing functions of* $t$.

The formula for $P'(t)$ in (1.12) implies that, if there exist $t_a \neq t_b$ in $[0, t_*)$ such that $\mu_{t_a} = \mu_{t_b}$, then $P(t)$ is not strictly convex: indeed, $P'(t)$ is constant on $[t_a, t_b]$. By analyticity, we then deduce that $P(t)$ must be affine on $(0, t_*)$. Therefore, we get an immediate corollary of Theorem 1.2:

**Corollary 1.3.** *If there exist* $t_a \neq t_b$ *in* $(0, t_*)$ *such that* $\mu_{t_a} = \mu_{t_b}$ *then* $P(t)$ *is affine on* $(0, t_*)$, *and* $\log J^sT$ *is* $\mu_t$ *a.e. cohomologous to its average* $\int \log J^sT\, d\mu_t$ *for all* $t \in (0, t_*)$.

We expect that there does not exist any Sinai billiard table such that $\log J^sT$ is $\mu_t$ a.e. cohomologous to a constant on $M'$ for some $t \in [0, t_*)$. If we only want to verify that $\mu_0 \neq \mu_1 = \mu_{\mathrm{SRB}}$, it is enough to show that $P''(1) \neq 0$. Note that in [BD], assuming sparse recurrence (see Definition 5.4), we showed that $\mu_0 = \mu_{\mathrm{SRB}}$ (i.e., $\mu_0 = \mu_1$) only if $\frac{1}{p} \log |\det(DT^{-p}|_{E^s}(x))| = P(0)$ for every nongrazing periodic orbit $T^m(x) = x$.

The proof of analyticity of $P(t)$ via analyticity of $\mathcal{L}_t$ in Theorem 1.2 gives:

**Corollary 1.4** (Uniform Rates of Mixing)**.** *The exponential rate of mixing of* $\mu_t$ *for* $C^1$ *observables is uniformly bounded away from* 1 *in any compact subinterval of* $(0, t_*)$.

In addition, the proof of the claim on $P''(t) = 0$ in Theorem 1.2 gives:[7]

**Corollary 1.5** (Central Limit Theorem)**.** *For any* $t \in (0, t_*)$ *such that* $P''(t) \neq 0$, *setting* $\chi_t := P'(t)$ *and* $\sigma_t := P''(t)$, *we have* $\lim_{k\to\infty} \mu_t\big(\frac{1}{\sqrt{k}} \sum_{j=0}^{k-1} (\log J^sT - \chi_t) \circ T^j \leq z\big) = \frac{1}{\sqrt{2\pi\sigma_t}} \int_{-\infty}^{z} e^{-v^2/(2\sigma_t^2)}\, dv$, *for any* $z \in \mathbb{R}$.

To end this section, we motivate heuristically the choice of the weight $1/|J^sT|^{1-t}$ in (1.11), by analogy with the theory for smooth hyperbolic $T$. For a transitive Anosov diffeomorphism $T$, the transfer operator whose maximal left and right eigenvectors on an anisotropic Banach space give rise to $\mu_t$ is $\widetilde{\mathcal{L}}_t(f) = \big(f/(|J^uT|^t J^sT)\big) \circ T^{-1}$ (see [GL] or [Ba, Chapter 7]). A coboundary argument, reflecting the fact that $C^1$ functions are interpreted as distributions via integration with respect to the SRB measure $\mu_{\mathrm{SRB}} = (2|\partial Q|)^{-1} \cos\varphi\, dr d\varphi$ here (see below Proposition 4.2), but with respect to Lebesgue in [GL, Ba], will replace $1/(|J^uT|^t J^sT)$ by $1/|J^sT|^{1-t}$: Indeed, (1.5) gives (on $M'$)

$$
\begin{aligned}
-\log\big(|J^uT|^t J^sT\big) &= -\log |J^sT J^uT|^t - \log |J^sT|^{1-t} \\
&= -t \log\left(\frac{E \cos\varphi}{(E\cos\varphi) \circ T}\right) - (1-t) \log J^sT\,. \qquad (1.14)
\end{aligned}
$$

The first term of (1.14) is a coboundary. Thus the operators $\tilde{\mathcal{L}}_t$ and $\mathcal{L}_t$ from (1.11) have isomorphic spectral data.

## 2. Topological Formulation $P_*(t,g)$ for $P(t,g)$. Variational Principle (Theorem 2.4)

### 2.1. Hyperbolicity and Distortion. $\mathcal{W}^s$, $\widehat{\mathcal{W}}^s$, $\mathcal{W}^s_{\mathbb{H}}$, $\widehat{\mathcal{W}}^s_{\mathbb{H}}$. Families $\mathcal{M}^n_{-k}$, $\mathcal{M}^{n,\mathbb{H}}_{-k}$.

For $n > 0$, following [BD], define $\mathcal{M}_0^n$ to be the set of maximal connected components of $M \setminus \mathcal{S}_n$, and $\mathcal{M}_{-n}^0$ to be the maximal connected components of $M \setminus \mathcal{S}_{-n}$. Set $\mathcal{M}^n_{-k} = \mathcal{M}^0_{-k} \bigvee \mathcal{M}^n_0$. Note that if $A \in \mathcal{M}^n_0$, then $T^kA \in \mathcal{M}^{n-k}_{-k}$ for each $0 \leq k \leq n$, and $T^kA$ is a union of elements of $\mathcal{M}^0_{-k}$ for each $k > n$.

To control distortion, we introduce homogeneity strips whose spacing depends on $t_0 \in (0,1)$ if $t \geq t_0$. Choose[8] $q = q(t_0) > 1$ such that $qt_0 \geq 2$. For fixed $k_0 \in \mathbb{N}$ define

$$
\mathbb{H}_k = \{(r,\varphi) \in M : (k+1)^{-q} \leq \frac{\pi}{2} - \varphi < k^{-q}\}\,, \quad \text{for } k \geq k_0, \qquad (2.1)
$$

[7] Our approach gives other limit theorems (large deviation estimates, invariance principles, see [DZ1, Sect. 6]).
[8] The standard choice for $t = 1$ is $q = 2$.

and similarly $\mathbb{H}_{-k}$ is defined approaching $\varphi = -\pi/2$. A connected component of $\mathbb{H}_k$, for some $|k| \geq k_0$, or of the set $\mathbb{H}_0 = \{(r,\varphi) : k_0^{-q} \leq \min\{\frac{\pi}{2} - \varphi, -\frac{\pi}{2} - \varphi\}\}$ is called a *homogeneity strip.* We let $\mathcal{H}$ denote the partition of $M$ into homogeneity strips. Let $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{|k|\geq k_0} \partial \mathbb{H}_k)$ and, for $n \in \mathbb{Z}$, let $\mathcal{S}_n^{\mathbb{H}} = \cup_{i=0}^{-n} T^i \mathcal{S}_0^{\mathbb{H}}$ denote the *extended singularity set for $T^n$.*

Fix[9] $\delta_0 \in (0,1)$. Let $\mathcal{W}^s$ denote the set of all nontrivial connected subsets $W$ of local stable manifolds of $T$ of length at most $\delta_0$. Such curves have curvature bounded above by a fixed constant [CM, Prop 4.29], and $T^{-n}\mathcal{W}^s = \mathcal{W}^s$ for all $n \geq 1$, up to subdivision of curves according to the length scale $\delta_0$. Let $\mathcal{W}^s_{\mathbb{H}} \subset \mathcal{W}^s$ denote the set of nontrivial connected subsets $W$ of elements of $\mathcal{W}^s$ with the property that $T^n W$ belongs to a single homogeneity strip for each $n \geq 0$. Such curves are called[10] *homogeneous stable manifolds.*

We call a $C^2$ curve $W \subset M$ (cone) stable if at each point $x$ in $W$, the tangent vector $\mathcal{T}_x W$ to $W$ lies in $\mathcal{C}^s$. We denote by $\widehat{\mathcal{W}}^s$ the set of (cone) stable curves with second derivative bounded by a constant chosen sufficiently large ([CM, Prop 4.29]) so that $T^{-n}\widehat{\mathcal{W}}^s \subset \widehat{\mathcal{W}}^s$ for all $n \geq 1$, up to subdivision of curves according to $\delta_0$. Finally, $\widehat{\mathcal{W}}^s_H \subset \widehat{\mathcal{W}}^s$ is the set of elements of $\widehat{\mathcal{W}}^s$ contained in a single homogeneity strip, while $\mathcal{W}^s_H$ is the set of elements of $\mathcal{W}^s$ that are contained in a single homogeneity strip. Such curves are called *weakly homogeneous* (cone) stable curves and stable manifolds, respectively. Obviously, $\mathcal{W}^s_{\mathbb{H}} \subset \mathcal{W}^s_H \subset \mathcal{W}^s \subset \widehat{\mathcal{W}}^s$ and $\mathcal{W}^s_H \subset \widehat{\mathcal{W}}^s_H$.

For every $W \in \widehat{\mathcal{W}}^s$, let $C^1(W)$ denote the space of $C^1$ functions on $W$, and for every $\eta \in (0,1)$ let $C^\eta(W)$ denote the closure[11] of $C^1(W)$ for the $\eta$-Hölder norm defined by

$$|\psi|_{C^\eta(W)} = \sup_W |\psi| + H^\eta_W(\psi)\,,\ H^\eta_W(\psi) = \sup_{\substack{x,y\in W\\ x\neq y}} \frac{|\psi(x)-\psi(y)|}{d(x,y)^\eta}\,. \tag{2.2}$$

The following lemma extends standard distortion bounds for homogeneous curves to all exponents $t > 0$. (See Lemma 6.2 for a further generalisation.)

**Lemma 2.1.** *There exists $\bar\delta_0 > 0$ and $C_d > 0$, depending on $k_0$ and $q$, such that for all $\delta_0 < \bar\delta_0$, all $n \geq 0$, and any $W \in T^{-n}\widehat{\mathcal{W}}^s$ such that $T^i W \in \widehat{\mathcal{W}}^s_{\mathbb{H}}$ for each $i = 0, \dots, n-1$, we have*

$$\left|1 - \frac{|J_W T^n(x)|^t}{|J_W T^n(y)|^t}\right| \leq 2^t C_d d(x,y)^{1/(q+1)}\,,\ \ \forall\, x,\, y \in W\,,\ \forall t > 0\,,$$

*where $J_W T^n(x) = |\det(DT^n_x|\mathcal{T}_x W)|$ denotes the Jacobian of $T^n$ along $W$.*

*Proof.* There exists $C_d < \infty$, independent of $\delta_0$, but depending on $k_0$ and $q$ such that

$$\left|1 - \frac{J_W T^n(x)}{J_W T^n(y)}\right| \leq C_d d(x,y)^{1/(q+1)}\,,\ \ \forall\, x,\, y \in W\,,\ \forall\, W \text{ as in the lemma.} \tag{2.3}$$

(For $q = 2$, see e.g. [CM, Lemma 5.27] or [DZ1, App. A]. The proofs there give (2.3) for all $q > 1$.)

For $t \leq 1$, the estimate is an immediate consequence of (2.3), since for all $A > 0$, we have $|1 - A^t| \leq |1 - A|$. Now choose $\bar\delta_0$ such that $C_d \bar\delta_0^{1/(q+1)} \leq 3/4$. Then, for $t > 1$, we set $A = \frac{J_W T^n(x)}{J_W T^n(y)}$. By (2.3), this implies that $1/4 \leq A \leq 2$ if $\delta_0 < \bar\delta_0$. For $A$ in this range, we have, again using (2.3), that $|1 - A^t| \leq 2^t |1 - A| \leq 2^t C_d d(x,y)^{1/(q+1)}$. □

[9] The index $k_0 = k_0(t_0, t_1)$ and the length scale $\delta_0 = \delta_0(t_0, t_1) < 1$ will be chosen in Definition 3.2.

[10] In [CM], these curves are called $H$-manifolds. This strong notion of homogeneity is needed to prove Hölder continuity of the conditional densities of the SRB measure decomposed along stable manifolds – needed to get valid test functions for our spaces — using the asymptotic limit of the ratio of stable Jacobians, forward iterates must be contained in a single homogeneity strip (so that the ration remains bounded).

[11] Using the closure of $C^1$ will give injectivity of the inclusion of the strong space in the weak one in Proposition 4.2.

Next, recalling that $\mathcal{S}_k^{\mathbb{H}} = \cup_{i=0}^{-k} T^i \mathcal{S}_0^{\mathbb{H}}$, define for $n \geq 1$,

$$
\begin{aligned}
\mathcal{M}_0^{n,\mathbb{H}} &= \text{maximal connected components of } M \setminus \left( T^{-n}\mathcal{S}_0 \cup \mathcal{S}_{n-1}^{\mathbb{H}} \right), \\
\mathcal{M}_{-n}^{0,\mathbb{H}} &= \text{maximal connected components of } M \setminus \left( \mathcal{S}_0 \cup T(\mathcal{S}_{-(n-1)}^{\mathbb{H}}) \right) , \qquad (2.4)\\
\mathcal{M}_{-k}^{n,\mathbb{H}} &= \mathcal{M}_{-k}^{0,\mathbb{H}} \bigvee \mathcal{M}_0^{n,\mathbb{H}} , \ k \geq 1 .
\end{aligned}
$$

We comment on the use of $\mathcal{S}_0^{\mathbb{H}}$ in (2.4). First notice (just like for the sets $\mathcal{M}_{-k}^n$ defined in the beginning of this subsection) that if $A \in \mathcal{M}_0^{n,\mathbb{H}}$, then $T^k A \in \mathcal{M}_{-k}^{n-k,\mathbb{H}}$ for each $0 \leq k \leq n$, and $T^k A$ is a union of elements of $\mathcal{M}_{-k}^{0,\mathbb{H}}$ for each $k > n$. Next, if $W \in \widehat{\mathcal{W}}_{\mathbb{H}}^s$ is such that $V = T^{-1}W$ is a single curve, then $J_W T^{-1}(x) \approx 1/\cos\varphi(T^{-1}x)$ while $J_V T(y) \approx \cos\varphi(y)$. Thus by (2.3), the definitions in (2.4) guarantee that for any $W \in \widehat{\mathcal{W}}_{\mathbb{H}}^s$ such that $W \subset A \in \mathcal{M}_{-n}^{0,\mathbb{H}}$, the Jacobian $J_W T^{-n}$ has bounded distortion on $W$, while $J_{T^{-n}W} T^n$ has bounded distortion on $T^{-n}W$ (which is contained in a single element of $\mathcal{M}_0^{n,\mathbb{H}}$).

We shall also need the following distortion bound.

**Lemma 2.2** (Distortion Relative to $\mathcal{M}_{-n}^{0,\mathbb{H}}$)**.** *There exists $C > 0$ such that for all $n \geq 1$, for all $U, V \in \widehat{\mathcal{W}}_{\mathbb{H}}^s$ such that $U, V \subset A \in \mathcal{M}_{-n}^{0,\mathbb{H}}$, and all*[12] *$u \in \bar{U} \setminus \mathcal{S}_{-n}$, $v \in \bar{V} \setminus \mathcal{S}_{-n}$,*

$$
\left| \log \frac{J_U T^{-n}(u)}{J_V T^{-n}(v)} \right| \leq C .
$$

The bound above is more general (and weaker) than the usual distortion bound along stable curves given by (2.3) or between stable curves given by [CM, Theorem 5.42] (or more generally [DZ1, Appendix A]) since we do not assume that the points $u$, $v$ in $\bar{A}$, with $A \in \mathcal{M}_{-n}^{0,\mathbb{H}}$, lie on the same stable or unstable curve.

*Proof.* Let $n \geq 1$, $u \in \bar{U}$, $v \in \bar{V}$, be as in the statement of the lemma. Define $u_i = T^{-i}u$, $v_i = T^{-i}v$ for $i = 0, \dots, n$, and notice that $u_i, v_i$ belong to the closure of the same element of $\mathcal{M}_{-n+i}^{i,\mathbb{H}}$. By the uniform hyperbolicity of $T$, for $i = 0, \dots, n$, if $A \in \mathcal{M}_{-n+i}^{i,\mathbb{H}}$, then $\mathrm{diam}^u(\bar{A}) \leq C\Lambda^{-i}$ and $\mathrm{diam}^s(\bar{A}) \leq C\Lambda^{-n+i}$, where $\mathrm{diam}^u(B)$ is the maximum length of an unstable curve in $B$, and $\mathrm{diam}^s(B)$ is the maximum length of a stable curve in $B$. Thus, due to the uniform transversality of $\mathcal{C}^s$ and $\mathcal{C}^u$, we have

$$
d(u_i, v_i) \leq \bar{C} \max\{\Lambda^{-i}, \Lambda^{-n+i}\} . \qquad (2.5)
$$

By the time-reversal of [CM, eq. (5.24)], we have that

$$
\log J_{U_i} T^{-1}(u_i) = \log \frac{\cos\varphi(u_i) + \tau(u_{i+1})(\mathcal{K}(u_i) - \mathcal{V}(u_i))}{\cos\varphi(u_i)} + \log \frac{\sqrt{1 + \mathcal{V}(u_{i+1})^2}}{\sqrt{1 + \mathcal{V}(u_i)^2}} , \qquad (2.6)
$$

where $\mathcal{V}(u_i) = \frac{d\varphi}{dr}(u_i) < 0$ is the slope of the tangent line to $U_i$ at $u_i$. Summing over $i$, the last term above telescopes and the sum is uniformly bounded away from 0 and $\infty$, giving,

$$
\left| \log \frac{J_U T^{-n}(u)}{J_V T^{-n}(v)} \right| \leq C + \sum_{i=0}^{n-1} \left| \log \frac{\cos\varphi(v_{i+1})}{\cos\varphi(u_{i+1})} \right| + \left| \log \frac{\cos\varphi(u_i) + \tau(u_{i+1})(\mathcal{K}(u_i) - \mathcal{V}(u_i))}{\cos\varphi(v_i) + \tau(v_{i+1})(\mathcal{K}(v_i) - \mathcal{V}(v_i))} \right| \qquad (2.7)
$$

Since $u_{i+1}$, $v_{i+1}$ lie in the same homogeneity strip for each $i$, using (2.1) we have

$$
\left| \log \frac{\cos\varphi(v_{i+1})}{\cos\varphi(u_{i+1})} \right| \leq C \frac{|\varphi(u_{i+1}) - \varphi(v_{i+1})|}{\cos\varphi(u_{i+1})} \leq C d(u_{i+1}, v_{i+1})^{1/(q+1)} . \qquad (2.8)
$$

[12] $\bar{U}$ denotes the closure of $U$ in $M$. The distortion bounds on $U$ and $V$ extend trivially to the boundaries of homogeneity strips, but not to real singularity lines, hence $\bar{U} \setminus \mathcal{S}_{-n}$.

Next, the terms in the second set on the right-hand side of (2.7) are bounded and the denominator in the expression is at least $\tau_{\min}\mathcal{K}_{\min} > 0$. Moreover, $\mathcal{K}$ is differentiable while $\tau$ is 1/2-Hölder continuous.[13] Thus following [CM, eq. (5.26)], we have

$$\sum_{i=0}^{n-1} \left|\log \frac{\cos\varphi(u_i) + \tau(u_{i+1})(\mathcal{K}(u_i) - \mathcal{V}(u_i))}{\cos\varphi(v_i) + \tau(v_{i+1})(\mathcal{K}(v_i) - \mathcal{V}(v_i))}\right| \le C \sum_{i=0}^{n-1} d(u_{i+1}, v_{i+1})^{1/2} + d(u_i, v_i) + |\Delta\mathcal{V}_i|,$$

where $\Delta\mathcal{V}_i = \mathcal{V}(u_i) - \mathcal{V}(v_i)$. By (2.5), the sums over all terms in (2.7) involving $d(u_i, v_i)$ are uniformly bounded in $n$. It remains to estimate $\sum_{i=0}^{n-1} |\Delta\mathcal{V}_i|$. By [CM, eq. (5.29)] and (2.5), we bound $|\Delta\mathcal{V}_i|$ by

$$\begin{aligned} C\Big(|\Delta\mathcal{V}_0|\Lambda^{-i} + \sum_{j=0}^{i} \Lambda^{-j} d(u_{i-j}, v_{i-j})^{1/2}\Big) &\le C\big(|\Delta\mathcal{V}_0|\Lambda^{-i} + \sum_{j=0}^{i} \Lambda^{-j}(\Lambda^{(-i+j)/2} + \Lambda^{(-n+i-j)/2})\big) \\ &\le C'\big(|\Delta\mathcal{V}_0|\Lambda^{-i} + \Lambda^{-i/2} + \Lambda^{(-n+i)/2}\big). \end{aligned}$$

Summing over $i$ completes the proof of the lemma. □

2.2. **Topological Formulation $P_*(t,g)$ of the Pressure $P(t,g)$. Theorem 2.4.** In view of our proof of uniqueness in §5.5 (which uses differentiability of the pressure), for a $C^1$ function $g : M \to \mathbb{R}$ and $n \ge 1$, we set $S_n g = \sum_{i=0}^{n-1} g \circ T^i$. The hyperbolicity of $T$ implies the following distortion bounds: There exists $C_* < \infty$ such that for all $n \ge 1$

$$(2.9) \qquad |e^{S_n g(x) - S_n g(y)} - 1| \le C_* |\nabla g|_{C^0}\, d(x, y)\,,\ \forall W \in \widehat{\mathcal{W}}^s \text{ such that } T^i W \in \widehat{\mathcal{W}}^s\,,\ \forall\, 0 \le i \le n\,.$$

Recalling (1.4), we define (aside from §5.5 we only need $g \equiv 0$),

$$(2.10) \qquad Q_n(t,g) = \sum_{A \in \mathcal{M}_0^{n,\mathbb{H}}} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)}\,,\ \ Q_n(t) = Q_n(t,0)\,,\ \ n \ge 1\,,$$

and

$$(2.11) \qquad P_*(t,g) = \limsup_{n\to\infty} \frac{1}{n} \log Q_n(t,g)\,,\ \ P_*(t) := P_*(t,0).$$

We will show the following result in Section 2.3:

**Proposition 2.3** (Topological Pressure)**.** *For all $t > 0$ and $g \in C^1$, we have*[14] *$P(t,g) \le P_*(t,g)$.*

For $t_* > 1$ given by (1.9), the analysis carried out in Sections 3–5 will yield:

**Theorem 2.4** ((Strong[15]) Variational Principle)**.** *If $t \in (0, t_*)$, then $P_*(t) = P(t)$ and the supremum is attained at the unique invariant measure $\mu_t$ from Theorem 1.1.*

*Proof.* This follows from Proposition 2.3, Theorem 4.1, Corollary 5.3, and Proposition 5.7. □

Theorem 5.8 will give the generalisation of the above strong form of the variational principle to $P(t,g) = P_*(t,g)$ for suitable $g$.

We first establish basic properties of $P_*(t,g)$:

**Proposition 2.5.** *For each $t > 0$ and $g \in C^1$ the limsup (2.11) defining $P_*(t,g)$ is a limit in $(-\infty, \infty)$. The function $t \mapsto P_*(t,g)$ is convex and strictly decreasing on $(0,\infty)$.*

**Remark 2.6.** *It is not hard to show, using Lemma 2.2, that for each $t > 0$, there exists $C_D > 0$ such that $Q_n(t) \le C_D^t \sum_{A \in \mathcal{M}_0^{n,\mathbb{H}}} \inf_{x \in A \cap M'} |J^s T^n(x)|^t$ for all $n \ge 1$, so that replacing the supremum by an infimum in the definition of $Q_n(t)$ does not change the value of $P_*(t)$.*

---

[13] We cannot take advantage of the smoother bounds on $\tau$ given by [CM, eq. (5.28)] since our points $u_i$ and $v_i$ may lie on different stable or unstable manifolds.

[14] Recall our convention that $\int_M \log J^u T\, d\mu = \infty$ if $\mu(M \setminus M') > 0$.

[15] By "strong" we mean that the supremum is a maximum, and it is attained at a unique measure.

*Proof of Proposition 2.5.* We first set $g = 0$. The partition $\mathcal{M}_0^1$ is finite, and each element of $\mathcal{M}_0^1$ is subdivided by curves in $\mathcal{S}_0^{\mathbb{H}}$ to comprise a union of elements of $\mathcal{M}_0^{1,\mathbb{H}}$, according to (2.4). Thus,

$$Q_1(t) = \sum_{A\in\mathcal{M}_0^{1,\mathbb{H}}} \sup_{x\in A\cap M'} |J^sT(x)|^t \le C \sum_{E\in\mathcal{M}_0^1}\sum_{k\ge k_0} \sup_{x\in\mathbb{H}_k} |\cos\varphi(x)|^t \le C\#\mathcal{M}_0^1 \sum_{k\ge k_0} k^{-qt},$$

and the sum converges since $qt \ge 2 > 1$. We next show that $Q_n(t)$ is submultiplicative:

$$\begin{aligned} Q_{n+k}(t) &= \sum_{A\in\mathcal{M}_0^{n,\mathbb{H}}} \sum_{\substack{B\in\mathcal{M}_0^{n+k,\mathbb{H}}\\ B\subset A}} \sup_{x\in B\cap M'} |J^sT^{n+k}(x)|^t \\ &\le \sum_{A\in\mathcal{M}_0^{n,\mathbb{H}}} \sup_{y\in A\cap M'} |J^sT^n(y)|^t \sum_{\substack{B\in\mathcal{M}_0^{n+k,\mathbb{H}}\\ B\subset A}} \sup_{x\in B\cap M'} |J^sT^k(T^nx)|^t,\ \forall k,n\ge 1. \end{aligned} \tag{2.12}$$

Notice that if $B, B' \subset A \in \mathcal{M}_0^{n,\mathbb{H}}$ are distinct elements of $\mathcal{M}_0^{n+k,\mathbb{H}}$, then $T^nB, T^nB' \in \mathcal{M}_{-n}^{k,\mathbb{H}} = \mathcal{M}_{-n}^{0,\mathbb{H}} \bigvee \mathcal{M}_0^{k,\mathbb{H}}$ are both contained in $T^nA \in \mathcal{M}_{-n}^{0,\mathbb{H}}$ and so must lie in distinct elements of $\mathcal{M}_0^{k,\mathbb{H}}$. Thus the inner sum in (2.12) is bounded by $Q_k(t)$ for each $A$, and the outer sum is bounded by $Q_n(t)$, proving submultiplicativity. If $g \neq 0$, it is easy to see that we also have $Q_{n+k}(t,g) \le Q_n(t,g)Q_k(t,g)$. Therefore, since $Q_1(t,g) < \infty$, the sequence in (2.11) converges to a limit in $[-\infty,\infty)$.

To see that $P_*(t,g) > -\infty$, let $x_p$ be a periodic point of period $p$ with no tangential collisions,[16] and let $\chi_p^-$ denote the negative Lyapunov exponent of $x_p$. Then, $Q_{np}(t,g) \ge |J^sT^{np}(x_p)|^t e^{S_{np}g(x_p)} = e^{npt\chi_p^-} e^{nS_pg(x_p)}$, and so $P_*(t,g) \ge t\chi_p^- e^{S_pg(x_p)} > -\infty$.

To prove convexity, pick $t, t' > 0$ and $\alpha \in [0,1]$. Then using the Hölder inequality,

$$\begin{aligned} Q_n(\alpha t + (1-\alpha)t', g) &= \sum_{A\in\mathcal{M}_0^{n,\mathbb{H}}} \sup_{x\in A\cap M'} |J^sT^n|^{\alpha t+(1-\alpha)t'} e^{\alpha S_ng(x)} e^{(1-\alpha)S_ng(x)} \\ &\le \Big(\sum_{A\in\mathcal{M}_0^{n,\mathbb{H}}} \sup_{x\in A\cap M'} |J^sT^n|^t e^{S_ng(x)}\Big)^\alpha \Big(\sum_{A\in\mathcal{M}_0^{n,\mathbb{H}}} \sup_{x\in A\cap M'} |J^sT^n|^{t'} e^{S_ng(x)}\Big)^{1-\alpha} = Q_n(t,g)^\alpha Q_n(t',g)^{1-\alpha}. \end{aligned}$$

Taking logarithms, dividing by $n$, and letting $n \to \infty$ proves convexity.

Next, fixing $t > 0$ and applying (1.2), we find for $s > 0$,

$$Q_n(t+s,g) = \sum_{A\in\mathcal{M}_0^{n,\mathbb{H}}} \sup_{x\in A\cap M'} |J^sT^n|^{t+s} e^{S_ng(x)} \le C_1^{-s}\Lambda^{-ns} Q_n(t,g),$$

so that $P_*(t+s,g) \le P_*(t,g) - s\log\Lambda$, that is, $P_*(t,g)$ is strictly decreasing in $t$. □

2.3. **Proof that $P_*(t,g) \ge P(t,g)$ (Proposition 2.3).** If $\mathcal{Q}$ is a partition of $M$ we let $\operatorname{Int}\mathcal{Q}$ denote the set of interiors of elements of $\mathcal{Q}$. In [BD], we worked with $\mathcal{P}$, the (finite) partition of $M$ into maximal connected sets on which $T$ and $T^{-1}$ are continuous, noticing that the set $\operatorname{Int}\mathcal{P}$ coincides with $\mathcal{M}_{-1}^1$, while the refinements $\mathcal{P}_{-k}^n = \bigvee_{i=-k}^n T^{-i}\mathcal{P}$ may also contain isolated points if three or more scatterers have a common tangential trajectory (see [BD, Fig.1]). (Note that $\mathcal{P}$ is a set-theoretical partition: zero measure sets do not need to be ignored.) We also observed that $\mathcal{P}$ is a generator for any $T$-invariant Borel probability measure $\mu$, since $\bigvee_{i=-\infty}^{\infty} T^{-i}\mathcal{P}$ separates[17] points in the compact metric space $M$: if $x \neq y$ there exists $k \in \mathbb{Z}$ such that $T^k(x)$ and $T^k(z)$ lie in different elements of $\mathcal{P}$. Let $\bar{\mathcal{P}}$ be the partition of $M$ into maximal connected sets on which $T$ is continuous. Then $\bar{\mathcal{P}} = \mathcal{P} \bigvee T(\bar{\mathcal{P}})$, so $\bar{\mathcal{P}}$ is also a generator for $T$. We have $\operatorname{Int}\bar{\mathcal{P}} = \mathcal{M}_0^1$. More generally, $\operatorname{Int}(\bigvee_{k=0}^{n-1} T^{-k}\bar{\mathcal{P}}) = \mathcal{M}_0^n$ for $n \ge 1$.

---

[16]Such a periodic point always exists. For example, since two adjacent scatterers are in convex opposition, there is a period 2 orbit whose trajectory is normal to both scatteres.

[17]In fact, all points $x \neq y$ may be separated while the definition of a generator in [Pa] allows a zero measure set of pathological pairs.

*Proof of Proposition 2.3.* If a $T$-invariant probability measure $\mu$ gives positive weight to $M'$ or, more generally, if $\int_M \log J^uT\,d\mu = \infty$, then $P_\mu(t,g) = -\infty$, so $P_\mu(t,g) < P_*(t,g)$. We can thus assume without loss of generality that $\mu$ is a $T$-invariant probability measure with $\int_M \log J^uT\,d\mu < \infty$, in particular $\mu(\mathcal{S}_n) = 0$ for each $n \in \mathbb{Z}$. Then

$$H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}\bar{\mathcal{P}}\right) = H_\mu(\mathcal{M}_0^n)\,, \tag{2.13}$$

since the boundary of any element of $\bigvee_{k=0}^{n-1} T^{-k}\bar{\mathcal{P}}$ is contained in $\mathcal{S}_n$.

Since $\bar{\mathcal{P}}$ is a generator, we have $h_\mu(T) = h_\mu(T,\bar{\mathcal{P}})$ for any $T$-invariant probability measure $\mu$ on $M$ (see e.g. [W, Theorem 4.17]). Then, using (1.6), we find, adapting the classical argument (see e.g. [W, Prop 9.10]), that

$$\begin{aligned}
h_\mu(T,\bar{\mathcal{P}}) - t\int_M \log J^uT\,d\mu &= \lim_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{k=0}^{n-1}T^{-k}\bar{\mathcal{P}}\right) + \lim_{n\to\infty}\frac{1}{n}\int_M t\sum_{k=0}^{n-1}\log J^sT\circ T^k\,d\mu \\
&\le \lim_{n\to\infty}\frac{1}{n}\Big(\sum_{A\in\mathcal{M}_0^n}\mu(A)\big[-\log\mu(A) + \sup_{A\cap M'} t\log J^sT^n\big]\Big) \\
&\le \lim_{n\to\infty}\frac{1}{n}\sum_{A\in\mathcal{M}_0^n}\mu(A)\log\frac{\sup_{A\cap M'}|J^sT^n|^t}{\mu(A)} \le \lim_{n\to\infty}\frac{1}{n}\log\sum_{A\in\mathcal{M}_0^n}\sup_{A\cap M'}|J^sT^n|^t\,,
\end{aligned}$$

where we used (2.13) in the second line, and the convexity of the logarithm in the third line. Finally, notice that each element of $\mathcal{M}_0^n$ is a union of elements of $\mathcal{M}_0^{n,\mathbb{H}}$, modulo the boundaries of homogeneity strips. But since the distortion bound Lemma 2.2 extends to the boundaries of homogeneity strips, we have

$$\sup_{A\cap M'}|J^sT^n|^t = \sup_{\substack{B\in\mathcal{M}_0^{n,\mathbb{H}}\\ B\subset A}}\sup_{B\cap M'}|J^sT^n|^t \le \sum_{\substack{B\in\mathcal{M}_0^{n,\mathbb{H}}\\ B\subset A}}\sup_{B\cap M'}|J^sT^n|^t\,,$$

for each $A\in\mathcal{M}_0^n$. Using this bound in the previous estimate and applying Proposition 2.5 implies $h_\mu(T) - t\int_M \log J^uT\,d\mu \le P_*(t)$ for every $T$-invariant probability measure $\mu$.

If $g\neq 0$, we may write

$$\begin{aligned}
h_\mu(T,\bar{\mathcal{P}}) + \int(t\log J^sT + g)\,d\mu &= \lim_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{k=0}^{n-1}T^{-k}\bar{\mathcal{P}}\right) + \lim_{n\to\infty}\frac{1}{n}\int S_n(t\log J^sT + g)\,d\mu \\
&\le \lim_{n\to\infty}\frac{1}{n}\log\sum_{A\in\mathcal{M}_0^n}\sup_{A\cap M'}|J^sT^n|^t e^{S_ng}\,,
\end{aligned}$$

and this last expression is bounded by $P_*(t,g)$ by the same reasoning as above, using that the analogue of Lemma 2.2 holds for $e^{S_ng}$: Recalling (2.9), for all $n>0$, all $A\in\mathcal{M}_0^{n,\mathbb{H}}$, and $x,y\in A$, since the diameter of $T^iA$ is bounded by (2.5),

$$e^{S_ng(x)-S_ng(y)} \le 1 + \bar{C}\,C_*\cdot|\nabla g|_{C^0}\,. \tag{2.14}$$

$\square$

## 3. Growth Lemmas

In this section, after preliminaries in §3.1, introducing in particular the contraction rate $\theta$ and sets $\mathcal{G}_n(W)$ appearing when iterating the transfer operator $\mathcal{L}_t$, we prove a series of growth and complexity lemmas which will allow us to control the sums over $\mathcal{G}_n(W)$ for $W\in\widehat{\mathcal{W}}^s$. This culminates in the lower bound of Proposition 3.14, which implies exact exponential growth (Proposition 3.15) of $Q_n(t,g)$. (This exact exponential growth is essential to control the peripheral spectrum of $\mathcal{L}_t$.)

Since $J^sT$ is not bounded away from zero, we shall use different strategies for $t \in (0,1]$ and $t > 1$. Several important growth lemmas are proved for $t \in (0,1]$ in Sections 3.3 and 3.5. We then use the results for $t \le 1$ to bootstrap an analogous set of lemmas for $t > 1$ in Sections 3.4, 3.5, and 3.6.

3.1. **One-Step Expansion: $\theta(t_1)$. Choice of $q(t_0)$, $k_0(t_0, t_1)$, $\delta_0(t_0, t_1)$. $\mathcal{G}_n(W)$, $\mathcal{I}_n(W)$.** We begin by proving an adaptation of the one-step expansion (see e.g. [CM, Lemma 5.56]) for our choice of potential and homogeneity strips. Using the notation from (2.6), recall $t_* > 1$ from (1.9), and the adapted metric from [CM, Section 5.10]:

$$\|dx\|_* = \frac{\mathcal{K} + \mathcal{V}}{\sqrt{1+\mathcal{V}^2}} \|dx\| \,.$$

**Lemma 3.1** (One-Step Expansion)**.** *For $t_1 \in (1, t_*)$, fix*[18] *$\theta \in (\Lambda^{-1}, \Lambda^{-1/2})$ such that $\theta^{t_1} < e^{P_*(t_1)}$. Then for each $\bar t_0 \in (0,1)$ and $q > 2/\bar t_0$, there exist $\bar k_0(\bar t_0, t_1, q) \ge 1$ and $\bar\delta_0(\bar t_0, t_1, q) > 0$ such that*

$$\sum_i |J_{V_i}T|^t_{C^0(V_i),*} < \theta^t \,, \quad \forall W \in \widehat{\mathcal{W}}^s \text{ with } |W| < \bar\delta_0 \,, \ \forall t \ge \bar t_0 \,, \tag{3.1}$$

*where the $V_i$ range over the maximal*[19] *connected weakly $(q, \bar k_0)$-homogeneous components of $T^{-1}W$, and $|J_{V_i}T|_{C^0(V_i),*}$ denotes the maximum on $V_i$ of the Jacobian of $T$ along $V_i$ for the metric $\|\cdot\|_*$.*

*Proof.* Note that $|J_{V_i}T|_{C^0(V_i),*} \le \Lambda^{-1}$ and, if $V_i \subset \mathbb{H}_k$, then $|J_{V_i}T|_{C^0(V_i),*} \le Ck^{-q}$ for some $C > 0$ [CM, eq. (5.36)]. There exists $\bar\delta > 0$ such that if $W \in \widehat{\mathcal{W}}^s$ with $|W| \le \bar\delta$, then $T^{-1}W$ has at most $\frac{\tau_{\max}}{\tau_{\min}} + 1$ connected components, and all but at most one component experience nearly tangential collisions (see [CM, Sect. 5.10]).

For $\bar t_0 \in (0,1)$ and $q > 2/\bar t_0$, choose $\bar k_0 = \bar k_0(\bar t_0, t_1, q)$ such that

$$\Lambda^{-\bar t_0} + \frac{\tau_{\max}}{\tau_{\min}} \sum_{k \ge \bar k_0} C^{\bar t_0} k^{-q\bar t_0} \le \Lambda^{-\bar t_0} + \frac{\tau_{\max}}{\tau_{\min}} C^{\bar t_0}\, \bar k_0^{-1} < \theta^{\bar t_0} \,. \tag{3.2}$$

For all $W \in \widehat{\mathcal{W}}^s$, we have $|T^{-1}W| \le C'|W|^{1/2}$ [CM, Exercise 4.50] for some $C' > 0$ independent of $W$. Next, choose $\bar\delta_0(\bar t_0, t_1)$ so small that $C'\bar\delta_0^{1/2} \le \bar k_0^{-q}$, so that if $|W| \le |\bar\delta_0|$, then each component of $T^{-1}W$ making a nearly tangential collision lies in a union of homogeneity strips $\mathbb{H}_k$ for $k \ge \bar k_0$.

Then if $|W| \le \bar\delta_0$, the quantity $\sum_i |J_{V_i}T|^{\bar t_0}_{C^0(V_i),*}$ is bounded by the left-hand side of (3.2), proving (3.1) for $t = \bar t_0$. Finally, for all $t \ge \bar t_0$,

$$\sup_{\substack{W \in \widehat{\mathcal{W}}^s \\ |W| \le \bar\delta_0(\bar t_0)}} \sum_i |J_{V_i}T|^t_{C^0(V_i),*} \le \Lambda^{-t+\bar t_0} \sup_{\substack{W \in \widehat{\mathcal{W}}^s \\ |W| \le \bar\delta_0(\bar t_0)}} \sum_i |J_{V_i}T|^{\bar t_0}_{C^0(V_i),*} \le \Lambda^{-t+\bar t_0}\theta^{\bar t_0} \le \theta^t \,. \tag{3.3}$$

□

We now choose the parameters defining $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}^s_{\mathbb{H}}$ and $\mathcal{W}^s$, $\mathcal{W}^s_{\mathbb{H}}$ depending on $t_0 \in (0,1)$, $t_1 \in (1, t_*)$:

**Definition 3.2.** *Given $t_0 \in (0,1)$, $t_1 \in (1, t_*)$, we fix $q(t_0) > 1$ such that $qt_0/2 \ge 2$, and fix $\theta = \theta(t_1)$, $k_0 = k_0(t_0, t_1, q) := \bar k_0(\frac{t_0}{2}, t_1, q)$, $\delta_0 = \delta_0(t_0, t_1, q) := \bar\delta_0(\frac{t_0}{2}, t_1, q)$ as in Lemma 3.1. Reduce $\delta_0$ if needed so that $C_d \delta_0^{\frac{1}{q+1}} \le 3/4$, with $C_d$ from (2.3). This choice of $(t_0, t_1)$, $\theta$, $q$, $\delta_0$, and $k_0$ determines the set of stable curves $\widehat{\mathcal{W}}^s$, $\widehat{\mathcal{W}}^s_{\mathbb{H}}$ and stable manifolds $\mathcal{W}^s$, $\mathcal{W}^s_{\mathbb{H}}$.*

Our proofs use sets $\mathcal{G}_n(W)$, $\mathcal{I}_n(W)$ associated with $\delta_0$ and $k_0$, and, for $\delta < \delta_0$, also $\mathcal{G}^\delta_n(W)$, $\mathcal{I}^\delta_n(W)$:

For $W \in \widehat{\mathcal{W}}^s$, we let $\mathcal{G}_1(W)$ denote the maximal, weakly homogeneous, connected components of $T^{-1}W$, with long pieces subdivided to have length between $\delta_0/2$ and $\delta_0$. Inductively, we define[20]

[18] This is possible by definition of $t_*$ and Proposition 2.3. It implies $\theta^t < e^{P_*(t)}$ for $t \le t_1$ since $P_*(t)$ is decreasing.

[19] $W$ is not necessarily weakly homogeneous, but each $V_i$ is, using parameters $q$ and $\bar k_0$ for the homogeneity strips.

[20] This definition of $\mathcal{G}_n(W)$ is as in [DZ1, DZ2], but different from [BD] where homogeneity was not required.

$\mathcal{G}_n(W) = \cup_{W_i \in \mathcal{G}_{n-1}(W)} \mathcal{G}_1(W_i)$. Thus $\mathcal{G}_n(W)$ is the countable collection of subcurves of $T^{-n}W$ subdivided according to the extended singularity set $\mathcal{S}^{\mathbb{H}}_{-n}$, and $T^j(V)$ is weakly homogeneous for all $0 \le j \le n-1$ and all $V \in \mathcal{G}_n(W)$, in particular $\mathcal{G}_n(W) \subset \widehat{\mathcal{W}}^s_{\mathbb{H}}$. For each $n \ge 1$, let $L_n(W)$ denote the elements of $\mathcal{G}_n(W)$ whose length is at least $\delta_0/3$. Let $\mathcal{I}_n(W)$ denote those elements $W_i \in \mathcal{G}_n(W)$ such that $T^k W_i$ is never contained in an element of $L_{n-k}(W)$ for all $k = 0, \dots, n-1$.

Finally, for $\delta < \delta_0$, define $\mathcal{G}^\delta_n(W)$ like $\mathcal{G}_n(W)$, but subdividing long pieces into pieces of length between $\delta/2$ and $\delta$. Similarly, denote by $L^\delta_k(W)$ those elements of $\mathcal{G}^\delta_k(W)$ having length at least $\delta/3$, and by $\mathcal{I}^\delta_n(W)$ those elements $W_i \in \mathcal{G}^\delta_n(W)$ such that $T^k W_i$ has never been contained in an element of $L^\delta_{n-k}(W)$ for all $k = 0, \dots, n-1$.

3.2. **Initial Lemmas for all $t > 0$.** We start with two easy lemmas. (In the present paper, the parameter $\varsigma$ appearing in Lemmas 3.3 and 3.4 will be zero or $1/p$, for $p > q+1$ chosen in (4.1).)

**Lemma 3.3.** *Fix $t_0 \in (0,1)$. There exists $C_0 = C_0(t_0) > 0$ such that for all $g \in C^0$, every $t \ge t_0$, all $t_1 \in (1, t_*)$, and all $0 \le \varsigma < \frac{2t - t_0}{2 - t_0}$ if $t_0 < 2$, all $0 \le \varsigma < 1$ if $t_0 \ge 2$, we have*

$$\sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \le C_0 \theta^{n(t-\varsigma)} e^{n|g|_{C^0}} \, , \; \forall W \in \widehat{\mathcal{W}}^s \, , \; \forall n \ge 1 \, . \tag{3.4}$$

*Proof.* The case $\varsigma = 0$, $g = 0$ can be proved by induction on $n$ using (3.1), since elements of $\mathcal{I}_n(W)$ have been short at each intermediate step. This is the same as in [DZ1, Lemma 3.1] (the exponent $t$ changes nothing), and the[21] constant $C_0^{[\varsigma=0]}$ comes from switching from the metric induced by the adapted norm $\| \cdot \|_*$ to the standard Euclidean norm at the last step.

For $\varsigma > 0$, $g = 0$, we use a Hölder inequality,

$$\sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma} |J_{W_i} T^n|^t_{C^0(W_i)} \le \Big( \sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|}{|W|} |J_{W_i} T^n|_{C^0(W_i)} \Big)^\varsigma \Big( \sum_{W_i \in \mathcal{I}_n(W)} |J_{W_i} T^n|^{\frac{t-\varsigma}{1-\varsigma}}_{C^0(W_i)} \Big)^{1-\varsigma} .$$

Since $|J_{W_i} T|_{C^0(W_i)} \le e^{C_d} \frac{|T^n W_i|}{|W_i|}$ by (2.3), the first sum is bounded by $e^{C_d \varsigma}$. Then, since $\frac{t-\varsigma}{1-\varsigma} \ge \frac{t_0}{2}$, Definition 3.2 and Lemma 3.1 for $\bar t_0 = t_0/2$, together with the case $\varsigma = 0$, imply the second sum is bounded by $(C_0^{[\varsigma=0]})^{1-\varsigma} \theta^{n(t-\varsigma)}$. This completes the proof of the lemma in the case $g = 0$. For nonzero $g$, use $|e^{S_n g}|_{C^0(W_i)} \le e^{n|g|_{C^0}}$ for all $W_i$ to bootstrap from the bound for $g = 0$. □

**Lemma 3.4.** *Fix $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$. Let $\tilde t_1 > t_0$. There exists $C_2 = C_2(t_0, t_1, \tilde t_1) > 0$ such that, for all $g \in C^0$ and all $\varsigma \in [0,1]$, we have*

$$\sum_{W_i \in \mathcal{G}_n(W)} \frac{|W_i|^\varsigma}{|W|^\varsigma} |J_{W_i} T^n|^{t+\varsigma}_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \le C_2 \, Q_n(t,g) \, , \; \forall W \in \widehat{\mathcal{W}}^s \, , \; \forall n \ge 1 \, , \; \forall t \in [t_0, \tilde t_1] \, . \tag{3.5}$$

*Proof.* The case $\varsigma = 0$ and $g = 0$ is trivial since by definition each $W_i \in \mathcal{G}_n(W)$ is contained in a single element of $\mathcal{M}_0^{n,\mathbb{H}}$. Since there can be at most $2/\delta_0$ elements of $\mathcal{G}_n(W)$ in one element of $\mathcal{M}_0^{n,\mathbb{H}}$ (with $\delta_0 = \delta_0(t_0, t_1)$ from Definition 3.2), the lemma holds with $C_2[0] = 2\delta_0^{-1} e^{\tilde t_1 C}$, where $C$ is from Lemma 2.2 (recall also footnote (21)). Next, for $\varsigma > 0$ and $g = 0$, notice that by (2.3),

$$|W_i| |J_{W_i} T^n|_{C^0(W_i)} \le e^{C_d \delta_0^{1/(q+1)}} |T^n W_i| \le e^{C_d \delta_0^{1/(q+1)}} |W| \, ,$$

so that the sum for $\varsigma > 0$ is bounded by the sum for $\varsigma = 0$ times $e^{\varsigma C_d \delta_0^{1/(q+1)}}$. If $g \ne 0$, then again using Lemma 2.2 on each $W_i$, we have

$$|J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \le e^{tC} \sup_{W_i \cap M'} ||J^s T^n|^t e^{S_n g}| \, ,$$

and the required bound follows with $C_2 = C_2[0] e^{\varsigma C_d \delta_0^{1/(q+1)}}$. □

[21] The sets $\widehat{\mathcal{W}}^s$ and $\mathcal{I}_n(W)$ become smaller if $t_1$ is larger; while $\theta$ increases if $t_1$ is larger, this does not affect $C_0$.

3.3. **Growth Lemmas for** $t \in (0,1]$**.** In this section, we prove two growth and complexity lemmas for $t \in (0,1]$. The first one shows that we can make the contribution from the sum over short pieces small compared to the sum over all pieces in $\mathcal{G}_n(W)$ by choosing a small length scale.

**Lemma 3.5.** *Let $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$. For any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $n_1 \geq 1$ such that for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$, all $n \geq n_1$, and all $g \in C^0$ with*

$$2|g|_{C^0} < -t_0 \log \theta\,, \quad \textit{i.e.} \quad e^{|g|_{C^0}} \theta^{t_0/2} < 1\,, \tag{3.6}$$

*we have*

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \leq \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)}\,,\ \forall t \in [t_0, 1]\,.$$

*In particular, taking $\varepsilon = 1/4$ gives $\delta_1 < \delta_0$ and $n_1 \geq 1$ such that for all $n \geq n_1$, for all $g \in C^0$ satisfying* (3.6)*, for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$, we have*

$$\sum_{W_i \in L_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \geq \tfrac{3}{4} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n} g|_{C^0(W_i)}\,,\ \forall t \in [t_0, 1]\,. \tag{3.7}$$

*Proof.* Let $\varepsilon > 0$ and choose $\bar\varepsilon > 0$ so that $6C_1^{-1}\bar\varepsilon/(1-\bar\varepsilon) < \varepsilon$ (where $C_1$ is from (1.2)). Next, choose $n_1$ such that $C_0 \theta^{tn_1} \Lambda^{-n_1(1-t)} < \bar\varepsilon$ (where $C_0$ is from Lemma 3.3 for $\varsigma = 0$). Recalling again that $|T^{-1}U| \leq C|U|^{1/2}$ for any $U \in \widehat{\mathcal{W}}^s$, we may choose $\delta_1 > 0$ such that, if $|U| < \delta_1$, then each homogeneous connected component of $T^{-n}U$ has length shorter than $\delta_0$ for each $n \leq 2n_1$. Then using Lemma 3.3 with $\varsigma = 0$, if $U \in \widehat{\mathcal{W}}^s$ with $|U| \leq \delta_1$,

$$\sum_{W_i \in \mathcal{G}_n(U)} |J_{W_i} T^n|^t_{C^0(W_i)} \leq C_0 \theta^{tn}, \quad \text{for all } n \leq 2n_1. \tag{3.8}$$

Now for $n \geq n_1$, write $n = kn_1 + \ell$, for some $0 \leq \ell < n_1$. Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$. Looking only at times $mn_1$, $m = 0, \ldots, k-1$, we group elements $W_i \in \mathcal{G}_n(W)$ with $|W_i| < \delta_1/3$ according to the largest $m$ such that $T^{(k-m)n_1+\ell} W_i \subset V_j \in L^{\delta_1}_{mn_1}(W)$. This is similar to using[22] the *most recent long ancestor,* except that we only look at times that are multiples of $n_1$. We denote by $\bar I^{\delta}_{(k-m)n_1+\ell}(V_j)$ the set of $W_i \in \mathcal{G}_n(W)$ identified with $V_j \in L^{\delta_1}_{mk}(W)$ in this way. Since $|W| \geq \delta_1/3$, every element of $\mathcal{G}_n^{\delta_1}(W)$ must have a long ancestor.

Note that since $T^{(k-m')n_1+\ell} W_i$ is contained in an element of $\mathcal{G}_{m'k}(W)$ that is shorter than $\delta_1/3$ for $m' < m$, we may apply (3.8) inductively $k-m$ times. Thus,

$$\begin{aligned}
\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t_{C^0(W_i)} &\leq \sum_{m=0}^{k-1} \sum_{V_j \in L^{\delta_1}_{mn_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} \sum_{W_i \in \bar I^{\delta_1}_{(k-m)n_1+\ell}(V_j)} |J_{W_i} T^{(k-m)n_1+\ell}|^t_{C^0(W_i)} \\
&\leq \sum_{m=0}^{k-1} \sum_{V_j \in L^{\delta_1}_{mn_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} C_0 \theta^{tn_1(k-m)}\,.
\end{aligned}$$

Next, notice that for $t \in (0,1]$, $V \in \widehat{\mathcal{W}}^s$ and each $k \geq 1$, using $|V| = \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |T^k(W_i)|$,

$$\sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |J_{W_i} T^k|^t_{C^0(W_i)} = \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} |J_{W_i} T^k|_{C^0(W_i)} |J_{W_i} T^k|^{t-1}_{C^0(W_i)} \tag{3.9}$$

[22]The most recent long ancestor for $W_i \in \mathcal{G}_n(W)$ corresponds to the maximal $m \leq n$ such that $T^{n-m} W_i \subset V_j$ and $V_j \in L_m(W)$, not to be confused with the first long ancestor, see (4.15).

$$\geq C_1\Lambda^{k(1-t)} \sum_{W_i \in \mathcal{G}_k^{\delta_1}(V)} \frac{|T^k W_i|}{|W_i|} \geq C_1 \Lambda^{k(1-t)} |V| \delta_1^{-1} .$$

Also note that by the proof of Lemma 2.1, we have

$$\frac{\sup_{x\in W_i} |J_{W_i}T^n(x)|^t}{\inf_{y\in W_i} |J_{W_i}T^n(y)|^t} \leq 1 + C_d \delta_0^{1/(q+1)} \leq 2 \,, \tag{3.10}$$

since $t \leq 1$. Putting these estimates together, we obtain,

$$\begin{aligned}
&\frac{\displaystyle\sum_{\substack{W_i\in\mathcal{G}_n^{\delta_1}(W)\\ |W_i|<\delta_1/3}} |J_{W_i}T^n|^t_{C^0(W_i)}}{\displaystyle\sum_{W_i\in\mathcal{G}_n^{\delta_1}(W)} |J_{W_i}T^n|^t_{C^0(W_i)}} \\
&\qquad \leq \sum_{m=0}^{k-1} \frac{2\displaystyle\sum_{V_j\in L^{\delta_1}_{mn_1}(W)} |J_{V_j}T^{mn_1}|^t_{C^0(V_j)} C_0\theta^{tn_1(k-m)}}{\displaystyle\sum_{V_j\in L^{\delta_1}_{mn_1}(W)} |J_{V_j}T^{mn_1}|^t_{C^0(V_j)} \sum_{W_i\in\mathcal{G}^{\delta_1}_{(k-m)n_1+\ell}(V_j)} |J_{W_i}T^{(k-m)n_1+\ell}|^t_{C^0(W_i)}} \\
&\qquad \leq \sum_{m=0}^{k-1} \frac{2\displaystyle\sum_{V_j\in L^{\delta_1}_{mn_1}(W)} |J_{V_j}T^{mn_1}|^t_{C^0(V_j)} C_0\theta^{tn_1(k-m)}}{\displaystyle\sum_{V_j\in L^{\delta_1}_{mn_1}(W)} |J_{V_j}T^{mn_1}|^t_{C^0(V_j)} C_1\Lambda^{(k-m)n_1(1-t)}|V_j|\delta_1^{-1}} \\
&\qquad \leq 6C_1^{-1}\sum_{m=0}^{k-1} \bar\varepsilon^{k-m} \leq 6C_1^{-1}\frac{\bar\varepsilon}{1-\bar\varepsilon}\,, \qquad (3.11)
\end{aligned}$$

where in the second inequality we used (3.9) on each $V_j \in L^{\delta_1}_{mn_1}(W)$. This ends the case $g = 0$.

If $g \neq 0$, letting $\varepsilon > 0$ and $\bar\varepsilon > 0$ be as above, we take $n_1$ such that $C_0\theta^{tn_1}e^{2n_1|g|_{C^0}}\Lambda^{-n_1(1-t)} < \bar\varepsilon$, and we choose $\delta_1 > 0$ such that, if $U \in \widehat{\mathcal{W}}^s$ satisfies $|U| < \delta_1$, then

$$\sum_{W_i\in\mathcal{G}_n(U)} |J_{W_i}T^n|^t_{C_0(W_i)} |e^{S_ng}|_{C^0(W_i)} \leq C_0\theta^{tn}e^{n|g|_{C^0}}\,, \qquad \text{for all } n \leq 2n_1, \tag{3.12}$$

which is the analogue of (3.8). (For fixed $t_0$, note that $n_1$ and $\delta_1$ depend only on $\varepsilon$, uniformly in $g$ satisfying (3.6).) The proof above can then be followed line by line, inserting $e^{S_ng}$. Thus (3.9) becomes,

$$\sum_{W_i\in\mathcal{G}^{\delta_1}_k(V)} |J_{W_i}T^k|^t_{C_0(W_i)} |e^{S_kg}|_{C^0(W_i)} \geq C_1\Lambda^{k(1-t)}e^{-k|g|_{C^0}}|V|\delta_1^{-1}\,. \tag{3.13}$$

Inserting this lower bound in (3.11) and applying Lemma 3.3 with $\varsigma = 0$ yields,

$$\frac{\displaystyle\sum_{\substack{W_i\in\mathcal{G}_n^{\delta_1}(W)\\ |W_i|<\delta_1/3}} |J_{W_i}T^n|^t_{C^0(W_i)}|e^{S_ng}|_{C^0(W_i)}}{\displaystyle\sum_{W_i\in\mathcal{G}_n^{\delta_1}(W)} |J_{W_i}T^n|^t_{C^0(W_i)}|e^{S_ng}|_{C^0(W_i)}} \tag{3.14}$$

$$\leq \sum_{m=0}^{k-1} \frac{2 \sum_{V_j \in L^{\delta_1}_{mn_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} C_0 \theta^{tn_1(k-m)} e^{n_1(k-m)|g|_{C^0}}}{\sum_{V_j \in L^{\delta_1}_{mn_1}(W)} |J_{V_j} T^{mn_1}|^t_{C^0(V_j)} C_1 \Lambda^{(k-m)n_1(1-t)} e^{-n_1(k-m)|g|_{C^0}} |V_j| \delta_1{}^{-1}}$$

$$\leq 6C_1^{-1} \sum_{m=0}^{k-1} \bar{\varepsilon}^{k-m} \leq 6C_1^{-1} \frac{\bar{\varepsilon}}{1-\bar{\varepsilon}} < \varepsilon \, .$$

□

The second lemma proves the analogue of Lemma 3.5 for elements of $\mathcal{M}_0^{n,\mathbb{H}}$, in anticipation of Proposition 3.14. For $A \in \mathcal{M}_0^{n,\mathbb{H}}$, let $B_{n-1}(A)$ denote the element of $\mathcal{M}_{-n+1}^{0,\mathbb{H}} \bigvee \mathcal{H}$ containing $T^{n-1}A \in \mathcal{M}_{-n+1}^{1,\mathbb{H}}$, recalling that $\mathcal{H}$ is the partition of $M$ into homogeneity strips $\mathbb{H}_k$. We introduce this additional intersection with $\mathcal{H}$ (omitted from the definition of $\mathcal{M}_{-n+1}^{0,\mathbb{H}}$) since it will be convenient to work with homogeneous partition elements in what follows. For $\delta > 0$, define

$$\mathcal{A}_n(\delta) = \{A \in \mathcal{M}_0^{n,\mathbb{H}} : \mathrm{diam}^u(B_{n-1}(A)) \geq \delta/3\} \, . \tag{3.15}$$

The following result shows that most of the weights contributing to $Q_n(t,g)$ come from elements of $\mathcal{A}_n(\delta)$ if $\delta$ is chosen small enough.

**Lemma 3.6.** *Let $t_0 > 0$ and $t_1 \in (1, t_*)$. For any $\upsilon \geq 0$, there exist $\delta_2 > 0$ and $c_0 = c_0(\upsilon) > 0$ with $c_0(\upsilon') \geq c_0(\upsilon)$ if $\upsilon' \in [0, \upsilon]$, such that for any $g \in C^1$ satisfying (3.6) with $|\nabla g|_{C^0} \leq \upsilon$,*

$$\sum_{A \in \mathcal{A}_n(\delta_2)} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)} \geq c_0 \, Q_n(t,g) \, , \ \forall n \in \mathbb{N} \, , \ \forall t \in [t_0, 1] \, .$$

*Proof.* Assume first $g = 0$. We begin by relating $J^s T^n$ on $A \in \mathcal{M}_0^{n,\mathbb{H}}$ with $J^u T^{-n+1}$ on $T^{n-1}A$. By (1.5), if $x \in A$ and $y = T^n x$, we have

$$J^s T^n(x) = \frac{(E \cos\varphi) \circ T^{-n}(y)}{(E\cos\varphi)(y)} J^u T^{-n}(y) \, .$$

Here, $J^u T^{-n} = \det(DT^{-n}|_{E^u})$, where $E^u$ is the unstable direction for $T$ (not $T^{-1}$), so that $J^u T^{-n}$ is a contraction. Next, since[23] $J^u T^{-1}(y) = C^{\pm 1} \cos\varphi(y)$, and the function $E$ is uniformly bounded away from 0, we conclude,

$$J^s T^n(x) = C^{\pm 1} \cos\varphi(T^{-n}y) J^u T^{-n+1}(Ty) \, . \tag{3.16}$$

For brevity, for any set $A \subset M$, we will denote

$$|J^s T^n|_A^t := \sup_{x \in A \cap M'} |J^s T^n(x)|^t \ \text{ and similarly, } \ |J^u T^{-n}|_A^t := \sup_{x \in A \cap M'} |J^u T^{-n}(x)|^t \, . \tag{3.17}$$

Next, we consider the evolution of elements of $\mathcal{M}_{-k}^{0,\mathbb{H}}$ under iteration by $T^j$ for $j \geq 1$. If $B \in \mathcal{M}_{-k}^{0,\mathbb{H}}$, then we subdivide $T^j B$ according to singularity curves and homogeneity strips at each iterate, much as we would consider the evolution of an unstable curve $U$ under $T^j$. We write $T^j B = \cup_{B' \in G_j(B)} B'$, where $G_j(B)$ is the maximal decomposition of $T^j B$ into elements of $\mathcal{M}_{-k-j}^{0,\mathbb{H}} \bigvee \mathcal{H}$, recalling that $\mathcal{H}$ denotes the partition of $M$ according to homogeneity strips. This last intersection with $\mathcal{H}$ is necessary since we will work with homogeneous elements $B' \subset T^j B$ (to maintain bounded distortion for $J^u T^{-j}$ on $B'$). Let $L_j^\delta(B)$ denote those elements $B' \in G_j(B)$ with $\mathrm{diam}^u(B') \geq \delta/3$.

[23] We use the notation $A = C^{\pm 1} B$ to denote $C^{-1} B \leq A \leq CB$.

Now by Definition 3.2 and applying the time reversal of the proof of Lemma 3.3 (with $\varsigma = 0$), there exists $\delta_2 > 0$ such that if $\max\{\mathrm{diam}^u(B), \mathrm{diam}^s(B)\} \le \delta_2$, then

$$\sum_{B' \in G_j(B)} |J^u T^{-j}|_{B'}^t \le C_0 \theta^{tj}, \qquad \text{for all } j \le n_1, \tag{3.18}$$

where $n_1 = n_1(1/4)$ is from (3.7) in Lemma 3.5. For convenience, we choose $\delta_2 \le \delta_1(1/4)$. Also, if $B \in \mathcal{M}_{-k}^{0,\mathbb{H}}$, then $\mathrm{diam}^s(B) \le C\Lambda^{-k}$ for some uniform constant $C > 0$. We choose $n_2 \ge n_1$ so that $\mathrm{diam}^s(B) \le \delta_2$ if $B \in \mathcal{M}_{-k}^{0,\mathbb{H}}$ for $k \ge n_2$.

We fix $n \ge n_2 + 1$ and prove the lemma for such $n$. For $B \in \mathcal{M}_{-n+1}^{0,\mathbb{H}} \bigvee \mathcal{H}$, let $B_{-j}$ denote the element of $\mathcal{M}_{-n+1+j} \bigvee \mathcal{H}$ containing $T^{-j}B$. We call $B_{-j}$ the most recent $u$-long ancestor of $B$ if $j$ is the minimal integer $k \le n - n_2$ such that $\mathrm{diam}^u(B_{-k}) \ge \delta_2$. If no such $j$ exists, we say that $B$ has been $u$-short since time $n_2$. (It follows from the definition of $n_2$, that $\mathrm{diam}^s(B_{-j}) \le \delta_2$ for all $j \le n - n_2 - 1$.) Let $\mathbb{L}_{-n+1+j}^{\delta_2}$ denote those elements of $\mathcal{M}_{-n+1+j} \bigvee \mathcal{H}$ which are $u$-long, and let $\mathbb{S}_{-n+1+j}^{\delta_2}$ denote those elements which are $u$-short (in the length scale $\delta_2$). Similarly, let $\mathbb{I}_j^{\delta_2}(B_{-j})$ denote the collection of $B \in \mathcal{M}_{-n+1} \bigvee \mathcal{H}$ whose most recent $u$-long ancestor is $B_{-j}$. Note that $\mathbb{I}_j^{\delta_2}(B_{-j}) \subset G_j(B_{-j})$.

Thus if $k \ge n_2$ and $B' \in \mathcal{M}_{-k}^{0,\mathbb{H}}$, then estimating inductively as in the proof of Lemma 3.3,

$$\sum_{B \in \mathbb{I}_j^{\delta_2}(B')} |J^u T^{-j}|_B^t \le C_0 \delta_2^{-1} \theta^{tj} \text{ for all } j \ge 0, \tag{3.19}$$

where the factor $\delta_2^{-1}$ is due to the fact that $B'$ itself may be $u$-long, in which case it would be artificially subdivided into $\sim \delta_2^{-1}$ pieces of $u$-diameter less than $\delta_2$ before being iterated.

Let $\mathcal{A}_n^c(\delta_2) = \mathcal{M}_0^{n,\mathbb{H}} \setminus \mathcal{A}_n(\delta_2)$. By (3.16),

$$\sum_{A \in \mathcal{A}_n^c(\delta_2)} |J^s T^n|_A^t = C^{\pm 1} \sum_{A \in \mathcal{A}_n^c(\delta_2)} |\cos\varphi|_A^t |J^u T^{-n+1}|_{T^{n-1}A}^t .$$

Note that if $B \in \mathcal{M}_{-n+1}^{0,\mathbb{H}} \bigvee \mathcal{H}$ with $\mathrm{diam}^u(B) < \delta_2/3$, then any $A \in \mathcal{M}_0^{n,\mathbb{H}}$ for which $B = B_{n-1}(A)$ belongs to $\mathcal{A}_n^c(\delta_2)$. Also,

$$T^{n-1}A \in \mathcal{M}_{-n+1}^{1,\mathbb{H}} = \mathcal{M}_{-n+1}^{0,\mathbb{H}} \bigvee \mathcal{H} \bigvee \mathcal{M}_0^1$$

so that for fixed $B$, the number of $A$ such that $B_{n-1}(A) = B$ is at most $\#\mathcal{M}_0^1$. Moreover, since all such $A$ are by definition contained in $T^{-n+1}(B_{n-1}(A)) = T^{-n+1}B$, and $T^{-n+1}B \in \mathcal{M}_0^{n-1,\mathbb{H}} \bigvee T^{-n+1}\mathcal{H}$, all $A$ corresponding to one $B$ are contained in the same homogeneity strip, so that $|\cos\varphi|_A$ is comparable on all such $A$. Thus,

$$\sum_{A \in \mathcal{A}_n^c(\delta_2)} |J^s T^n|_A^t = C^{\pm 1} \sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos\varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t , \tag{3.20}$$

and similarly,

$$\sum_{A \in \mathcal{A}_n(\delta_2)} |J^s T^n|_A^t = C^{\pm 1} \sum_{B \in \mathbb{L}_{-n+1}^{\delta_2}} |\cos\varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t . \tag{3.21}$$

Next, we group elements of $\mathbb{S}_{-n+1}^{\delta_2}$ by most recent $u$-long ancestor in $\mathcal{M}_{-n+1+j}^{0,\mathbb{H}}$, as described above. By (3.18), there is no need to consider long ancestors for $j < n_1$. Note that if $B' \in \mathcal{M}_{-n+1+j}^{0,\mathbb{H}} \bigvee \mathcal{H}$ and $B \in \mathbb{I}_j^{\delta_2}(B')$, then $T^{-n+1}B$ lies in the same homogeneity strip as $T^{-n+1+j}B'$, so that $\cos\varphi$ is comparable on each of these sets. Thus, by (3.18)–(3.19),

$$\sum_{B \in \mathbb{S}_{-n+1}^{\delta_2}} |\cos\varphi|_{T^{-n+1}B}^t |J^u T^{-n+1}|_B^t \tag{3.22}$$

$$
\begin{aligned}
&= \sum_{j=n_1}^{n-n_2-1} \sum_{B' \in \mathbb{L}^{\delta_2}_{-n+1+j}} \sum_{B \in \mathbb{I}^{\delta_2}_j(B')} |\cos\varphi|^t_{T^{-n+1}B} |J^uT^{-j}|^t_B |J^uT^{-n+1+j}|^t_{B'} \\
&\quad + \sum_{B' \in \mathbb{S}^{\delta_2}_{-n_2}} \sum_{B \in \mathbb{I}^{\delta_2}_{n-n_2}(B')} |\cos\varphi|^t_{T^{-n+1}B} |J^uT^{-j}|^t_B |J^uT^{-n+1+j}|^t_{B'} \\
&\le \sum_{j=n_1}^{n-n_2-1} \sum_{B' \in \mathbb{L}^{\delta_2}_{-n+1+j}} C\delta_2^{-1}\theta^{tj} |\cos\varphi|^t_{T^{-n+1+j}B'} |J^uT^{-n+1+j}|^t_{B'} \\
&\quad + \sum_{B' \in \mathbb{S}^{\delta_2}_{-n_2}} C\theta^{t(n-n_2-1)} |\cos\varphi|^t_{T^{-n_2}B'} |J^uT^{-n_2}|^t_{B'} \,,
\end{aligned}
$$

where the final sum over $B' \in \mathbb{S}^{\delta_2}_{-n_2}$ represents those $B \in \mathbb{S}^{\delta_2}_{-n+1}$ which have had no $u$-long ancestor since before time $n_2$.

To proceed, we will need the following sublemma, linking the contribution from $\mathbb{L}^{\delta_2}_{-n+1+j}$ to the contribution from $\mathbb{L}^{\delta_2}_{-n+1}$.

**Sublemma 3.7.** *Let $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$. There exists $C > 0$ such that for all $t \in [t_0, 1]$, each $n_1 \le j \le n - n_2 - 1$, and all $B' \in \mathbb{L}^{\delta_2}_{-n+1+j}$,*

$$
|\cos\varphi|^t_{T^{-n+1+j}B'} |J^uT^{-n+1+j}|^t_{B'} \le C\delta_2^{-1}\Lambda^{j(t-1)} \sum_{B \in L_j(B')} |\cos\varphi|^t_{T^{-n+1}B} |J^uT^{-n+1}|^t_B \,,
$$

*where $L_j(B')$ denotes the collection of elements $B \in G_j(B')$ with $diam^u(B) \ge \delta_2/3$.*

*Proof.* Since $B' \in \mathbb{L}^{\delta_2}_{-n+1+j}$, there exists an unstable curve $U \subset B'$ with $|U| \ge \delta_2/3$. Let $G'_j(B')$ denote those elements $B \in G_j(B')$ such that $T^jU \cap B \neq \emptyset$. Letting $\mathcal{G}^{\delta_2}_j(U)$ denote the $j$th generation of homogeneous elements of $T^jU$, using the time reversed definition of $\mathcal{G}^{\delta_2}_j(W)$ for stable curves from Section 3.1. If $U_i \in \mathcal{G}^{\delta_2}_j(U)$ has $|U| \ge \delta_2/3$, and $B \cap U \neq \emptyset$ for some $B \in G'_j(B')$, then necessarily, $\mathrm{diam}^u(B) \ge \delta_2/3$. Let $L'_j(B') \subset L_j(B')$ denote this collection of long elements. Then letting $L^{\delta_2}_j(U) \subset \mathcal{G}^{\delta_2}_j(U)$ denote those elements of $\mathcal{G}^{\delta_2}_j(U)$ with length at least $\delta_2/3$, we estimate,

$$
\begin{aligned}
(3.23) \qquad & \sum_{B \in L_j(B')} |\cos\varphi|^t_{T^{-n+1}B} |J^uT^{-n+1}|^t_B \\
&\ge C |\cos\varphi|^t_{T^{-n+1+j}B'} |J^uT^{-n+1+j}|^t_{B'} \sum_{B \in L'_j(B')} |J^uT^{-j}|^t_B \\
&\ge C' |\cos\varphi|^t_{T^{-n+1+j}B'} |J^uT^{-n+1+j}|^t_{B'} \delta_2 \sum_{U_i \in L^{\delta_2}_j(U)} |J_{U_i}T^{-j}|^t_{C^0(U_i)} \\
&\ge C'\delta_2 |\cos\varphi|^t_{T^{-n+1+j}B'} |J^uT^{-n+1+j}|^t_{B'} \frac{3}{4} \sum_{U_i \in \mathcal{G}^{\delta_2}_j(U)} |J_{U_i}T^{-j}|^t_{C^0(U_i)} \\
&\ge C''\delta_2 |\cos\varphi|^t_{T^{-n+1+j}B'} |J^uT^{-n+1+j}|^t_{B'} \Lambda^{j(1-t)} \sum_{U_i \in \mathcal{G}^{\delta_2}_j(U)} |J_{U_i}T^{-j}|_{C^0(U_i)} \\
&\ge C''\delta_2 |\cos\varphi|^t_{T^{-n+1+j}B'} |J^uT^{-n+1+j}|^t_{B'} \Lambda^{j(1-t)} \frac{|U|}{\delta_2} \,,
\end{aligned}
$$

where in the first inequality we have used the fact that $\cos\varphi$ is comparable on $T^{-n+1}B$ and $T^{-n+1+j}B'$, in the second inequality we have applied the time reversal of Lemma 2.2 and the factor $\delta_2$ appears since there may be up to $\sim \delta_2^{-1}$ elements of $L^{\delta_2}_j(U)$ in each element $B \in L'_j(B')$ (due to artificial subdivisions in the definition of $\mathcal{G}_j^{\delta_2}(U)$), and in the third inequality we have applied the time reversal of Lemma 3.5 and (3.7) from Lemma 3.5 since $\delta_2 \le \delta_1$ and $j \ge n_1$. Since $|U| \ge \delta_2/3$, this completes the proof of the sublemma. $\square$

Using the sublemma, we now estimate the right hand side of (3.22), summing over $B' \in \mathbb{L}^{\delta_2}_{-n+1+j}$ and noting that if $B \in L_j(B')$, then $B \in \mathbb{L}^{\delta_2}_{-n+1}$ and each such $B$ is associated with a unique $B'$:

$$\begin{aligned}\sum_{B \in \mathbb{S}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B} |J^u T^{-n+1}|^t_B \le & \sum_{B' \in \mathbb{S}^{\delta_2}_{-n_2}} C\theta^{t(n-n_2-1)} |\cos\varphi|^t_{T^{-n_2}B'} |J^u T^{-n_2}|^t_{B'} \\ & + \sum_{j=n_1}^{n-n_2-1} C\delta_2^{-2}\theta^{tj}\Lambda^{j(t-1)} \sum_{B\in\mathbb{L}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B \\ & \le C_{n_2}\theta^{tn} + C\delta_2^{-2} \sum_{B\in\mathbb{L}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B\,,\end{aligned}$$

for some constant $C_{n_2} > 0$ depending only on $n_2$.

Note that the sum over $\mathbb{L}^{\delta_2}_{-n+1}$ grows at a rate of at least $C\Lambda^{n(1-t)}$ by the proof of the sublemma. Thus we may choose $n_3 \ge n_2$ large enough that $C_{n_2}\theta^{tn} \le C\Lambda^{n(1-t)}$ for all $n \ge n_3$, which implies,

$$\sum_{B \in \mathbb{S}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B} |J^u T^{-n+1}|^t_B \le C\delta_2^{-2} \sum_{B\in\mathbb{L}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B\,.$$

Using this estimate with (3.20), (3.21) and the fact that $\mathcal{A}_n(\delta_2) \cup \mathcal{A}_n^c(\delta_2) = \mathcal{M}_0^{n,\mathbb{H}}$ yields,

$$\begin{aligned}Q_n(t) &= \sum_{A\in\mathcal{A}_n^c(\delta_2)} |J^sT^n|^t_A + \sum_{A\in\mathcal{A}_n(\delta_2)} |J^sT^n|^t_A \\ &\le C(\delta_2^{-2}+1) \sum_{B\in\mathbb{L}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B \le C(\delta_2^{-2}+1)\sum_{A\in\mathcal{A}_n(\delta_2)} |J^sT^n|^t_A\,,\end{aligned}$$

completing the proof of the lemma for $n \ge n_3$ and $g = 0$. The statement for general $n$ (and $g = 0$) follows, possibly reducing the constant $c_0$, since there are only finitely many $n$ to correct for.

If $g \ne 0$, the proof remains as is until (3.18) with the same choices of $n_2$ and $\delta_2$ (these choices are independent of $g$), so that (3.18) holds with $|e^{S_jg}|_{T^{-j}B'}$ inserted in the left-hand side and $e^{j|g|_{C^0}}$ in the right. The analogous modification is made to (3.19). Then (3.20) becomes

$$\sum_{A\in\mathcal{A}_n^c(\delta_2)} |J^sT^n|^t_A |e^{S_ng}|_A = C^{\pm 1} \sum_{B\in\mathbb{S}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B |e^{S_{n-1}g}|_{T^{-n+1}B}\,, \tag{3.24}$$

where $C$ depends on $|\nabla g|_{C^0}$ via (2.14), with the analogous modification to (3.21). Then (3.22) is modified in the obvious way for $n \ge n_2+1$,

$$\begin{aligned}&\sum_{B\in\mathbb{S}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B|e^{S_{n-1}g}|_{T^{-n+1}B} \\ &\le \sum_{j=n_1}^{n-n_2-1}\sum_{B'\in\mathbb{L}^{\delta_2}_{-n+1+j}} C\delta_1^{-1}\theta^{tj}e^{j|g|_{C^0}}|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'}|e^{S_{n-1-j}g}|_{T^{-n+1+j}B'} \\ &+ \sum_{B'\in\mathbb{S}^{\delta_2}_{-n_2}} C\theta^{t(n-n_2-1)}e^{(n-n_2-1)|g|_{C^0}}|\cos\varphi|^t_{T^{-n_2}B'}|J^uT^{-n_2}|^t_{B'}|e^{S_{n_2}g}|_{T^{-n_2}B'}\,.\end{aligned} \tag{3.25}$$

A suitable analogue of Sublemma 3.7 yields $C > 0$ such that for each $n_1 \le j \le n - n_2 - 1$ and $B' \in \mathbb{L}^{\delta_2}_{-n+1+j}$,

$$
\begin{aligned}
&|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'}|e^{S_{n-1-j}g}|_{T^{-n+1+j}B'} \\
&\qquad \le C\delta_2^{-1}\Lambda^{j(t-1)}e^{j|g|_{C^0}} \sum_{B\in L_j(B')} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B|e^{S_{n-1}g}|_{T^{-n+1}B}\,,
\end{aligned}
$$

where we have used the lower bound (3.13) rather than (3.9) in (3.23). This provides the contraction required to complete the proof of Lemma 3.6 since $\theta^t e^{|g|_{C^0}} < 1$ by (3.6). $\Box$

3.4. **Defining $s_1 > 1$. Growth Lemmas for $t \in (1, s_1)$.** In this section, we bootstrap from our results for $t \le 1$ to conclude a parallel set of results for $t \in (1, s_1)$, for $s_1 > 1$ from Definition 3.9 below. To do this, we will apply Propositions 3.14 and 3.15 from Section 3.5 for $t \le 1$ whose proofs rely only on the lemmas in Section 3.3. In Section 3.6, we show how to extend this to all $t < t_*$.

The easy lemma below will be crucial to define $s_1$:

**Lemma 3.8.** *We have $P_*(1) = 0$. Moreover, the limit $\chi_1 := \lim_{s\to 1^-} \frac{P_*(s)}{1-s}$ exists and $\chi_1 \ge \log\Lambda > 0$.*

In fact, $\chi_1 = \int_M \log J^uT\, d\mu_{\mathrm{SRB}}$, which follows from Theorems 1.2 and 2.4.

*Proof.* Proposition 3.14 for $t = 1$ together with [DZ1, Lemma 3.2] prove that $Q_n(1)$ is uniformly bounded for all $n$, so that $P_*(1) \le 0$. Since Proposition 2.3 gives $P_*(1) \ge P(1) = 0$, we have established that $P_*(1) = 0$. Next, the convexity of $P_*(t)$ (Proposition 2.5) on $(0, \infty)$ implies that left (and right) derivatives exist at every $t > 0$. Thus, since $P_*(1) = 0$, the limit below exists

$$
\lim_{s\to 1^-} \frac{P_*(s)}{1-s} = \lim_{s\to 1^-} \frac{P_*(s) - P_*(1)}{1-s}\,. \tag{3.26}
$$

The proof that $P(t)$ is strictly decreasing in Proposition 2.5 implies $\chi_1 \ge \log\Lambda > 0$. $\Box$

**Definition 3.9.** *Recalling $\theta(t_1) \in (\Lambda^{-1}, \Lambda^{-1/2})$ from Definition 3.2, we define $s_1 := \dfrac{\chi_1}{\chi_1 + \log\theta} > 1$.*

Note that $s_1$ is just the intersection point between the tangent line to $P_*(t)$ at $t = 1$ (which is the largest $t$ where we have established the lower bound (3.9) on the sum over $\mathcal{G}_n(W)$) and the line $y = t\log\theta$. If $t < s_1$ then $\theta^t < e^{P_*(t)}$, which can be viewed as a pressure gap condition. Note finally that establishing Theorem 2.4. in a neighbourhood of $t = 1$ will give $s_1 \le t_*$.

A key to many results for $0 < t \le 1$ is the lower bound on the rate of growth given by (3.9) in the proof of Lemma 3.5. Our next lemma obtains this lower bound for $t \ge 1$, interpolating via a Hölder inequality.

**Lemma 3.10.** *Let $t_0 \in (0, 1)$ and $t_1 \in (1, t_*)$. Let $\bar t_1 \ge 1$. For any $\kappa > 0$, there exist $C_\kappa > 0$, $\eta_\kappa > 0$ such that for all $g \in C^0$ and $\delta > 0$, and all $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta/3$,*

$$
\sum_{W_i\in\mathcal{G}_n^\delta(W)} |J_{W_i}T^n|^t_{C^0(W_i)}|e^{S_ng}|_{C^0(W_i)} \ge C_\kappa \delta^{-1/\eta_\kappa} e^{-n(\chi_1+\kappa)(t-1)-n|g|_{C^0}}\,,\ \forall n\ge 1\,,\ \forall t\in[1,\bar t_1]\,. \tag{3.27}
$$

*Proof.* Assume first $g = 0$. For $t \ge 1$, we have for any $s \in (0, 1)$, taking $\eta(s) \in (0, 1]$ such that $\eta t + (1-\eta)s = 1$, that $\sum_i a_i = \sum_i a_i^{\eta t+(1-\eta)s} \le (\sum_i a_i^t)^\eta (\sum_i a_i^s)^{1-\eta}$ for any positive numbers $a_i$. It follows that for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta/3$ and all $n \ge 1$,

$$
\sum_{W_i\in\mathcal{G}_n^\delta(W)} |J_{W_i}T^n|^t \ge \frac{\left(\sum_{W_i\in\mathcal{G}_n^\delta(W)} |J_{W_i}T^n|\right)^{1/\eta}}{\left(\sum_{W_i\in\mathcal{G}_n^\delta(W)} |J_{W_i}T^n|^s\right)^{(1-\eta)/\eta}} \ge \left(\frac{C_1}{3}\right)^{1/\eta} \left(C_2[0]\tfrac{\delta}{\delta_0}\tfrac{2}{c_2}e^{nP_*(s)}\right)^{(\eta-1)/\eta}\,, \tag{3.28}
$$

where we have used (3.9) for the lower bound in the numerator, and Lemma 3.4 with $\varsigma = 0$ and $\tilde t_1 = 1$ combined with Proposition 3.15 for the upper bound in the denominator. Since $\eta = (1-s)/(t-s)$, $e^{nP_*(s)(\eta-1)/\eta} = e^{-n(t-1)P_*(s)/(1-s)}$. For fixed $\kappa > 0$, Lemma 3.8 allows us to choose $s = s(\kappa) \in (0,1)$ (and hence $\eta_\kappa = \eta(s) > 0$) such that $P_*(s)/(1-s) \le \chi_1 + \kappa$, completing the proof for $g = 0$ since $\eta(s) > (1-s)/\bar t_1$. For $g \neq 0$, (3.27) follows since $|e^{S_n g}|_{C^0(W_i)} \ge e^{-n|g|_{C^0}}$ for each $W_i$. □

By definition, $\theta^t e^{\chi_1(t-1)} < 1$ if $t < s_1$. Thus for $\bar t_1 \in (1, s_1)$ there exists $\kappa_1 = \kappa(\bar t_1) > 0$ such that

$$\theta^{\bar t_1} e^{(\chi_1+\kappa_1)(\bar t_1 - 1)} < 1\,, \quad \text{and thus } \theta^t e^{(\chi_1+\kappa_1)(t-1)} < 1\,, \ \forall t \le \bar t_1\,. \tag{3.29}$$

Our next lemma is the analogue of Lemma 3.5 for $t > 1$.

**Lemma 3.11.** *Let $t_0 \in (0,1)$, $t_1 \in (1, t_*)$ and $\bar t_1 \in (1, s_1)$. Let $\kappa_1 = \kappa(\bar t_1)$ satisfy (3.29). Then for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $n_1 \ge 1$, such that*[24] *for all $W \in \widehat{\mathcal W}^s$ with $|W| \ge \delta_1/3$,*

$$\sum_{\substack{W_i \in \mathcal G_n^{\delta_1}(W)\\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \le \varepsilon \sum_{W_i \in \mathcal G_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)}\,, \ \forall t \in [1, \bar t_1]\,, \forall n \ge n_1\,,$$

*for all $g \in C^0$ satisfying (3.6) and such that, in addition,*

$$2|g|_{C^0} < -\bar t_1 \log \theta - (\chi_1 + \kappa_1)(\bar t_1 - 1)\,, \ \ i.e. \ \ \theta^{\bar t_1} e^{(\chi_1+\kappa_1)(\bar t_1 - 1) + 2|g|_{C^0}} < 1\,. \tag{3.30}$$

Let $[t_0, \bar t_1] \subset (0, s_1)$. For all $g \in C^1$ satisfying (3.6) and (3.30), Lemma 3.5 and Lemma 3.11 for $\varepsilon = 1/4$ give $n_1 \ge 1$ and $\delta_1 > 0$ such that for all $n \ge n_1$ and all $W \in \widehat{\mathcal W}^s$ with $|W| \ge \delta_1/3$,

$$\sum_{W_i \in L_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \ge \tfrac34 \sum_{W_i \in \mathcal G_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)}\,, \ \forall t \in [t_0, \bar t_1]\,. \tag{3.31}$$

*Proof of Lemma 3.11.* For $\varepsilon > 0$, choose $\bar\varepsilon > 0$ such that $2C_{\kappa_1}^{-1}\bar\varepsilon/(1-\bar\varepsilon) < \varepsilon$ (with $C_{\kappa_1}(t_0, t_1, \bar t_1)$ from (3.29)). Assume first that $g = 0$. Choose $n_1$ such that $C_0 \theta^{t n_1} e^{n_1(\chi_1+\kappa_1)(t-1)} \le \bar\varepsilon$. Next, choose $\delta_1 > 0$ such that (3.8) holds for all $n \le 2n_1$. Grouping elements of $\mathcal G_n^{\delta_1}(W)$ as in the proof of Lemma 3.5, we follow the estimates there (omitting (3.9)) until (3.11). In (3.11), we apply Lemma 3.10 to each $V_j \in L_{mn_1}^{\delta_1}(W)$ appearing in the denominator to obtain,

$$\sum_{W_i \in \mathcal G^{\delta_1}_{(k-m)n_1+\ell}(V_j)} |J_{W_i} T^{(k-m)n_1+\ell}|^t_{C^0(W_i)} \ge C_\kappa e^{-(k-m)n_1(\chi_1+\kappa_1)(t-1)}\,,$$

so that the left hand side of (3.11) is bounded by $2C_\kappa^{-1}\sum_{m=0}^{k-1}\bar\varepsilon^{k-m} \le \varepsilon$ by definition of $\bar\varepsilon$.

If $g \neq 0$, choose $n_1$ such that $C_0 \theta^{t n_1} e^{n_1(\chi_1+\kappa_1)(t-1) + n_1 2|g|_{C^0}} \le \bar\varepsilon$. Then choose $\delta_1 > 0$ such that (3.12) holds for this value of $n_1$. (The choices $n_1$ and $\delta_1$ are uniform for $g$ satisfying (3.6) and (3.30).) The argument then follows precisely the proof of Lemma 3.5, but applying the lower bound (3.27) (noting that $|g|_{C^0}$ is bounded for $g$ satisfying (3.30)), rather than (3.13), to each $V_j \in L_{mn_1}^{\delta_1}(W)$ appearing in the denominator of (3.14). □

Our final lemma of this section is the analogue of Lemma 3.6 for $t > 1$. Define $\mathcal A_n(\delta)$ as in (3.15).

**Lemma 3.12.** *Let $t_0 \in (0,1)$, $t_1 \in (1, t_*)$ and $\bar t_1 \in (1, s_1)$. Let $\delta_2 > 0$ be as in Lemma 3.6. For any $\upsilon > 0$, there exists $c_0(\upsilon) > 0$, with $c_0(\upsilon') \ge c_0(\upsilon) > 0$ if $\upsilon' \in [0, \upsilon]$, such that, for any $g \in C^1$ satisfying (3.6), (3.30), and $|\nabla g|_{C^0} \le \upsilon$, we have*

$$\sum_{A \in \mathcal A_n(\delta_2)} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)} \ge c_0 Q_n(t, g)\,, \ \forall n \in \mathbb N\,, \ \forall t \in [1, \bar t_1]\,.$$

*Proof.* The calculations in the proof of Lemma 3.6 for $g = 0$ are valid for all $t > 0$ up through (3.22). To proceed, we replace Sublemma 3.7 by the following.

[24] We take $\delta_1 < \delta_1(\varepsilon)$ and $n_1 \ge n_1(\varepsilon)$ with $\delta_1(\varepsilon)$ and $n_1(\varepsilon)$ from Lemma 3.5.

**Sublemma 3.13.** *Let $t_0 \in (0,1)$, $t_1 \in (1,t_*)$ and $\bar t_1 \in (1,s_1)$. There exists $C>0$ such that for all $t \in [1,\bar t_1]$, each $n_1 \le j \le n-n_2-1$ and all $B' \in \mathbb{L}^{\delta_2}_{-n+1+j}$,*

$$|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'} \le C\delta_2^{-1}e^{j(\chi_1+\kappa_1)(t-1)} \sum_{B\in L_j(B')} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B\,,$$

*where $L_j(B')$ denotes the collection of elements $B \in G_j(B')$ with $diam^u(B) \ge \delta_2/3$.*

*Proof.* The proof of this sublemma only requires one adjustment to the estimate in (3.23). Using the same notation as in Sublemma 3.7, we have

$$\begin{aligned}\sum_{B\in L_j(B')} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B &\ge C|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'}\sum_{B\in L_j'(B')}|J^uT^{-j}|^t_B\\
&\ge C'|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'}\delta_2\sum_{U_i\in L_j^{\delta_2}(U)}|J_{U_i}T^{-j}|^t_{C^0(U_i)}\\
&\ge C'\delta_2|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'}\tfrac{3}{4}\sum_{U_i\in\mathcal{G}_j^{\delta_2}(U)}|J_{U_i}T^{-j}|^t_{C^0(U_i)}\\
&\ge C''\delta_2|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'}C_\kappa e^{-j(\chi_1+\kappa_1)(t-1)}\,,\end{aligned}$$

where the only new justifications are that we use the time reversal of (3.31) in the third inequality since $\delta_2 \le \delta_1$ and $|U| \ge \delta_2/3$, and in the fourth inequality, we apply the time reversal of Lemma 3.10 since $j \ge n_1$. □

Using Sublemma 3.13, we estimate the right hand side of (3.22) as in the proof of Lemma 3.6, summing over $B' \in \mathbb{L}^{\delta_2}_{-n+1+j}$ and recalling that if $B \in L_j(B')$, then $B \in \mathbb{L}^{\delta_2}_{-n+1}$ and each such $B$ is associated with a unique $B'$:

$$\begin{aligned}\sum_{B\in\mathbb{S}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B \;&\le \sum_{B'\in\mathbb{S}^{\delta_2}_{-n_2}} C\theta^{t(n-n_2-1)}|\cos\varphi|^t_{T^{-n_2}B'}|J^uT^{-n_2}|^t_{B'}\\
&\quad+\sum_{j=n_1}^{n-n_2-1}C\delta_2^{-2}\theta^{tj}e^{j(\chi_1+\kappa_1)(t-1)}\sum_{B\in\mathbb{L}^{\delta_2}_{-n+1}}|\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B\\
&\le C_{n_2}\theta^{tn}+C\delta_2^{-2}\sum_{B\in\mathbb{L}^{\delta_2}_{-n+1}}|\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B\,,\end{aligned}\tag{3.32}$$

for some constant $C_{n_2}>0$ depending only on $n_2$, where we have used the fact that $\theta^te^{(\chi_1+\kappa_1)(t-1)}<1$ to sum over $j$.

The sum over $B \in \mathbb{L}^{\delta_2}_{-n+1}$ shrinks at a rate bounded below by $Ce^{-n(\chi_1+\kappa_1)(1-t)}$ by the proof of Sublemma 3.13. Thus we may choose $n_3 \ge n_2$ large enough that $C_{n_2}\theta^{tn} \le Ce^{-n(\chi_1+\kappa_1)(t-1)}$ for all $n \ge n_3$, which implies,

$$\sum_{B\in\mathbb{S}^{\delta_2}_{-n+1}} |\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B \le C\delta_2^{-2}\sum_{B\in\mathbb{L}^{\delta_2}_{-n+1}}|\cos\varphi|^t_{T^{-n+1}B}|J^uT^{-n+1}|^t_B\,.$$

The proof of the Lemma 3.6 proceeds without further changes from this point, ending the proof of Lemma 3.12 if $g=0$.

If $g \ne 0$, choosing $\delta_2$ as in the proof of Lemma 3.6 implies that (3.24) and (3.25) remain as written. The only change required in the proof is to use the lower bound (3.27) with $\kappa=\kappa_1$ to prove the analogue of Sublemma 3.7: There exists $C>0$ such that for all $n_1 \le j \le n-n_2-1$ and $B' \in \mathbb{L}^{\delta_2}_{-n+1+j}$,

$$|\cos\varphi|^t_{T^{-n+1+j}B'}|J^uT^{-n+1+j}|^t_{B'}|e^{S_{n-1-j}g}|_{T^{-n+1+j}B'}$$

$$\leq C\delta_2^{-1} e^{j(\chi_1+\kappa_1)(t-1)} e^{j|g|_{C^0}} \sum_{B\in L_j(B')} |\cos\varphi|^t_{T^{-n+1}B} |J^u T^{-n+1}|^t_B |e^{S_{n-1}g}|_{T^{-n+1}B}\,.$$

We then proceed as in (3.32) using the contraction provided by (3.30) to sum over $j$. □

3.5. **Lower Bounds on Complexity. Exact Exponential Growth of $Q_n(t,g)$.** In order to conclude that the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is $e^{P_*(t)}$ and to control the peripheral spectrum of $\mathcal{L}_t$, we shall establish the exact exponential growth of $Q_n(t)$.

The lower bound on the spectral radius of $\mathcal{L}_t$ is a consequence of the following lemma, guaranteeing that the weighted complexity of long elements of $\widehat{\mathcal{W}}^s$ grows at the rate $Q_n(t,g)$.

**Proposition 3.14.** *Let $t_0 \in (0,1)$, $t_1 \in (1,t_*)$ and $\bar t_1 \in (1,s_1)$. For any $\upsilon \geq 0$, there exists $c_1 = c_1(\upsilon) > 0$, with $c_1(\upsilon') \geq c_1(\upsilon)$ if $\upsilon' \in [0,\upsilon]$, such that, for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,*

$$\sum_{W_i\in\mathcal{G}_n(W)} |J_{W_i}T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \geq c_1 Q_n(t,g)\,,\ \forall n\geq 1\,,\ \forall t\in[t_0,\bar t_1]\,, \tag{3.33}$$

*for any $g \in C^1$ with $|\nabla g|_{C^0} \leq \upsilon$ and such that* (3.6) *and* (3.30) *hold.*

*Proof.* As usual we first consider $g = 0$. The main idea of the proof is to show that for each curve $W \in \mathcal{W}^s$ with $|W| \geq \delta_1/3$, the image $T^{-n}W$ intersects a positive fraction of elements of $\mathcal{M}_n^{0,\mathbb{H}}$, weighted by $|J^sT^n|^t$, for $n$ large enough. The mixing property of $\mu_{\mathrm{SRB}}$ is instrumental here.

To do this, we recall the construction of locally maximal homogeneous Cantor rectangles from [CM, Section 7.12] (and similar to those used in [BD, Section 5.3] where we worked[25] with $\mathcal{W}^s$ instead of $\mathcal{W}^s_{\mathbb{H}}$). We call $D \subset M$ a *solid rectangle* if $D$ is a closed, simply connected region whose boundary consists of two homogeneous unstable and two stable manifolds. Given such a rectangle $D$, the maximal Cantor rectangle $R(D)$ in $D$ is the union of all points in $D$ whose homogeneous stable and unstable manifolds completely cross $D$. Note that $R(D)$ is closed and contains the boundary of $D$ [CM, Section 7.11], but is not simply connected due to the effect of the singularities, which create, for any $\varepsilon > 0$, a dense set of points with stable and unstable manifolds shorter than $\varepsilon$.

In what follows, we restrict to Cantor rectangles with sufficiently high density, i.e.,

$$\inf_{x\in R} \frac{m_{W^u}(W^u(x)\cap R)}{m_{W^u}(W^u(x)\cap D(R))} \geq 0.99\,, \tag{3.34}$$

where $m_{W^u}$ denotes arclength measure along an unstable manifold. We say that a homogeneous stable curve $W \in \widehat{\mathcal{W}}^s_{\mathbb{H}}$ *properly crosses* a maximal homogeneous Cantor rectangle $R = R(D)$ satisfying (3.34) if $W$ crosses both unstable sides of $D$, and, in addition, for every $x \in R$, the point $W \cap W^u(x)$ divides the curve $W^u(x) \cap D(R)$ in a ratio between 0.1 and 0.9, and on either side of $W \cap W^u(x)$, the density of $R$ in $W^u(x) \cap D(R)$ is at least 0.9. Reversing the roles of stable and unstable manifolds, we obtain the analogous definition of an unstable curve properly crossing a Cantor rectangle.

By [CM, Lemma 7.87], we choose a finite number of locally maximal homogeneous Cantor rectangles $\mathcal{R}(\delta_2) = \{R_1,\dots,R_k\}$ satisfying (3.34) and its analogue along stable manifolds, with the property that any homogeneous stable or unstable curve of length at least $\delta_2/3$ properly crosses at least one of them. Let $\delta_2'$ be the minimum diameter of the rectangles in $\mathcal{R}(\delta_2)$ and note that $\delta_2'$ is a function only of $\delta_2$.

Now fix $n \geq 1$ and let $\mathcal{A}_n^i \subset \mathcal{A}_n(\delta_2)$ denote those elements $A \in \mathcal{A}_n(\delta_2)$ such that $B_{n-1}(A)$ contains an homogeneous unstable curve of length at least $\delta_2$ that properly crosses $R_i$. Due to Lemma 3.6 for $t \leq 1$ and Lemma 3.12 for $t > 1$, there exists $i^*$ such that

$$\sum_{A\in\mathcal{A}_n^{i^*}} \sup_{x\in A\cap M'} |J^sT^n(x)|^t \geq \frac{c_0}{k} Q_n(t)\,. \tag{3.35}$$

[25]The construction in [CM, Section 7.12] uses $\mathcal{W}^s_{\mathbb{H}}$, but since each $V$ in $\mathcal{W}^s$ are unions of manifolds $W_i$ in $\mathcal{W}^s_{\mathbb{H}}$, if the $W_i$ cross properly, so does $V$.

Fix an arbitrary homogeneous $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$, and let $R_j \in \mathcal{R}(\delta_2)$ denote the Cantor rectangle that is properly crossed by $W$ (recalling that $\delta_2 \le \delta_1$). By the mixing property of $\mu_{\mathrm{SRB}}$ and [CM, Lemma 7.90], there exists $N_1 = N_1(\delta_2) \ge 1$ such that $T^{-N_1} R_i$ has a homogeneous connected component that properly crosses $R_{i^*}$, for all $i = 1, \dots, k$. In particular, $T^{-N_1} R_j$ properly crosses $R_{i^*}$, so an element of $\mathcal{G}_{N_1}(W)$ properly crosses $R_{i^*}$.

Let $W_1 \in \mathcal{G}_{N_1}(W)$ denote the component of $T^{-N_1} W$ that properly crosses $R_{i^*}$ and note that $W_1$ crosses $B_{n-1}(A)$ for all $A \in \mathcal{A}_n^{i^*}$. Since $W_1$ is homogeneous and $N_1 \ge 1$, $W_1$ cannot cross a singularity line in $T^{-1}\mathcal{S}_0$ (since then the curve would have been subdivided at time $N_1 - 1$), and so for each such $A$, $W_1$ crosses an element $B'_A \in \mathcal{M}^{1,\mathbb{H}}_{-n+1}$, $B'_A \subset B_{n-1}(A)$. Let $V'_A = W_1 \cap B'_A$ and let $V_A = T^{-n+1} V'_A$. Then $V_A$ is a homogeneous component belonging to an element of $\mathcal{G}_{n-1+N_1}(W)$. By Lemma 2.2, recalling the notation $|J^s T^k|^t_{A'} = \sup_{x \in A' \cap M'} |J^s T^k(x)|^t$ from (3.17),

$$|J_{V_A} T^{n-1}|^t_{C^0(V_A)} = e^{\pm tC} |J^s T^{n-1}|^t_{T^{-n+1} B_{n-1}(A)} \,,$$

since $T^{-n+1} B_{n-1}(A) \in \mathcal{M}_0^{n-1,\mathbb{H}} \bigvee T^{-n+1}\mathcal{H}$. By definition, $T^{-n+1} B_{n-1}(A)$ contains $A$. Thus,

$$|J^s T^{n-1}|^t_A \le e^{tC} |J_{V_A} T^{n-1}|^t_{C^0(V_A)} \,. \tag{3.36}$$

Next, we wish to compare $J^s T$ on $T^{n-1}A$ with $J_{V'_A} T$. Since $V'_A \subset W_1 \subset T^{-N_1} W$, we have that $TV'_A$ is a stable curve, and so is $TW_1$, so that $J_{V'_A} T = e^{\pm C_d} J_{W_1} T = e^{\pm C_d} k^{-q}$, where $k$ is the index of the homogeneity strip containing $W_1$. But since $|W_1| \ge \delta'_2$ (since $W_1$ properly crosses $R_{i^*}$), we have $k \le (\delta'_2)^{-1/(q+1)}$ and so $J_{W_1} T \ge C(\delta'_2)^{q/(q+1)}$. Since $J^s T \le e^{C_d}$, we have using (3.36), that $|J^s T^n|^t_A \le C(\delta'_2)^{-tq/(q+1)} |J_{V_A} T^n|^t_{C^0(V_A)}$. Then summing over $A \in \mathcal{A}_n^{i^*}$, we obtain,

$$\sum_{A \in \mathcal{A}_n^{i^*}} |J^s T^n|^t_A \le C(\delta'_2)^{-tq/(q+1)} \sum_{V_i \in \mathcal{G}_n(TW_1)} |J_{V_i} T^n|^t_{C^0(V_i)} \,. \tag{3.37}$$

Next, we express the sum over $\mathcal{G}_{n+N_1-1}(W)$ in two ways. On the one hand, by Lemma 3.4,

$$\begin{aligned} \sum_{V_j \in \mathcal{G}_{n+N_1-1}(W)} |J_{V_j} T^{n+N_1-1}|^t_{C^0(V_j)} &\le \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \sum_{V_j \in \mathcal{G}_{N_1-1}(W_i)} |J_{V_j} T^{N_1-1}|^t_{C^0(V_j)} \\ &\le C_2[0] Q_{N_1-1}(t) \sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \,. \end{aligned} \tag{3.38}$$

On the other hand, letting $W'_1$ be the element of $\mathcal{G}_{N_1-1}(W)$ containing $TW_1$,

$$\begin{aligned} \sum_{V_j \in \mathcal{G}_{n+N_1-1}(W)} |J_{V_j} T^{n+N_1-1}|^t_{C^0(V_j)} &\ge e^{-tC_d} |J_{W'_1} T^{N_1-1}|^t_{C^0(W'_1)} \sum_{V_i \in \mathcal{G}_n(W'_1)} |J_{V_i} T^n|^t_{C^0(V_i)} \\ &\ge e^{-tC_d} C'(\delta'_2)^{t\left(\frac{2q+1}{q+1}\right)^{N_1-1}} \sum_{V_i \in \mathcal{G}_n(W'_1)} |J_{V_i} T^n|^t_{C^0(V_i)} \,, \end{aligned} \tag{3.39}$$

where the lower bound on $|J_{W'_1} T^{N_1-1}|^t_{C^0(W'_1)}$ comes from the fact that $|W'_1| \ge \delta'_2$ and for a stable curve $V$ such that $V$ and $T^{-1}V$ are both homogeneous, $|T^{-1}V| \le C|V|^{\frac{q+1}{2q+1}}$, and this bound can be iterated $N_1 - 1$ times as in [BD, eq. (5.3)].

Combining (3.37), (3.38) and (3.39), and recalling (3.35) yields,

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} \ge (C_2[0])^{-1} Q_{N_1-1}(t)^{-1} C''(\delta'_2)^{t\left(\frac{2q+1}{q+1}\right)^{N_1}} \frac{c_0}{k} Q_n(t) \,,$$

which completes the proof of the proposition if $g = 0$.

If $g \neq 0$, starting as above, we choose the finite family of Cantor rectangles $\mathcal{R}(\delta_2)$ in the same way, and find an index $i^*$ such that the analogue of (3.35)

$$\sum_{A \in \mathcal{A}_n^{i^*}} \sup_{x \in A \cap M'} |J^s T^n(x)|^t e^{S_n g(x)} \geq \frac{c_0}{k} Q_n(t, g) ,$$

holds, using Lemma 3.6 if $t \leq 1$ and Lemma 3.12 if $t > 1$. Fixing $W \in \mathcal{W}^s$, choosing $N_1$ as above, and using the same notation introduced there, we obtain the modification of (3.36),

$$||J^s T^{n-1}|^t e^{S_{n-1} g}|_A \leq e^{tC}(1 + \bar{C} C_* |\nabla g|_{C^0}) ||J_{V_A} T^{n-1}|^t e^{S_{n-1} g}|_{C^0(V_A)} ,$$

applying (2.14). Next, (3.37) needs only the multiplication by $e^{S_n g}$ to each term on both sides, up to replacing the constant $C$ by $C(1 + \bar{C}\, C_* \cdot |\nabla g|_{C^0})$. The upper bound (3.38) requires only a change of constant

to $C_2[0] Q_{N_1 - 1}(t, g)$, using Lemma 3.4 with $\varsigma = 0$, while the lower bound (3.39) requires the added factor $e^{-(N_1 - 1)|g|_{C^0}}$ on the right hand side. Since $N_1$ is fixed (depending only on $\mathcal{R}(\delta_2)$), these bounds are combined as in the case $g = 0$ to complete the proof of the proposition. □

The following important consequence of Proposition 3.14 will be used to characterize the peripheral spectrum of $\mathcal{L}_t$.

**Proposition 3.15** (Exact Exponential Growth of $Q_n(t, g)$)**.** *Let $t_0 \in (0, 1)$, $t_1 \in (1, t_*)$ and $\bar{t}_1 \in (1, s_1)$. For any $\upsilon > 0$ there exists $c_2(\upsilon) > 0$, with $c_2(\upsilon') \geq c_2(\upsilon)$ if $\upsilon' \in [0, \upsilon]$, such that for any $g \in C^1$ with $|\nabla g|_{C^0} \leq \upsilon$ and such that* (3.6) *and* (3.30) *hold, we have*

$$e^{nP_*(t,g)} \leq Q_n(t, g) \leq \frac{2}{c_2} e^{nP_*(t,g)} , \ \forall t \in [t_0, \bar{t}_1] , \ \forall n \geq 1 . \tag{3.40}$$

*Proof.* The lower bound follows immediately from submultiplicativity of $Q_n(t, g)$ (obtained in the proof of Proposition 2.5 for any $t > 0$ and $g \in C^1$) since then $P_*(t, g) = \inf_n \frac{1}{n} \log Q_n(t, g)$.

To obtain the upper bound for $g = 0$, we first prove the following supermultiplicative property: There exists $c_2 > 0$ such that for all $t \in [t_0, \bar{t}_1]$ and for any $j, n \geq 1$,

$$Q_{n+j}(t) \geq c_2 Q_n(t) Q_j(t) . \tag{3.41}$$

Let $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$. For $n$, $j \geq 1$, by Lemma 3.4 with $\varsigma = 0$,

$$\sum_{W_i \in \mathcal{G}_{n+j}^{\delta_1}(W)} |J_{W_i} T^{n+j}|^t_{C^0(W_i)} \leq C_2[0]\, Q_{n+j}(t) .$$

On the other hand, if $n \geq n_1$, then using Lemma 2.1,

$$\begin{aligned}
\sum_{W_i \in \mathcal{G}_{n+j}^{\delta_1}(W)} |J_{W_i} T^{n+j}|^t_{C^0(W_i)} &\geq C \sum_{V_k \in \mathcal{G}_n^{\delta_1}(W)} |J_{V_k} T^n|^t_{C^0(V_k)} \sum_{W_i \in \mathcal{G}_j^{\delta_1}(V_k)} |J_{W_i} T^j|^t_{C^0(W_i)} \\
&\geq C \sum_{V_k \in L_n^{\delta_1}(W)} |J_{V_k} T^n|^t_{C^0(V_k)} \sum_{W_i \in \mathcal{G}_j^{\delta_1}(V_k)} |J_{W_i} T^j|^t_{C^0(W_i)} \\
&\geq C \sum_{V_k \in L_n^{\delta_1}(W)} |J_{V_k} T^n|^t_{C^0(V_k)} c_1 Q_j(t) \\
&\geq C c_1 Q_j(t) \tfrac{3}{4} \sum_{V_k \in \mathcal{G}_n^{\delta_1}(W)} |J_{V_k} T^n|^t_{C^0(V_k)} \ \geq \ C' c_1^2 Q_j(t) Q_n(t) ,
\end{aligned}$$

where in the third and fifth inequalities, we have used Proposition 3.14 and in the fourth inequality we have applied (3.31). This proves (3.41) for $n \geq n_1$, and the case $n \leq n_1$ follows by adjusting the constant $c_2$. (Note that $c_2$ is uniform in $t$.) The proof of the upper bound on $Q_n(t)$ then proceeds precisely as in the proof of [BD, Proposition 4.6]. The case of nonzero $g$ is identical. □

3.6. **Growth Lemmas and Exact Exponential Growth for** $t \in (s_1, t_*)$**.** The main result of this section is Proposition 3.18 which extends Propositions 3.14 and 3.15 to all $t < t_*$. The constant $t_* > 1$ is defined by (1.9), while $s_1 > 1$ is introduced in Definition 3.9. What we have proved up to now suffices to establish all the results of Sections 4–6 for $t \in (0, s_1)$. In particular Theorem 2.4 holds in a neighbourhood of $t = 1$, so we know that $s_1 \le t_*$.

Recall that $t_1 \in (1, t_*)$ is fixed in Definition 3.2, determining $\theta \in (\Lambda^{-1}, \Lambda^{-1/2})$ and our main statements are for $t \in [t_0, t_1]$. If $s_1 \ge t_1$, there is nothing to do. Otherwise, $\theta^{s_1} < e^{P(s_1)} \le e^{P_*(s_1)}$ by Proposition 2.3. Since $P_*(t)$ is convex and decreasing, the left-hand slopes are lower semi-continuous, so we may choose $\bar t_1 \in (1, s_1)$ so that the intersection point $s_2(\bar t_1)$ between the tangent line to $P_*(t)$ (from the left) at $t = \bar t_1$ and the line $t \log \theta$ satisfies $s_2 > s_1$. Indeed, we have

$$s_2 = s_2(\bar t_1) := \frac{P_*(\bar t_1) + \chi_2 \bar t_1}{\chi_2 + \log \theta}\,, \quad \text{where } \chi_2 = \chi_2(\bar t_1) := \lim_{s \to \bar t_1^-} \frac{P_*(s) - P_*(\bar t_1)}{\bar t_1 - s} \ge \log \Lambda\,, \tag{3.42}$$

where (by convexity of $P_*(t)$) the limit defining $\chi_2$ exists and $P_*(t)$ lies above its tangents, so that $\theta^t < e^{P_*(t)}$ for all $t < s_2$.

Our next lemma is an analogue of Lemma 3.10, interpolating now from $\bar t_1$ to $s_2$.

**Lemma 3.16.** *Fix $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$, and let $\bar t_1 \in (1, s_1)$ and $s_2(\bar t_1) > s_1$ be as above. For any $\bar t_2 \in (s_1, s_2)$ and any $\kappa > 0$, there exist $C_\kappa > 0$, $\eta_\kappa > 0$ such that for all $g \in C^0$, all $\delta > 0$, all $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta/3$, and all $n \ge 1$,*

$$\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \ge C_\kappa \delta^{-1/\eta_\kappa} e^{-n(\chi_2 + \kappa)(t - \bar t_1) + n P_*(\bar t_1) - n|g|_{C^0}}\,, \ \forall t \in [\bar t_1, \bar t_2]\,. \tag{3.43}$$

*Proof.* We adapt the proof of Lemma 3.10. First assume $g = 0$. For $t \ge s_1$, let $s \in (1, \bar t_1)$, and $\eta(s) \in (0, 1]$ such that $\eta t + (1 - \eta) s = \bar t_1$. Then again using the Hölder inequality, for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta/3$ and all $n \ge 1$,

$$\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^t \ge \frac{\left(\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^{\bar t_1}\right)^{1/\eta}}{\left(\sum_{W_i \in \mathcal{G}_n^\delta(W)} |J_{W_i} T^n|^s\right)^{(1-\eta)/\eta}} \ge \frac{\left(c_1 e^{n P_*(\bar t_1)}\right)^{1/\eta}}{\left(C_2[0] \frac{\delta}{\delta_0} \frac{2}{c_0} e^{n P_*(s)}\right)^{(1-\eta)/\eta}}\,, \tag{3.44}$$

where we have used Propositions 3.14 and 3.15 for the lower bound in the numerator, and Lemma 3.4 with $\varsigma = 0$ and Proposition 3.15 for the upper bound in the denominator. Since $\eta = (\bar t_1 - s)/(t - s)$,

$$e^{-n(P_*(s) - P_*(\bar t_1))/\eta} e^{n P_*(s)} = e^{-n(t-s) \frac{P_*(s) - P_*(\bar t_1)}{\bar t_1 - s}} e^{n P_*(s)}\,.$$

For fixed $\kappa > 0$, by (3.42), we may choose $s = s(\kappa) \in (1, \bar t_1)$ and $\eta_\kappa > 0$ such that $(t - s) \frac{P_*(s) - P_*(\bar t_1)}{\bar t_1 - s} \le (t - \bar t_1)(\chi_2 + \kappa)$, completing the proof for $g = 0$ since $P_*(s) \ge P_*(\bar t_1)$. For $g \ne 0$, the lemma follows, again using the bound $|e^{S_n g}|_{C^0(W_i)} \ge e^{-n|g|_{C^0}}$. □

By definition, $\theta^t e^{\chi_2(t - \bar t_1) - P_*(\bar t_1)} < 1$ if $t < s_2$. Thus for $\bar t_2 \in (s_1, s_2)$, there exists $\kappa_2 = \kappa(\bar t_2) > 0$ such that

$$\theta^{\bar t_2} e^{(\chi_2 + \kappa_2)(\bar t_2 - \bar t_1) - P_*(\bar t_1)} < 1\,, \quad \text{and thus } \theta^t e^{(\chi_2 + \kappa_2)(t - \bar t_1) - P_*(\bar t_1)} < 1\,, \ \forall t \le \bar t_2\,. \tag{3.45}$$

Our next lemma extends Lemma 3.11 for $t \in [\bar t_1, \bar t_2]$.

**Lemma 3.17.** *Let $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$. Let $\bar t_2 \in (s_1, s_2)$, and let $\kappa_2 = \kappa(\bar t_2)$ satisfy (3.45). Then for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $n_1 \ge 1$, such that for all $W \in \widehat{\mathcal{W}}^s$ with $|W| \ge \delta_1/3$, and for all $n \ge n_1$,*

$$\sum_{\substack{W_i \in \mathcal{G}_n^{\delta_1}(W) \\ |W_i| < \delta_1/3}} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \le \varepsilon \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)}\,, \ \forall t \in [1, \bar t_2]\,,$$

*for all $g \in C^0$ satisfying* (3.6), (3.30) *and such that, in addition,*

$$2|g|_{C^0} < -\bar t_2 \log\theta - (\chi_2+\kappa_2)(\bar t_2 - \bar t_1) + P_*(\bar t_1)\,, \quad \text{i.e.} \quad \theta^{\bar t_2} e^{(\chi_2+\kappa_2)(\bar t_2-\bar t_1) - P_*(\bar t_1) + 2|g|_{C^0}} < 1\,. \tag{3.46}$$

*Proof.* The proof of Lemma 3.17 proceeds with the analogous modifications used in the proof of Lemma 3.11, using the lower bound (3.43) in place of (3.27). The proof goes through due to the contraction provided by (3.45) and (3.46). □

Let $[t_0, \bar t_2] \subset (0, s_2)$. For all $g \in C^1$ satisfying (3.6), (3.30), (3.46), Lemmas 3.5, 3.11 and 3.17 for $\varepsilon = 1/4$ give $n_1 \geq 1$ and $\delta_1 > 0$ such that for all $n \geq n_1$ and all $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,

$$\sum_{W_i \in L_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \geq \tfrac{3}{4} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)}\,, \; \forall t \in [t_0, \bar t_2]\,. \tag{3.47}$$

At this point it is clear that Lemma 3.12 (with the same constant $\delta_2 > 0$, but possibly smaller $c_0 > 0$), and Propositions 3.14 and 3.15 (with possibly smaller constants $c_1, c_2 > 0$) hold with $\bar t_1$ replaced by $\bar t_2 \in (s_1, s_2)$.

The interpolation can now be continued inductively. Suppose we have created a sequence $1 < \bar t_1 < s_1 < \bar t_2 < s_2 < \ldots < \bar t_n < s_n < t_1 < t_*$ so that Propositions 3.14 and 3.15 hold with $\bar t_1$ replaced by $\bar t_n$. Then since $s_n < t_1$, we have $\theta^{s_n} < e^{P_*(s_n)}$ and we may define

$$\chi_{n+1} = \lim_{s \to \bar t_n^-} \frac{P_*(s) - P_*(\bar t_n)}{\bar t_n - s} \geq \log\Lambda, \quad \text{and} \quad s_{n+1} = \frac{P_*(\bar t_n) + \chi_n \bar t_n}{\chi_n + \log\theta} > s_n\,,$$

where $s_{n+1} > s_n$ by choice of $\bar t_n$. Following the proof of Lemma 3.16 with $\bar t_1, \bar t_2, \chi_2$ replaced by $\bar t_n, \bar t_{n+1}, \chi_{n+1}$, it follows that the conclusion of the lemma holds for all $t \in [\bar t_n, \bar t_{n+1}]$. Analogous modifications to Lemma 3.17 imply that Lemma 3.12 and the propositions of Section 3.5 hold with $\bar t_1$ replaced by $\bar t_{n+1} \in (s_n, s_{n+1})$.

Finally, the sequence $(s_n)$ cannot accumulate on any $s_\infty \leq t_1$. For if it does, then by definition of $\theta$, it follows that $\theta^{s_\infty} < e^{P_*(s_\infty)}$, so we may repeat the construction above, finding a point of intersection $s' > s_\infty$ between $t \log\theta$ and the left hand tangent to $P_*(t)$ at some $\bar t_n < s_\infty$. It follows that this sequence of interpolations can be chosen so that $t_1 < \bar t_n < t_*$ for some $n \geq 1$. At this point we stop, and since we have made only finitely many choices of the required constants, we have extended the analogues of Propositions 3.14 and 3.15 to all $t_1 < t_*$:

**Proposition 3.18.** *Let $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$. For any $\upsilon \geq 0$, there exists $c_1(\upsilon)$, $c_2(\upsilon)$, with $\inf_{[0,\upsilon]} c_i \geq c_i(\upsilon) > 0$, $i = 1, 2$, such that, for any $g \in C^1$ with $|\nabla g|_{C^0} \leq \upsilon$ and such that $|g|_{C^0}$ is sufficiently small (depending on the number of interpolations required to reach $t_1$),*

a) *for any $W \in \widehat{\mathcal{W}}^s$ with $|W| \geq \delta_1/3$,*

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|^t_{C^0(W_i)} |e^{S_n g}|_{C^0(W_i)} \geq c_1 Q_n(t, g)\,, \; \forall n \geq 1\,, \; \forall t \in [t_0, t_1]\,;$$

b) *for all $n \geq 1$, we have* $e^{nP_*(t,g)} \leq Q_n(t,g) \leq \dfrac{2}{c_2} e^{nP_*(t,g)}\,, \; \forall t \in [t_0, t_1]\,.$

## 4. Spectral Properties of $\mathcal{L}_t$ (Theorem 4.1)

4.1. **Definition of Norms and Spaces $\mathcal{B}$ and $\mathcal{B}_w$.** For fixed $t_0 > 0$ and $t_1 \in (\max\{t_0, 1\}, t_*)$, we choose $\theta(t_1) \in (\Lambda^{-1}, \Lambda^{-1/2})$ satisfying $\theta^{t_1} < e^{P_*(t_1)}$, $q > \min\{1, 2/t_0\}$, $k_0 = k_0(t_0, t_1)$ (for the homogeneity strips (2.1)), and $\delta_0 = \delta_0(t_0, t_1)$ from Definition 3.2. These choices affect the definitions of $\mathcal{W}^s$ and $\mathcal{W}^s_{\mathbb{H}}$, as well as conditions (4.1) and (4.2) below on the parameters $\alpha$, $\beta$, $\gamma$, $p$, $\varepsilon_0$, determining spaces $\mathcal{B} = \mathcal{B}(t_0, t_1)$ and $\mathcal{B}_w = \mathcal{B}_w(t_0, t_1)$ on which $\mathcal{L}_t$ will be bounded for all $t \geq t_0$. An additional condition on the parameter $p$ depending on $t_1 < t_*$ will be needed to obtain the Lasota–Yorke bound (4.8) (see Lemma 4.7) and thus the spectral gap of $\mathcal{L}_t$ on $\mathcal{B}$ for all $t \in [t_0, t_1]$.

First we define notions of distance[26] between stable curves and test functions as follows.

Since the slopes of stable curves are uniformly bounded away from the vertical, we view each $W \in \widehat{\mathcal{W}}^s$ as the graph of a function of the $r$-coordinate over an interval $I_W$,

$$W = \{G_W(r) : r \in I_W\} = \{(r, \varphi_W(r)) : r \in I_W\}\,.$$

By the uniform bound on the curvature of $W \in \widehat{\mathcal{W}}^s$, we have $B := \sup_{W\in\widehat{\mathcal{W}}^s} |\varphi_W''| < \infty$.

Next, given $W_1, W_2 \in \widehat{\mathcal{W}}^s$ with functions $\varphi_{W_1}$, $\varphi_{W_2}$, we define

$$d_{\mathcal{W}^s}(W_1, W_2) = |I_{W_1} \triangle I_{W_2}| + |\varphi_{W_1} - \varphi_{W_2}|_{C^1(I_{W_1}\cap I_{W_2})}\,,$$

if $W_1$ and $W_2$ lie in the same homogeneity strip, and $d_{\mathcal{W}^s}(W_1, W_2) = 3B + 1$ otherwise.

Finally, if $d_{\mathcal{W}^s}(W_1, W_2) < 3B + 1$, then for $\psi_1 \in C^0(W_1)$, $\psi_2 \in C^0(W_2)$, define

$$d(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{C^0(I_{W_1}\cap I_{W_2})}\,,$$

while if $d_{\mathcal{W}^s}(W_1, W_2) \geq 3B + 1$ and $\psi_1 \in C^0(W_1)$, $\psi_2 \in C^0(W_2)$, we set $d(\psi_1, \psi_2) = \infty$.

We next define the norms, introducing parameters $\alpha$, $\beta$, $\gamma$, $p$, and $\varepsilon_0$. Choose[27]

$$\alpha \in \Big(0, \frac{1}{q+1}\Big]\,, \quad p > q+1, \quad \beta \in \Big(\frac{1}{p}, \alpha\Big), \quad \gamma \in \Big(0, \min\Big\{\frac{1}{p}, \alpha - \beta, \frac{1}{6q+7}\Big\}\Big)\,. \tag{4.1}$$

(This implies $\alpha \leq 1/3$, $\gamma < 1/p$, and $\min(\beta, t) > \frac{1}{p}$.) Finally for $C_{vert} < \infty$ to be determined in (4.21), let $\varepsilon_0$ satisfy

$$0 < C_{vert}\, \varepsilon_0^{1/(q+1)} \leq \frac{3}{4}\,. \tag{4.2}$$

For $f \in C^1(M)$, recalling $C^\eta(W)$ and $\mathcal{W}^s_H$ from Section 2.1, define the weak norm of $f$ by[28]

$$|f|_w = \sup_{W\in\mathcal{W}^s_H}\ \sup_{|\psi|_{C^\alpha(W)}\leq 1} \int_W f\,\psi\, dm_W\,,$$

define the stable norm of $f$ by

$$\|f\|_s = \sup_{W\in\mathcal{W}^s_H}\ \sup_{|\psi|_{C^\beta(W)}\leq |W|^{-1/p}} \int_W f\,\psi\, dm_W\,, \tag{4.3}$$

and the unstable norm of $f$ by

$$\|f\|_u = \sup_{\varepsilon\leq\varepsilon_0}\ \sup_{\substack{W_1,W_2\in\mathcal{W}^s_H\\ d_{\mathcal{W}^s}(W_1,W_2)\leq\varepsilon}}\ \sup_{\substack{|\psi_i|_{C^\alpha(W_i)}\leq 1\\ d(\psi_1,\psi_2)=0}} \varepsilon^{-\gamma}\left|\int_{W_1} f\,\psi_1\, dm_{W_1} - \int_{W_2} f\,\psi_2\, dm_{W_2}\right|\,.$$

Finally, define the strong norm of $f$ to be

$$\|f\|_{\mathcal{B}} = \|f\|_s + c_u\|f\|_u$$

for a constant $c_u = c_u(\beta, \gamma, p) > 0$ (so that $c_u$ depends on $[t_0, t_1]$) to be chosen in (4.9). Define $\mathcal{B}$ to be the completion of $C^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm, and $\mathcal{B}_w$ to be the completion of $C^1(M)$ in the $|\cdot|_w$ norm.

---

[26]The triangle inequality is not satisfied, but this is of no consequence for our purposes.

[27]The condition $\gamma \leq \frac{1}{6q+7}$ is used in Lemma 4.3.

[28]Using weakly homogeneous curves implies that Lebesgue measure belongs to $\mathcal{B}$, see Remark 4.5.

4.2. **Statement of the Spectral Result. Embeddings.** The equilibrium measure in Theorem 1.1 and its properties will be obtained by letting the transfer operator $\mathcal{L}_t$ act on $\mathcal{B}$.

**Theorem 4.1** (Spectrum of $\mathcal{L}_t$ on $\mathcal{B}$)**.** *For each $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$ there exists a Banach space $\mathcal{B} = \mathcal{B}(t_0, t_1)$ such that for each $t \in [t_0, t_1]$, the operator $\mathcal{L}_t$ is bounded on $\mathcal{B}$ with spectral radius equal to $e^{P_*(t)}$ and, recalling $\theta(t_1)$ from Lemma 3.1, essential spectral radius not larger than*

$$\max\{\Lambda^{(-\beta+1/p)}, \theta^{(t-1/p)} e^{-P_*(t)}, \Lambda^{-\gamma}\} e^{P_*(t)} < e^{P_*(t)} .$$

*Moreover $\mathcal{L}_t$ has a spectral gap: the only eigenvalue of modulus $e^{P_*(t)}$ is $e^{P_*(t)}$ and it is simple.*

*Let $\nu_t$ denote the unique element of $\mathcal{B}$ with $\nu_t(1) = 1$ satisfying $\mathcal{L}_t \nu_t = e^{P_*(t)} \nu_t$, and let $\tilde{\nu}_t$ denote the maximal eigenvector for the dual, $\mathcal{L}_t^* \tilde{\nu}_t = e^{P_*(t)} \tilde{\nu}_t$. Then the distribution $\mu_t$ defined by $\mu_t(\psi) = \frac{\tilde{\nu}_t(\psi \nu_t)}{\tilde{\nu}_t(\nu_t)}$ is in fact a $T$-invariant probability measure. This measure is mixing, correlations for $C^\alpha$ observables decay exponentially with rate $\upsilon$ for any*

$$\upsilon > \upsilon_0(t) := \sup\{|\lambda| \mid \lambda \in sp(e^{-P_*(t)} \mathcal{L}_t) \setminus \{1\}\} , \tag{4.4}$$

*and correlations for Hölder observables of arbitrary exponent decay exponentially.*

Recall that $\theta < 1/\sqrt{\Lambda} < 1$. Note that since $qt_0 > 1$ while $\beta < 1/(q+1)$ and $\gamma \le \min\{1/(q+1), 1/(q+1) - \beta\}$, our bound on the essential spectral radius tends to $e^{P_*(t_0)}$ as $t_0 \to 0$. Similarly, as $t_1 \to t_*$ we need to let $p \to \infty$ to ensure $\theta^{t_1 - 1/p} < e^{P_*(t_1)}$ (see Lemma 4.7) and our bound on the essential spectral radius tends to $e^{P_*(t_*)}$ as $t_1 \to t_*$.

As usual, Hennion's theorem is the key to prove the above theorem. It requires two ingredients: the compact embedding proposition below and the Lasota–Yorke estimates in Proposition 4.6.

**Proposition 4.2** (Embeddings)**.** *For any $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$, the continuous inclusions*

$$C^1(M) \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^1(M))^*$$

*hold, so that $C^1(M) \subset (\mathcal{B}_w)^* \subset \mathcal{B}^* \subset (C^1(M))^*$. In addition, the inclusions $C^1(M) \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{B}_w$ are injective, and the embedding of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ is compact.*

The embedding $\mathcal{B}_w \subset (C^1(M))^*$ is understood in the following sense: For $f \in C^1(M)$, we identify $f$ with the measure $f d\mu_{\mathrm{SRB}} \in (C^1(M))^*$. Then, for $f \in \mathcal{B}_w$ there exists $C_f < \infty$ such that, letting $f_n \in C^1(M)$ be a sequence converging to $f$ in the $\mathcal{B}_w$ norm, for every $\psi \in C^1(M)$ the limit $f(\psi) := \lim_{n \to \infty} \int f_n \psi \, d\mu_{\mathrm{SRB}}$ exists and satisfies $|f(\psi)| \le C_f |\psi|_{C^1(M)}$. See Lemma 4.14 for a strengthening of this embedding.

*Proof of Proposition 4.2.* The proof of the claims in the first sentence is the same as the proof of [BD, Prop. 4.2, Lemma 4.4]. The injectivity of the first inclusion is obvious, while the injectivity of the second follows from our definition of $C^\beta(W)$: if $|f|_w = 0$ then $\|f\|_u = 0$ since the class of test functions is the same, but also $\|f\|_s = 0$ since $C^1(W)$ is dense in $C^\beta(W)$, proving injectivity. The proof of the compact embedding follows exactly the lines of that of [BD, Prop. 6.1], using $\widehat{\mathcal{W}}^s$. The only differences are that, in the unstable norm, $|\psi|_{C^\beta(W)} \le |\log |W||^{\tilde{\gamma}}$ there is replaced by $|\psi|_{C^\beta(W)} \le |W|^{-1/p}$, while the logarithmic modulus of continuity $|\log \epsilon|^{-\zeta}$ there is replaced by a Hölder modulus of continuity $\epsilon^\gamma$. □

To show that the transfer operator $\mathcal{L}_t$ is bounded on $\mathcal{B}$, we require the following lemma.

**Lemma 4.3.** *For any $f \in C^1(M)$ and any $t \ge t_0$, the image $\mathcal{L}_t f$ belongs to the closure of $C^1(M)$ in the strong norm $\| \cdot \|_{\mathcal{B}}$, for $\mathcal{B} = \mathcal{B}(t_0, t_1)$.*

We prove Lemma 4.3 in Section 4.4.

**Remark 4.4** (Lemma 4.9 in [BD]). *We remark that the proof of* [BD, Lemma 4.9]*, which is the analogue of the present Lemma 4.3, was omitted there. The reference given there to* [DZ1, Lemma 3.8] *is not correct since $J^sT$ is not piecewise Hölder. However, its statement is correct as the proof in Section 4.4 and Remark 4.11 demonstrate.*

**Remark 4.5** (Lebesgue measure belongs to $\mathcal{B}$). *Since we identify $f \in C^1(M)$ with the measure $f d\mu_{SRB}$, Lebesgue measure is identified with the function $f = 1/\cos\varphi$, which is not in $C^1(M)$. However, it follows from* [DZ2, Lemma 3.5]*, that $1/\cos\varphi$ can be approximated by $C^1$ functions in the $\mathcal{B}$ norm, so that Lebesgue measure belongs to $\mathcal{B}$. (The proof requires that our norms integrate on weakly homogeneous stable manifolds, rather than on all $W \in \mathcal{W}^s$ as was done in* [BD]*.)*

4.3. **Lasota–Yorke Inequalities.** Using the exact bounds for $Q_n(t)$ from Proposition 3.15, we prove the following proposition (under more general conditions than Theorem 4.1).

**Proposition 4.6.** *Fix $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$ and let $\mathcal{B} = \mathcal{B}(t_0, t_1)$, $\theta = \theta(t_1)$. Fix $t_2 \in (t_0, \infty)$. There exists $C = C(t_0, t_2) < \infty$ and, for every $n \geq 0$, there exists $C_n = C_n(t_0, t_2) < \infty$ such that, for any $t \in [t_0, t_2]$, the operator $\mathcal{L}_t$ extends continuously to $\mathcal{B}_w$ and $\mathcal{B}$ and*

$$|\mathcal{L}_t^n f|_w \leq C Q_n(t) |f|_w \,, \ \forall f \in \mathcal{B}_w \,, \tag{4.5}$$

$$\|\mathcal{L}_t^n f\|_s \leq C Q_n(t) \left[(\Lambda^{(-\beta+1/p)n} + \theta^{(t-1/p)n} Q_n(t)^{-1})\right] \|f\|_s + C_n |f|_w \,, \ \forall\, f \in \mathcal{B} \,, \tag{4.6}$$

$$\|\mathcal{L}_t^n f\|_u \leq C Q_n(t) \left[n \Lambda^{-\gamma n} \|f\|_u + Q_n(t - 1/p) Q_n(t)^{-1} \|f\|_s\right] \,, \ \forall f \in \mathcal{B} \,. \tag{4.7}$$

*Moreover, if $t_2 = t_1$, then, up to taking $p$ large enough, for any*

$$\sigma \in (\max\{\Lambda^{-\beta+1/p}, \Lambda^{-\gamma}, \theta^{t-1/p} e^{-P_*(t)}\}, 1) \,,$$

*there exists $c_u = c_u(t_0, t_1) > 0$, and $\bar{C}_n > 0$, such that, for all $f \in \mathcal{B}$,*

$$\|\mathcal{L}_t^n f\|_{\mathcal{B}} \leq C e^{P_*(t)n} \left[\sigma^n \|f\|_{\mathcal{B}} + \bar{C}_n |f|_w\right] \,, \ \forall n \geq 1 \,. \tag{4.8}$$

Proving (4.8) will use the following lemma:

**Lemma 4.7.** *For any $t_1 \in (1, t_*)$ there exists $p > 1$ such that $\theta^{t-1/p} < e^{P_*(t)}$ for all $t \in (1/p, t_1]$.*

*Proof.* If $t \in (0,1]$ then $P_*(t) \geq 0$ so that $\theta^{t-1/p} < 1 \leq e^{P_*(t)}$ for all $\theta < 1$, all $p > 1$ and all $t \in (1/p, 1]$. For $t \in (1, t_1]$, since the slopes of $P_*(t)$ are at most $-\log\Lambda$ by the proof of Proposition 2.5, we have $\frac{P_*(t_1) - P_*(t)}{t_1 - t} < -\log\Lambda < \log\theta$, so that $\theta^t e^{-P_*(t)} < \theta^{t_1} e^{-P_*(t_1)}$. The choice of $\theta = \theta(t_1)$ in Definition 3.2 gives $\theta^{t_1} e^{-P_*(t_1)} < 1$. Choosing $p > 1$ such that $\theta^{t_1 - 1/p} e^{-P_*(t_1)} < 1$ ends the proof. □

*Proof of Proposition 4.6.* We first show that (4.5), (4.6), and (4.7) imply that if $t_2 = t_1 < t_*$ and $p$ is large enough, then $\mathcal{L}_t$ satisfies the Lasota–Yorke inequality (4.8) for $f \in \mathcal{B}(t_0, t_1)$: Choosing $p$ according to Lemma 4.7, observe that $\theta^{t-1/p} e^{-P_*(t)} < 1$ implies $\theta^{t-1/p} Q_n(t)^{-1} \leq \theta^{(t-1/p)n} e^{-P_*(t)n} < 1$ for all $n \geq 1$, since $Q_n(t) \geq e^{P_*(t)n}$ by Proposition 3.15. Next, recalling that $P_*(t)$ is strictly decreasing by Proposition 2.5, and fixing

$$\varepsilon_1 := P_*(t - 1/p) - P_*(t) \in (0, P_*(t_0 - 1/p) - P_*(t_1)) \,,$$

we find, using both the lower and upper bounds from Proposition 3.18(b),

$$Q_n(t - 1/p) Q_n(t)^{-1} \leq \frac{2}{c_2} e^{P_*(t-1/p)n} e^{-P_*(t)n} \leq \frac{2}{c_2} e^{\varepsilon_1 n} \,, \ \forall n \geq 1 \,.$$

Next, fix $1 > \sigma > \max\{\Lambda^{-\beta+1/p}, \Lambda^{-\gamma}, \theta^{t-1/p} e^{-P_*(t)}\}$ and choose $N \geq 1$ such that

$$\frac{2C}{c_2} \max\{N \Lambda^{-\gamma N}, 2(\Lambda^{-(\beta-1/p)N} + \theta^{(t-1/p)N} e^{-P_*(t)N})\} \leq \sigma^N \,.$$

Choosing $c_u > 0$ to satisfy

$$c_u \le \frac{c_2^2 \sigma^N}{8C e^{2(P_*(t_0 - 1/p) - P_*(t_1))N}}\,, \tag{4.9}$$

we estimate, using once more the upper bound for $Q_n(t)$ from Proposition 3.18(b),

$$\begin{aligned}\|\mathcal{L}_t^N f\|_{\mathcal{B}} &= \|\mathcal{L}_t^N f\|_s + c_u \|\mathcal{L}_t^N f\|_u \\ &\le e^{P_*(t)N}\left[\frac{\sigma^N}{2}\|f\|_s + c_u \sigma^N \|f\|_u + \frac{4c_u}{c_2^2} e^{\varepsilon_1 N}\|f\|_s\right] + C_N |f|_w \\ &\le e^{P_*(t)N}\left[\sigma^N \|f\|_{\mathcal{B}} + e^{-P_*(t)N} C_N |f|_w\right].\end{aligned}$$

Iterating this equation and using the first claim of (4.5) (recalling one more time the upper bound for $Q_n(t)$ from Proposition 3.18(b)) yields (4.8) for $n = \ell N$, with $\ell \ge 1$. The general case follows since (4.6) and (4.7) imply $\|\mathcal{L}_t^k f\|_{\mathcal{B}} \le \bar{C}\|f\|_{\mathcal{B}}$ for $k \le N$.

By Lemma 4.3, it suffices to prove the bounds (4.5), (4.6), and (4.7) for $f \in C^1(M)$, and they also imply that $\mathcal{L}_t$ extends to a bounded operator on $\mathcal{B}$ and $\mathcal{B}_w$. This is similar to the proof of [DZ1, Proposition 2.3] and is the content of Sections 4.3.1–4.3.3. □

4.3.1. *Proof of Weak Norm Bound* (4.5). Let $f \in C^1(M)$, $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ such that $|\psi|_{C^\alpha(W)} \le 1$. Then for $n \ge 0$, we have

$$\begin{aligned}\int_W \mathcal{L}_t^n f\, \psi\, dm_W &= \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f \psi \circ T^n |J^s T^n|^t \, dm_{W_i} \\ &\le \sum_{W_i \in \mathcal{G}_n(W)} |f|_w |\psi \circ T^n|_{C^\alpha(W)} ||J^s T^n|^t|_{C^\alpha(W)}\,.\end{aligned} \tag{4.10}$$

The contraction along stable manifolds implies for $x, y \in W_i \in \mathcal{G}_n(W)$, recalling (2.2),

$$|\psi(T^n x) - \psi(T^n y)| \le H^\alpha_W(\psi) d(T^n x, T^n y)^\alpha \le H^\alpha_W(\psi) |J^s T^n|^\alpha_{C^0(W_i)} d(x,y)^\alpha\,. \tag{4.11}$$

This implies $H^\alpha_{W_i}(\psi \circ T^n) \le |J^s T^n|^\alpha_{C^0(W_i)} H^\alpha_W(\psi)$ and $|\psi \circ T^n|_{C^\alpha(W_i)} \le C_1^{-1} |\psi|_{C^\alpha(W)}$, with $C_1$ from (1.2).

Moreover, since $\alpha \le 1/(q+1)$, the distortion bound of Lemma 2.1 implies

$$||J^s T^n|^t|_{C^\alpha(W_i)} \le (1 + 2^t C_d) |J^s T^n|^t_{C^0(W_i)}\,,\ \forall\, W_i \in \mathcal{G}_n(W)\,. \tag{4.12}$$

Using (4.11) and (4.12) in (4.10), we obtain,

$$\int_W \mathcal{L}_t^n f\, \psi\, dm_W \le \sum_{W_i \in \mathcal{G}_n(W)} |f|_w C_1^{-1}(1 + 2^t C_d) |J^s T^n|^t_{C^0(W_i)} \le C |f|_w Q_n(t)\,,$$

where in the last inequality, we have used Lemma 3.4 with $\varsigma = 0$. Taking the suprema over $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \le 1$ and $W \in \mathcal{W}^s$ yields (4.5).

4.3.2. *Proof of Stable Norm Bound* (4.6). Let $f \in C^1(M)$, $W \in \mathcal{W}^s$, and $\psi \in C^\beta(W)$ be such that $|\psi|_{C^\beta(W)} \le |W|^{-1/p}$. For $n \ge 0$ and $W_i \in \mathcal{G}_n(W)$, we define the average $\overline{\psi}_i = |W_i|^{-1} \int_{W_i} \psi \circ T^n \, dm_{W_i}$. Then as in (4.10), we write,

$$\begin{aligned}\int_W \mathcal{L}_t^n f\, \psi\, dm_W = &\sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f\,(\psi \circ T^n - \overline{\psi}_i) |J^s T^n|^t \, dm_{W_i} \\ &+ \sum_{W_i \in \mathcal{G}_n(W)} \overline{\psi}_i \int_{W_i} f\, |J^s T^n|^t \, dm_{W_i}\,.\end{aligned} \tag{4.13}$$

Note that by (4.11),

$$|\psi \circ T^n - \overline{\psi}_i|_{C^\beta(W_i)} \le 2|J^sT^n|^\beta_{C^0(W_i)}|\psi|_{C^\beta(W)} \le 2|J^sT^n|^\beta_{C^0(W_i)}|W|^{-1/p}.$$

Therefore, replacing $\alpha$ by $\beta$ in (4.12), the definition of the strong stable norm gives

$$\begin{aligned}
&\sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i} f\,(\psi \circ T^n - \overline{\psi}_i)|J^sT^n|^t\, dm_{W_i} \\
&\qquad \le \sum_{W_i \in \mathcal{G}_n(W)} 2(1+2^tC_d)\|f\|_s \frac{|W_i|^{1/p}}{|W|^{1/p}}|J^sT^n|^{t+\beta}_{C^0(W_i)} \\
&\qquad \le 2(1+2^tC_d)C_1^{-1}\Lambda^{-n(\beta-1/p)}\|f\|_sC_2[0]\,Q_n(t)\,,
\end{aligned} \tag{4.14}$$

where in the second inequality we have used Lemma 3.4 with $\varsigma = 1/p$ (recall $\beta > 1/p$).

For the second sum in (4.13), note that $|\overline{\psi}_i| \le |W|^{-1/p}$. If $|W| \ge \delta_0/3$, then we simply estimate

$$\sum_{W_i \in \mathcal{G}_n(W)} \overline{\psi}_i \int_{W_i} f\,|J^sT^n|^t\, dm_{W_i} \le \frac{3}{\delta_0^{1/p}}|f|_w(1+2^tC_d)\sum_{W_i \in \mathcal{G}_n(W)} |J^sT^n|^t_{C^0(W_i)} \le C|f|_wQ_n(t)\,,$$

by Lemma 3.4 with $\varsigma = 0$.

If $|W| < \delta_0/3$, we handle the estimate differently, splitting the sum into two parts as follows. We decompose the elements of $\mathcal{G}_n(W)$ by *first long ancestor* as follows: Recalling the sets $\mathcal{I}_n(W)$ defined in §3.1, we call $V_j \in \mathcal{G}_k(W)$ the first long ancestor of $W_i \in \mathcal{G}_n(W)$ if

$$T^{n-k}W_i \subset V_j\,, \quad |V_j| \ge \delta_0/3\,, \quad \text{and } TV_j \text{ is contained in an element of } \mathcal{I}_{k-1}(W)\,. \tag{4.15}$$

We denote by $P_k(W)$ the set of such $V_j \in \mathcal{G}_k(W)$ that are long for the first time at time $k$. Note that $W_i$ has no long ancestor if and only if $W_i \in \mathcal{I}_n(W)$.

Grouping the terms in the second sum in (4.13) by whether they belong to $\mathcal{I}_n(W)$ or not, we apply the weak norm to those elements that have a first long ancestor, and the strong stable norm to those that do not. Thus,

$$\begin{aligned}
&\sum_{W_i \in \mathcal{I}_n(W)} \overline{\psi}_i \int_{W_i} f\,|J^sT^n|^t\, dm_{W_i} \le \sum_{W_i \in \mathcal{I}_n(W)} |\overline{\psi}_i|\|f\|_s|W_i|^{1/p}|J^sT^n|^t_{C^\beta(W_i)} \\
&\le (1+2^tC_d)\|f\|_s \sum_{W_i \in \mathcal{I}_n(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}}|J^sT^n|^t_{C^0(W_i)} \le (1+2^tC_d)\|f\|_sC_0\theta^{n(t-1/p)}\,,
\end{aligned} \tag{4.16}$$

where in the last estimate we applied Lemma 3.3 with $\varsigma = 1/p$ since[29] $1/p \le \min\{1/2, t/2\}$.

For the terms that have a first long ancestor in $P_k(W)$, we again apply Lemma 3.3 from time 0 (since $|W| < \delta_0/3$) to time $k$, setting $\mathcal{G}_0(V) = \{V\}$,

$$\begin{aligned}
&\sum_{k=1}^n \sum_{V_j \in P_k(W)} \sum_{W_i \in \mathcal{G}_{n-k}(V_j)} \overline{\psi}_i \int_{W_i} f\,|J^sT^n|^t\, dm_{W_i} \\
&\quad \le \sum_{k=1}^n \sum_{V_j \in P_k(W)} |f|_w|V_j|^{-1/p}(1+2^tC_d)\frac{|V_j|^{1/p}}{|W|^{1/p}}|J^sT^k|^t_{C^0(V_j)} \sum_{W_i \in \mathcal{G}_{n-k}(V_j)} |J^sT^{n-k}|^t_{C^0(W_i)} \\
&\quad \le \sum_{k=1}^n \sum_{V_j \in P_k(W)} |f|_w3\delta_0^{-1/p}(1+2^tC_d)\frac{|V_j|^{1/p}}{|W|^{1/p}}|J^sT^k|^t_{C^0(V_j)}CC_2[0]\,\mathcal{Q}_{n-k}(t) \\
&\quad \le \sum_{k=1}^n |f|_w3\delta_0^{-1/p}(1+2^tC_d)CC_2[0]\,C_0\theta^{k(t-1/p)}\mathcal{Q}_{n-k}(t)\,,
\end{aligned}$$

[29]This bound holds since $p > q+1$ in the definition of the norms, yet $q \ge 2/t$ from (2.1).

applying Lemma 3.4 for $\varsigma = 0$ in the second inequality and Lemma 3.3 (for $\varsigma = 1/p$) in the third.

Putting these estimates together with (4.14) in (4.13) yields[30]

$$\int_W \mathcal{L}_t^n f\, \psi\, dm_W \le C Q_n(t)\big(\Lambda^{-n(\beta-1/p)} + \theta^{n(t-1/p)} Q_n(t)^{-1}\big)\|f\|_s + C \max_{0\le j\le n} \mathcal{Q}_j(t)|f|_w\,,$$

and taking the appropriate suprema proves (4.6) ($C_n$ depends on $t$ only through $[t_0, t_1]$).

4.3.3. *Proof of Unstable Norm Bound* (4.7). Let $\varepsilon < \varepsilon_0$ and let $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$. For $n \ge 1$ and $\ell = 1, 2$, we partition $T^{-n}W^\ell$ into matched pieces $U_j^\ell$ and unmatched pieces $V_i^\ell$ like in [DZ1] as follows.

To each homogeneous connected component $V$ of $T^{-n}W^1$, we associate a family of vertical segments $\{\gamma_x\}_{x\in V}$ of length at most $C_1^{-1}\Lambda^{-n}\varepsilon$ such that if $\gamma_x$ is not cut by an element of $\mathcal{S}_n^{\mathbb{H}}$, its image $T^n\gamma_x$ will have length $C\varepsilon$ and will intersect $W^2$. According to [CM, Sect. 4.4], for such a segment, $T^i\gamma_x$ will be an unstable curve for $i = 1, \dots, n$ and so will remain uniformly transverse to the stable cone and undergo the minimum expansion given by (1.2).

Repeating this procedure for each connected component of $T^{-n}W^1$, we obtain a partition of $W^1$ into subintervals for which $T^n\gamma_x$ is not cut and intersects $W^2$ and subintervals for which this is not the case. This also defines an analogous partition on $W^2$ and on the images $T^{-n}W^1$ and $T^{-n}W^2$. We call two curves in $T^{-n}W^1$ and $T^{-n}W^2$ *matched* if they are connected by the foliation $\gamma_x$ and their images under $T^n$ are connected by $T^n\gamma_x$. We further subdivide the matched pieces if necessary to ensure that they have length $\le \delta_0$ and that they remain homogeneous stable curves. Thus there are at most two matched pieces $U_j^\ell$ corresponding to each element of $\mathcal{G}_n(W^\ell)$. The rest of the connected components of $T^{-n}W^\ell$ we call *unmatched* and denote them by $V_i^\ell$. Once again, there are at most two unmatched pieces $V_i^\ell$ corresponding to each element of $\mathcal{G}_n(W^\ell)$.

Recalling the notation of Section 4.1, we have constructed a pairing on matched pieces $U_j^\ell$ defined over a common $r$-interval $I_j$ such that for each $j$,

$$U_j^\ell = G_{U_j^\ell}(I_j) = \{(r, \varphi_{U_j^\ell}(r)) : r \in I_j\}\,, \quad \ell = 1, 2. \tag{4.17}$$

Now let $\psi_\ell \in C^\alpha(W^\ell)$ with $|\psi_\ell|_{C^\alpha(W^\ell)} \le 1$ and $d(\psi_1, \psi_2) = 0$. Decomposing $W^1$ and $W^2$ into matched and unmatched pieces as above, we write,

$$\begin{aligned}\left|\int_{W^1} \mathcal{L}_t^n f\, \psi_1 - \int_{W^2} \mathcal{L}_t^n f\, \psi_2\right| &\le \sum_j \left|\int_{U_j^1} f\, \psi_1\circ T^n\, |J^sT^n|^t - \int_{U_j^2} f\, \psi_2 \circ T^n\, |J^sT^n|^t\right| \\ &\quad + \sum_{\ell,i}\left|\int_{V_i^\ell} f\, \psi_\ell \circ T^n\, |J^sT^n|^t\right|\,.\end{aligned} \tag{4.18}$$

We estimate the unmatched pieces first. For this we use the fact that unmatched pieces $V_i^\ell$ occur either because $T^nV_i^\ell$ is near the endpoints of $W^\ell$ or because a vertical segment $T^n\gamma_x$ intersects $\mathcal{S}_{-n}^{\mathbb{H}}$. In either case, due to the uniform transversality of the stable and unstable cones, we have $|T^nV_i^\ell| \le C\varepsilon$ for some uniform constant $C > 0$, independent of $n$ and $W^\ell$, since $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$. Thus, we estimate the sum over unmatched pieces using the strong stable norm,

$$\begin{aligned}\sum_{\ell,i}\left|\int_{V_i^\ell} f\, \psi_\ell\circ T^n\, |J^sT^n|^t\right| &\le \sum_{\ell,i} \|f\|_s |V_i^\ell|^{1/p} |\psi_\ell\circ T^n|_{C^\beta(V_i^\ell)}(1 + 2^tC_d)|J^sT^n|^t_{C^0(V^\ell)_i} \\ &\le \|f\|_s C_1^{-1}(1+2^tC_d)\sum_{\ell,i}|T^nV_i^\ell|^{1/p}|J^sT^n|^{t-1/p}_{C^0(V^\ell)_i} \\ &\le 4C_2[0]\, C_1^{-1}(1+2^tC_d)\|f\|_s\varepsilon^{1/p}Q_n(t-1/p)\,,\end{aligned} \tag{4.19}$$

[30] It is in fact possible to show $\max_{0\le j\le n}\mathcal{Q}_j(t) \le \max\{\mathcal{Q}_n(t), \mathcal{Q}_n(1)\}$, but we shall not use this.

where $C_1$ is from (1.2) and we have used (4.12) in the first inequality, (4.11) in the second, and Lemma 3.4 (for $\varsigma = 0$) in the third since there are at most two unmatched pieces corresponding to each element of $\mathcal{G}_n(W^\ell)$.

To perform the estimate over matched pieces in (4.18), we need the following sublemma.

**Sublemma 4.8.** *There exists $C > 0$, independent of $t$, $n$, $W^1$, and $W^2$ such that*

a) $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le Cn\Lambda^{-n}\varepsilon =: \varepsilon_1$, *for all $j$;*

b) $|\psi_1 \circ T^n |J^sT^n|^t - \widetilde{\psi}_2 |\widetilde{J}^sT^n|^t|_{C^\beta(U_j^1)} \le C2^t |J^sT^n|^t_{C^0(U_j^1)} \varepsilon^{\alpha-\beta}$, *for all $j$.*

*Proof.* Part (a) of the sublemma is [DZ1, Lemma 4.2]. To prove part (b), note that due to the uniform bound on slopes of stable curves, it follows

$$1 \le JG_W(r) := \sqrt{1 + (\varphi_W'(r))^2} \le \sqrt{1 + (\mathcal{K}_{\max} + \tau_{\min}^{-1})^2} =: C_g < \infty \,. \tag{4.20}$$

Therefore $1 \le |JG_{U_j^\ell}|_{C^0(I_j)} \le C_g$, and we have

$$\begin{aligned}
&|\psi_1 \circ T^n |J^sT^n|^t - \widetilde{\psi}_2 |\widetilde{J}^sT^n|^t|_{C^\beta(U_j^1)} \\
&\qquad \le C_g |(\psi_1 \circ T^n |J^sT^n|^t) \circ G_{U_j^1} - (\psi_2 |\widetilde{J}^sT^n|^t) \circ G_{U_j^2}|_{C^\beta(I_j)} \\
&\qquad \le C_g |\psi_2 \circ T^n|_{C^\beta(U_j^2)} ||J^sT^n|^t \circ G_{U_j^1} - |J^sT^n|^t \circ G_{U_j^2}|_{C^\beta(I_j)} \\
&\qquad\qquad + C_g ||J^sT^n|^t|_{C^\beta(U_j^1)} |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \\
&\qquad \le C_g C_1^{-1} ||J^sT^n|^t \circ G_{U_j^1} - |J^sT^n|^t \circ G_{U_j^2}|_{C^\beta(I_j)} \\
&\qquad\qquad + C_g(1 + 2^t C_d) ||J^sT^n|^t|_{C^0(U_j^1)} |\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \,,
\end{aligned}$$

where we have used (4.11) and (4.12) for the final inequality. We first observe that

$$|\psi_1 \circ T^n \circ G_{U_j^1} - \psi_2 \circ T^n \circ G_{U_j^2}|_{C^\beta(I_j)} \le C\varepsilon^{\alpha-\beta} \,,$$

by [DZ1, Lemma 4.4]. For brevity, set $J_\ell = J^sT^n \circ G_{U_j^\ell}$. By[31] [DZ1, eq. (4.16)], we have

$$\left| 1 - \frac{J_1(r)}{J_2(r)} \right| \le C_{vert}\, \varepsilon^{1/(q+1)} \,, \ \forall r \in I_j \,, \tag{4.21}$$

for some constant $C_{vert} > 0$ depending only on the uniform angle between the vertical direction and the stable and unstable cones. Thus, since $\varepsilon_0 > 0$ satisfies (4.2) and $\varepsilon \le \varepsilon_0$, this implies that $\frac{1}{4} \le \frac{J_1(r)}{J_2(r)} \le \frac{7}{4}$. Then, estimating as in Lemma 2.1, we have

$$|J_1^t(r) - J_2^t(r)| \le 2^t |J_1^t|_{C^0(I_j)} C_{vert}\, \varepsilon^{1/(q+1)} \,.$$

Following [DZ1, eq. (4.17) and (4.18)], yields,

$$H^\beta(J_1^t - J_2^t) \le C2^t |J_1^t|_{C^0(I_j)} \sup_{r,s \in I_j} \min\{\varepsilon^{1/(q+1)} |r-s|^{-\beta}, |r-s|^{1/(q+1)-\beta}\} \,,$$

where $H^\beta(\cdot)$ is the Hölder constant with exponent $\beta$ on $I_j$. This bound is maximized when $\varepsilon = |r-s|$, which yields $H^\beta(J_1^t - J_2^t) \le C2^t |J_1^t|_{C^0(I_j)} \varepsilon^{1/(q+1)-\beta}$. Putting these estimates together yields,

$$||J^sT^n|^t \circ G_{U_j^1} - |J^sT^n|^t \circ G_{U_j^2}|_{C^\beta(I_j)} \le C2^t |J^sT^n|^t_{C^0(U_j^1)} \varepsilon^{1/(q+1)-\beta} \,.$$

Together with the previous estimate on $\psi_\ell$, this completes the proof of the sublemma since $\alpha \le 1/(q+1)$. □

[31] The case $q = 2$ is treated there, the general case is similar.

Returning to (4.18), we split the estimate over matched pieces as follows. First, recalling (4.17), define on each $U_j^1$,

$$\widetilde{\psi}_2 = \psi_2 \circ T^n \circ G_{U_j^2} \circ G_{U_j^1}^{-1}, \quad \text{and} \quad \widetilde{J}^s T^n = J^s T^n \circ G_{U_j^2} \circ G_{U_j^1}^{-1} .$$

Then,

$$\begin{aligned} \left| \int_{U_j^1} f \, \psi_1 \circ T^n \, |J^s T^n|^t - \int_{U_j^2} f \, \psi_2 \circ T^n \, |J^s T^n|^t \right| &\le \left| \int_{U_j^1} f \, (\psi_1 \circ T^n \, |J^s T^n|^t - \widetilde{\psi}_2 |\widetilde{J}^s T^n|^t) \right| \\ &+ \left| \int_{U_j^1} f \, \widetilde{\psi}_2 |\widetilde{J}^s T^n|^t - \int_{U_j^2} f \, \psi_2 \circ T^n \, |J^s T^n|^t \right| . \end{aligned} \tag{4.22}$$

We estimate the first term on the right side using the strong stable norm and Lemma 4.8(b),

$$\left| \int_{U_j^1} f \, (\psi_1 \circ T^n \, |J^s T^n|^t - \widetilde{\psi}_2 |\widetilde{J}^s T^n|^t) \right| \le \|f\|_s \delta_0^{1/p} C 2^t |J^s T^n|^t_{C^0(U_j^1)} \varepsilon^{\alpha - \beta} .$$

Then, noting that $d(\psi_1 \circ T^n \, |J^s T^n|^t, \widetilde{\psi}_2 |\widetilde{J}^s T^n|^t) = 0$ by definition, and the $C^\alpha$ norm of each test function is bounded by $C 2^t |J^s T^n|^t_{C^0(I_j)}$, using (4.11) and (4.12), we estimate the second term on the right side of (4.22) using the strong unstable norm:

$$\begin{aligned} \left| \int_{U_j^1} f \, \widetilde{\psi}_2 |\widetilde{J}^s T^n|^t - \int_{U_j^2} f \, \psi_2 \circ T^n |J^s T^n|^t \right| &\le \|f\|_u d_{\mathcal{W}^s}(U_j^1, U_j^2)^\gamma C 2^t |J^s T^n|^t_{C^0(U_j^1)} \\ &\le C' \|f\|_u n^\gamma \Lambda^{-n\gamma} \varepsilon^\gamma |J^s T^n|^t_{C^0(U_j^1)} , \end{aligned} \tag{4.23}$$

where we used Lemma 4.8(a) in the second inequality. Putting these estimates into (4.22), then combining with (4.19) in (4.18), and summing over $j$ (since there are at most two matched pieces corresponding to each element of $\mathcal{G}_n(W^1)$), yields,

$$\begin{aligned} &\left| \int_{W^1} \mathcal{L}_t^n f \, \psi_1 - \int_{W^2} \mathcal{L}_t^n f \, \psi_2 \right| \\ &\qquad \le C \left( \|f\|_u n^\gamma \Lambda^{-n\gamma} \varepsilon^\gamma Q_n(t) + \|f\|_s (\varepsilon^{1/p} Q_n(t - 1/p) + \varepsilon^{\alpha-\beta} Q_n(t)) \right) . \end{aligned} \tag{4.24}$$

Dividing through by $\varepsilon^\gamma$ and taking the appropriate suprema over $W^\ell$ and $\psi_\ell$ proves (4.7) since $\gamma \le \min\{1/p, \alpha - \beta\}$.

4.4. **Proof of Lemma 4.3.** **($\mathcal{L}_t(C^1) \subset \mathcal{B}$).** We assume $0 \le t < 1$. The proof for $t \ge 1$ is similar, but simpler, since $\mathcal{L}_t f$ is bounded when $t \ge 1$. Without loss of generality, we also assume that $t_0 \le 1/2$, so that, by Definition 3.2, $q \ge 8$ and $p > 9$.

We introduce a mollification in order to approximate $\mathcal{L}_t f$ by functions in $C^1(M)$: Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a $C^\infty$ nonnegative, rotationally symmetric function supported on the unit disk with $\int_{\mathbb{R}^2} \rho \, d^2 z = 1$ and $|\rho|_{C^1} \le 2$. For $f \in C^1(M)$ and $\eta > 0$, define

$$g_\eta(x) = \int_{B_\eta(x)} \eta^{-2} \rho \left( \frac{d(x,z)}{\eta} \right) \mathcal{L}_t f(z) d^2 z ,$$

where $B_\eta(x)$ is the ball of radius $\eta$ centered at $x$. Viewing $M$ as a subset of $\mathbb{R}^2$, we set $\mathcal{L}_t f \equiv 0$ outside $M$ so that the integral is well-defined even when $B_\eta(x) \not\subset M$. We first develop bounds on $|g_\eta|_{C^0(M)}$ and $|g_\eta|_{C^1(M)}$, for any $t \ge 0$.

Since $t < 1$, the operator $\mathcal{L}_t f$ is unbounded in neighborhoods of $T\mathcal{S}_0$, so the bounds on $g_\eta$ will be greatest in such neighborhoods. Suppose $x$ and $\eta$ are such that $B_\eta(x) \cap T\mathcal{S}_0 \neq \emptyset$ and note that there can be at most $\tau_{\max}/\tau_{\min} + 1$ connected components of $B_\eta(x) \setminus T\mathcal{S}_0$. Fix one such component with boundary $S \in T\mathcal{S}_0$ such that $S$ is the accumulation of the sequence of sets, $B_\eta(x) \cap T\mathbb{H}_k$, $k \ge k_0$. On each such set, $|J^s T|^{1-t} = C^{\pm 1} k^{-q(1-t)}$. Also, due to the uniform transversality of $T\mathcal{S}_0^{\mathbb{H}}$ with the stable cone, we have $\mathrm{diam}^s(B_\eta(x) \cap T\mathbb{H}_k) \le Ck^{-2q-1}$, and $\mathrm{diam}^u(B_\eta(x) \cap T\mathbb{H}_k) \le C\eta$. Moreover, since the boundary of $T\mathbb{H}_k$ has distance approximately $k^{-2q}$ from $S$, we have $B_\eta(x) \cap T\mathbb{H}_k = \emptyset$ unless $k \ge C\eta^{-1/(2q)}$. Assembling these facts, we estimate,

$$|g_\eta| \le C \sum_{k \ge C\eta^{-1/(2q)}} \int_{B_\eta(x) \cap T\mathcal{S}_0} \eta^{-2} \mathcal{L}_t f \, d^2 z \le C|f|_\infty \sum_{k \ge C\eta^{-1/(2q)}} \eta^{-1} k^{-q-1-qt} \,.$$

We conclude that, for any $0 \le t < 1$,[32]

$$|g_\eta|_{C^0(M)} \le C|f|_\infty \eta^{\frac{t}{2}-\frac{1}{2}}, \quad \text{and similarly,} \quad |g_\eta|_{C^1(M)} \le C|f|_\infty \eta^{\frac{t}{2}-\frac{3}{2}} \,. \tag{4.25}$$

To prove Lemma 4.3, we must to control $g_\eta - \mathcal{L}_t f$ integrated along stable manifolds. To this end, we will need the following two lemmas. (The first one is classical and the second uses bounds on the auxiliary foliation constructed in [BDL, Section 6].)

**Lemma 4.9.** *Let $W \in \mathcal{W}^s$ be weakly homogeneous and for $\eta > 0$ let $W_u(\eta) \subset W$ denote the set of points in $W$ whose unstable manifold extends at least length $\eta$ on both sides of $W$. Then $m_W(W \setminus W_u(\eta)) \le C\eta$ for some constant $C > 0$ independent of $W$ and $\eta$.*

*Proof.* This is precisely [CM, Theorem 5.66]. See also the corrected proof in [BDT]. □

**Lemma 4.10.** *There exist constants $C, C_s > 0$ such that for any weakly homogeneous unstable curve $U$ and any $\varrho > 0$, there exists a set $U' \subset U$ with $m_U(U \setminus U') \le C\varrho$ such that*

$$\left| \frac{J^sT(x)}{J^sT(y)} - 1 \right| \le C_s \left( \varrho^{-\frac{q}{q+1}} k_U^{-q} d(x,y) + d(x,y)^{1/(q+1)} \right) , \quad \forall x, y \in U' \,,$$

*where $k_U$ is the index of the homogeneity strip containing $U$.*

*Proof.* Fixing a length $\varrho < k_U^{-q-1}$, we define a foliation of stable curves transverse to $U$, following the procedure[33] in [BDL, Sections 6.1, 6.2]: Choose $n \in \mathbb{N}$ arbitrarily large and define a smooth "seeding" foliation of homogeneous stable curves transverse to connected components of $T^nU$; elements of the seeding foliation are then pulled back under $T^{-n}$ and those that are not cut form a foliation of homogeneous stable curves of length at least $\varrho$ and transverse to $U$. Letting $U_n' \subset U$ denote the set covered by this surviving foliation, we have $m_U(U \setminus U_n') \le C\varrho$, for some $C > 0$ independent of $n$ [BDL, Section 6.1]. Moreover, expressing the foliation in local coordinates $(s,u)$ adapted to the stable and unstable directions defines a function $G(s,u)$ such that each stable curve can be expressed as $\{(s, G(s,u))\}_{s \in [-\varrho, \varrho]}$, and $G(0,u) = u$, so that the unstable manifold $U$ corresponds to the vertical segment $\{(0,u)\}_{u \in [0,|U|]}$. It follows that the slope $\mathcal{V}(u)$ of the tangent vector to the foliation at $(0,u)$ is just $\partial_s G(0,u)$. By [BDL, Lemma 6.5], $\partial_u \partial_s G \in C^0$ with $|\partial_u \partial_s G|_\infty \le C\varrho^{-q/(q+1)} k_U^{-q}$ (where we have adapted the exponent according to the spacing of our homogeneity strips).

Note that the foliation of stable curves constructed in this way has tangent vectors in $DT^{-n}C^s$. Since the bounds on $m_U(U \setminus U_n')$ and $|\partial_u \partial_s G|_\infty$ are independent of $n$, we conclude there exists a set $U' \subset U$ with $m_U(U \setminus U') \le C\varrho$ such that the stable manifolds passing through $U'$ have length at least $\varrho$ and satisfy $|\partial_u \partial_s G|_\infty \le C\varrho^{-q/(q+1)} k_U^{-q}$ (see also [BDL, Remark 1.1]).

Finally, for $u, v \in U'$ we estimate as in (2.7) (with $n = 1$), using (2.8) for $\log \frac{\cos \varphi(u)}{\cos \varphi(v)}$ and $|\mathcal{V}(u) - \mathcal{V}(v)| \le C\varrho^{-q/(q+1)} k_U^{-q} d(u,v)$ from the construction in [BDL]. Putting these estimates together proves the lemma. □

[32]For $t = 0$, any choice of $q > 1$ gives the same bound.

[33][BDL] constructs this as a foliation of unstable curves transverse to a stable curve. By the time reversal property of the billiard, the same construction holds with stable and unstable directions exchanged.

We record for future use that for any measurable set $V \subseteq W \in \mathcal{W}^s$,

$$
\begin{aligned}
\int_V \mathcal{L}_t f \, \psi \, dm_V &= \int_{T^{-1}V} f|J^sT|^t \psi \circ T \, dm_{T^{-1}V} \\
&\le |f|_\infty |\psi|_\infty |T^{-1}V|^{1-t}|V|^t \le C|f|_\infty|\psi|_\infty|V|^{(1+t)/2}\,,
\end{aligned}
\tag{4.26}
$$

where $|V|$ denotes the arc length measure of $V$, and we have used the Hölder inequality for the first inequality and the bound $|T^{-1}V| \le C|V|^{1/2}$ in the second.

*Approximating the strong stable norm.* Fix $\eta > 0$, and let $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \le |W|^{-1/p}$. If $|W| \le \eta$, then using (4.25) and (4.26), we write, simply, using $p > 9$,

$$
\int_W (\mathcal{L}_t f - g_\eta)\psi \, dm_W \le C|f|_\infty |W|^{-1/p}\big(|W|^{\frac{1+t}{2}} + |W|\eta^{\frac{t}{2}-\frac{1}{2}}\big) \le C|f|_\infty \eta^{\frac{t}{2}+\frac{1}{3}}\,.
\tag{4.27}
$$

In what follows, we assume $|W| > \eta$. Let $W_\eta^-$ denote the curve $W$ minus the $\eta$-neighborhood of its boundary. Treating the integral over the two components of $W \setminus W_\eta^-$ in the same way as (4.27), we estimate, using that $m_W(W \setminus W_\eta^-) \le 2\eta$,

$$
\int_{W\setminus W_\eta^-} (\mathcal{L}_t f - g_\eta)\psi \, dm_W \le C|f|_\infty \eta^{\frac{t}{2}+\frac{1}{3}}\,.
\tag{4.28}
$$

Next, since $W$ intersects at most $N = \tau_{\max}/\tau_{\min} + 1$ elements of $T\mathcal{S}_0$, the set $W \cap \big(\cup_{k \ge \eta^{-1/(2q+1)}} T\mathbb{H}_k\big)$ comprises at most $N$ intervals of length $C\eta^{2q/(2q+1)}$. We estimate as in (4.27) using $V = W \cap \big(\cup_{k\ge\eta^{-1/(2q+1)}} T\mathbb{H}_k\big)$ in (4.26), and that $p > q+1 \ge 9$

$$
\int_{W\cap\left(\cup_{k\ge\eta^{-1/(2q+1)}} T\mathbb{H}_k\right)} (\mathcal{L}_t f - g_\eta)\psi \, dm_W \le C|f|_\infty \eta^{\frac{t}{2}+\frac{3}{10}}\,.
\tag{4.29}
$$

Finally, we estimate $\mathcal{L}_t f - g_\eta$ on those portions of $W_\eta^-$ that intersect $T\mathbb{H}_k$ for $k \le \eta^{-1/(2q+1)}$. Let $x$ be such a point in $W_\eta^-$. Due to the restriction on $k$, the ball $B_\eta(x)$ lies in a bounded number of homogeneity strips, so we may use bounded distortion in conjunction with Lemma 4.10 to bound the difference in each such interval. Let $S_\eta = W \setminus W_u(\eta)$ denote the exceptional set of points in Lemma 4.9. We write $A_\eta(x)$ for the subset of $B_\eta(x)$ foliated by unstable manifolds of length at least $2\eta$, and let $E_\eta(x) = B_\eta(x) \setminus A_\eta(x)$. Then,

$$
\begin{aligned}
\mathcal{L}_t f(x) - g_\eta(x) &= \int_{B_\eta(x)} \eta^{-2}\rho(\tfrac{d(x,z)}{\eta})(\mathcal{L}_t f(x) - \mathcal{L}_t f(z))d^2z \\
&= \int_{A_\eta(x)} \eta^{-2}\rho(\tfrac{d(x,z)}{\eta})(\mathcal{L}_t f(x) - \mathcal{L}_t f(z))d^2z \\
&\qquad + \int_{E_\eta(x)} \eta^{-2}\rho(\tfrac{d(x,z)}{\eta})(\mathcal{L}_t f(x) - \mathcal{L}_t f(z))d^2z\,.
\end{aligned}
\tag{4.30}
$$

We first estimate the integral over $E_\eta(x)$ using the bound $\mathcal{L}_t f(z) \le C\mathcal{L}_t f(x)$ for $z \in B_\eta(x)$, since $B_\eta(x)$ lies in a bounded number of homogeneity strips. Then, using the fact that the unstable foliation is absolutely continuous, we disintegrate as follows,

$$
\int_{E_\eta(x)} \eta^{-2}\rho(\tfrac{d(x,z)}{\eta})(\mathcal{L}_t f(x) - \mathcal{L}_t f(z))d^2z \le C\mathcal{L}_t f(x)\eta^{-1}|S_\eta \cap B_\eta(x)|\,.
\tag{4.31}
$$

Next, we estimate the integral over $A_\eta(x)$. Since each point $y \in A_\eta(x) \cap W_\eta^-$ has an unstable manifold $U_y$ extending a length at least $\eta$ on either side of $W$, we set $\varrho = \eta^{1+\frac{1}{2q}}$ and denote by $A'_\eta(x)$ those points contained in sets $U'_y \subset U_y$ satisfying Lemma 4.10. It follows from that lemma and the absolute continuity of the unstable foliation that

$$
\int_{A_\eta(x)\setminus A'_\eta(x)} \eta^{-2}\rho(\tfrac{d(x,z)}{\eta})(\mathcal{L}_t f(x) - \mathcal{L}_t f(z))d^2z \le C\mathcal{L}_t f(x)\eta^{\frac{1}{2q}}\,,
\tag{4.32}
$$

where we have again used the bound $\mathcal{L}_t f(z) \le C\mathcal{L}_t f(x)$ on $B_\eta(x)$.

For $z \in A'_\eta(x)$, we bound the difference $\mathcal{L}_t f(x) - \mathcal{L}_t f(z)$ as follows. Let $y = [x, z]$ denote the point of intersection between the stable manifold of $x$ (which is $W$) and the unstable manifold of $z$, which is $U_y$. By definition, $z \in U'_y$ and it is always the case that $y \in U'_y$ since the stable manifold of $y$, $W$, has length at least $\eta > \varrho$. Thus,

$$
\begin{aligned}
&|\mathcal{L}_t f(x) - \mathcal{L}_t f(z)| \\
&\quad \le \left| \frac{f(T^{-1}x)}{|J^sT|^{1-t}(T^{-1}x)} - \frac{f(T^{-1}y)}{|J^sT|^{1-t}(T^{-1}y)} \right| + \left| \frac{f(T^{-1}y)}{|J^sT|^{1-t}(T^{-1}y)} - \frac{f(T^{-1}z)}{|J^sT|^{1-t}(T^{-1}z)} \right| \\
&\quad \le \mathcal{L}_t 1(x) [|f|_{C^1} d(T^{-1}x, T^{-1}y) + |f|_{C^0} C d(T^{-1}x, T^{-1}y)^{1/(q+1)} \\
&\qquad + |f|_{C^1} d(T^{-1}y, T^{-1}z) + |f|_{C^0} C_s (\eta^{-\frac{2q+1}{2q+2}} d(T^{-1}y, T^{-1}z) + d(T^{-1}y, T^{-1}z)^{1/(q+1)})] \,,
\end{aligned}
$$

where we have used Lemma 2.1 along $W$ and Lemma 4.10 along $U_y$ with $\varrho = \eta^{\frac{2q+1}{2q}}$. Next, $d(T^{-1}y, T^{-1}z) \le Cd(y, z) \le C\eta$, while for $x \in T\mathbb{H}_k$,

$$
d(T^{-1}x, T^{-1}y) \le Ck^q d(x, y) \le C\eta^{\frac{q+1}{2q+1}}
$$

since $k \le \eta^{-1/(2q+1)}$. Putting these estimates together we obtain,

$$
|\mathcal{L}_t f(x) - \mathcal{L}_t f(z)| \le |f|_{C^1} \mathcal{L}_t 1(x) C\eta^{\frac{1}{2q+2}} \qquad \text{for } z \in A'_\eta(x),
$$

and combining this with (4.31) and (4.32) in (4.30) yields,

$$
|\mathcal{L}_t f(x) - g_\eta(x)| \le C|f|_{C^1} \mathcal{L}_t 1(x) \eta^{\frac{1}{2q+2}} + C|f|_{C^0} \mathcal{L}_t 1(x) \eta^{-1} |S_\eta \cap B_\eta(x)| \,. \tag{4.33}
$$

We must integrate this bound over $W_\eta^- \cap (\cup_{k \le \eta^{-1/(2q+1)}} T\mathbb{H}_k)$. We estimate the integral of the first term in (4.33) simply using (4.26),

$$
C|f|_{C^1} \eta^{\frac{1}{2q+2}} \int_{W_\eta^- \cap (\cup_{k \le \eta^{-1/(2q+1)}} T\mathbb{H}_k)} \mathcal{L}_t 1 \, \psi \, dm_W \le C|f|_{C^1} \eta^{\frac{1}{2q+2}} \,. \tag{4.34}
$$

Finally, to bound the second term in (4.33), we write $I_\eta(x) = B_\eta(x) \cap W$ and

$$
|S_\eta \cap B_\eta(x)| = \int_{I_\eta(x)} 1_{S_\eta}(z) dm_W(z) = \int_{-\eta}^{\eta} 1_{S_\eta}(x + G_W(x; r)) JG_W(x; r) dr \,,
$$

where $G_W(x; r)$ denotes the (local) graph of the function defining $W$ in a neighborhood of $x$, as in (4.17), and we have centered the local $r$-interval at $r = 0$. Then,

$$
\begin{aligned}
&\int_{W_\eta^- \cap (\cup_{k \le \eta^{-1/(2q+1)}} T\mathbb{H}_k)} \mathcal{L}_t f(x) \frac{\psi(x)}{\eta} \int_{-\eta}^{\eta} 1_{S_\eta}(x + G_W(x; r)) JG_W(x; r) \, dr \, dm_W(x) \\
&\qquad \le |f|_{C^0} \frac{|W|^{-1/p}}{\eta} \int_{-\eta}^{\eta} \int_{W_\eta^-} \mathcal{L}_t 1(x) 1_{S_\eta}(x + G_W(x; r)) JG_W(x; r) \, dm_W(x) \, dr \\
&\qquad \le C|f|_{C^0} \frac{|W|^{-1/p}}{\eta} \int_{-\eta}^{\eta} |S_\eta|^{(1+t)/2} dr \ \le \ C|f|_{C^0} \eta^{\frac{1}{3} + \frac{t}{2}} \,,
\end{aligned} \tag{4.35}
$$

where we have used (4.20) and the fact that translations of $W_\eta^-$ up to length $\eta$ are subsets of $W$ in order to apply (4.26) for the second inequality, and Lemma 4.9, with $|W| \ge \eta$ and $p > 9$ for the final inequality.

Finally, using (4.34) and (4.35) in (4.33), and adding the contributions from (4.28) and (4.29) in addition to (4.27) yields,

$$
\int_W (\mathcal{L}_t f - g_\eta) \, \psi \, dm_W \le C|f|_{C^1} \eta^{\frac{1}{2q+2}} \,, \tag{4.36}
$$

for some $C > 0$ independent of $W$, since $\min\{\frac{t}{2} + \frac{3}{10}, \frac{1}{2q+2}\} = \frac{1}{2q+2}$ whenever $q > 1$ and $t > 0$. Taking the appropriate suprema over $\psi$ and $W$ yields the required estimate $\|\mathcal{L}_t f - g_\eta\|_s \le C|f|_{C^1}\eta^{\frac{1}{2q+2}}$.

*Approximating the unstable norm.* Let $\varepsilon \le \varepsilon_0$ and $W_1, W_2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon$. Let $\psi_i \in C^\alpha(W_i)$ with $|\psi_i|_{C^\alpha(W_i)} \le 1$, $i = 1, 2$, and $d(\psi_1, \psi_2) = 0$. We must estimate,

$$\int_{W_1} (\mathcal{L}_t f - g_\eta)\psi_1 \, dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta)\psi_2 \, dm_{W_2} .$$

We consider two cases.

*Case 1:* $\eta^{\frac{1}{2q+2}} < \varepsilon^{2\gamma}$. We apply (4.36) to each term separately and obtain

$$\varepsilon^{-\gamma} \left| \int_{W_1} (\mathcal{L}_t f - g_\eta)\psi_1 \, dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta)\psi_2 \, dm_{W_2} \right| \le C|f|_{C^1}\eta^{\frac{1}{4q+4}} .$$

*Case 2:* $\eta^{\frac{1}{2q+2}} \ge \varepsilon^{2\gamma}$. In this case, we write

$$\begin{aligned} &\int_{W_1} (\mathcal{L}_t f - g_\eta)\psi_1 \, dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta)\psi_2 \, dm_{W_2} \\ &\quad = \int_{W_1} \mathcal{L}_t f \, \psi_1 \, dm_{W_1} - \int_{W_2} \mathcal{L}_t f \, \psi_2 \, dm_{W_2} + \int_{W_2} g_\eta \, \psi_2 \, dm_{W_2} - \int_{W_1} g_\eta \, \psi_1 \, dm_{W_1} . \end{aligned} \tag{4.37}$$

We estimate the difference involving $\mathcal{L}_t f$ using the estimates in Section 4.3.3, but using the fact that $f \in C^1(M)$ to obtain stronger bounds. In particular, the integral over unmatched pieces from (4.19) is bounded by $C|f|_{C^0}\varepsilon$. The bound on the first term of (4.22) remains the same, but the bound on the second term from (4.23) is improved to $C|f|_{C^1}\varepsilon$. Putting these estimates together as in (4.24) and dividing[34] by $\varepsilon^\gamma$ implies,

$$\varepsilon^{-\gamma} \left| \int_{W_1} \mathcal{L}_t f \, \psi_1 \, dm_{W_1} - \int_{W_2} \mathcal{L}_t f \, \psi_2 \, dm_{W_2} \right| \le C|f|_{C^1}\varepsilon^{\alpha-\beta-\gamma} \le C|f|_{C^1}\eta^{\frac{\alpha-\beta-\gamma}{4\gamma(q+1)}} . \tag{4.38}$$

Next, we turn to the second difference in (4.37). Using the notation of Section 4.3.3, we split the integrals up into one integral over the common $r$-interval $I_1 \cap I_2$ and at most two integrals over $I_1 \triangle I_2$. The (at most two) curves $V_i^\ell \subset W_\ell$ corresponding to intervals in $I_1 \triangle I_2$ have length bounded by $C\varepsilon$ by definition of $d_{\mathcal{W}^s}(W_1, W_2)$. Thus using (4.25), we have

$$\int_{V_i^\ell} g_\eta \, \psi_i \, dm_{W_\ell} \le C|f|_{C^0}\eta^{\frac{t}{2}-\frac{1}{2}}\varepsilon \le C|f|_{C^0}\eta^{\frac{t}{2}-\frac{1}{2}+\frac{1-\gamma}{4\gamma(q+1)}}\varepsilon^\gamma . \tag{4.39}$$

On the curves $U_1, U_2$, which are the graphs of the functions $\varphi_{U_1}, \varphi_{U_2}$ over $I_1 \cap I_2$, we have,

$$\int_{U^1} g_\eta \, \psi_1 \, dm_{W_1} - \int_{U_2} g_\eta \, \psi_2 \, dm_{W_2} \le |JG_{U_1}(g_\eta\psi_1) \circ G_{U_1} - JG_{U_2}(g_\eta\psi_2) \circ G_{U_2}|_{C^0(I_1 \cap I_2)} , \tag{4.40}$$

where $G_{U_\ell}(r) = (r, \varphi_{U_1}(r))$. Then estimating as in the proof of Sublemma 4.8, we have

$$|JG_{U_1}(g_\eta\psi_1) \circ G_{U_1} - JG_{U_2}(g_\eta\psi_2) \circ G_{U_2}|_{C^0(I_1 \cap I_2)} \le C|g_\eta|_{C^1(M)}\varepsilon \le C|f|_\infty \eta^{\frac{t}{2}-\frac{3}{2}+\frac{1-\gamma}{4\gamma(q+1)}}\varepsilon^\gamma , \tag{4.41}$$

where we have used the fact that $d(\psi_1, \psi_2) = 0$ and $|\varphi'_{U_1} - \varphi'_{U_2}| \le \varepsilon$. Putting these estimates together with (4.38) in (4.37) yields,

$$\begin{aligned} &\varepsilon^{-\gamma} \left| \int_{W_1} (\mathcal{L}_t f - g_\eta)\psi_1 \, dm_{W_1} - \int_{W_2} (\mathcal{L}_t f - g_\eta)\psi_2 \, dm_{W_2} \right| \\ &\qquad \le C|f|_{C^1}\eta^{\frac{\alpha-\beta-\gamma}{4\gamma(q+1)}} + C|f|_{C^0}\eta^{\frac{t}{2}-\frac{3}{2}+\frac{1-\gamma}{4\gamma(q+1)}} , \end{aligned} \tag{4.42}$$

[34] We use here the strict inequality $\gamma < \alpha - \beta$.

and we use $\gamma \le \frac{1}{6q+7}$ from (4.1) to deduce that $-\frac{3}{2} + \frac{1-\gamma}{4\gamma(q+1)} \ge 0$. This completes Case 2, which, together with Case 1, implies the required bound $\|\mathcal{L}_t f - g_\eta\|_u \le |f|_{C^1(M)} \eta^\delta$, for some $\delta > 0$, ending the proof of Lemma 4.3.

**Remark 4.11** (Adapting the proof of Lemma 4.3 to the case $t = 0$)**.** *Homogeneity strips are not used in* [BD]*, so one requires a nonhomogeneous version of Lemma 4.9, but it is not hard to show directly that there exists $C > 0$ such that $m_W(W \setminus W_u(\eta)) \le C\sqrt{\eta}$ for any $W \in \mathcal{W}^s$ and $\eta > 0$, and this weaker bound suffices (see discussion of* (4.35) *below). Lemma 4.10 can be kept unchanged as it is only needed on unstable manifolds contained in a single homogeneity strip.*

*We show how to adapt the proof of Lemma 4.3 to the norm from* [BD, §4.1] *with $q = 2$, and parameters $\beta$, $\gamma$, and $\varsigma$: Eq* (4.27) *and* (4.28) *get better since the test function satisfies $|\psi| \le |\log |W||^\gamma$, so we find $\eta^{1/2}|\log \eta|^\gamma$. Similarly,* (4.29) *has the bound $\eta^{3/10}|\log \eta|^\gamma$. Eq* (4.30)*–*(4.34) *remain as written. Eq* (4.35) *proceeds as above until the last line, at which point we use $|S_\eta| \le C\sqrt{\eta}$, so that the final bound becomes $C|f|_\infty \eta^{1/4}|\log \eta|^\gamma$. Thus we arrive at* (4.36) *with a bound $C|f|_{C^1}\eta^{1/6}$. The factor $|\log \eta|^\gamma$ can be absorbed by the various exponents, all being greater than* 1/6*. So there is no extra restriction the parameter $\gamma$ from* [BD] *from the stable norm estimate.*

*For the unstable norm estimate, one distinguishes between the case $\eta^{1/6} < |\log \varepsilon|^{-2\varsigma}$, which yields a bound with $\eta^{1/12}$, and the case $\eta^{1/6} \ge |\log \varepsilon|^{-2\varsigma}$, which implies that $\varepsilon \le \exp(-\eta^{-\frac{\alpha-\beta}{12\varsigma}})$, which is superexponentially small in $\eta$, so that* (4.37) *remains the same, while* (4.38) *is bounded by $\varepsilon^{\alpha-\beta}|\log \varepsilon|^\varsigma \le \exp(-\eta^{-\frac{\alpha-\beta}{24\varsigma}})$. Similarly,* (4.39) *is bounded by $|\log \varepsilon|^{-\varsigma}$ times a factor superexponentially small in $\eta$. (We have a power of $\varepsilon$ which is factored into $|\log \varepsilon|^{-\varsigma}$ times $\varepsilon^{1-\delta}$, for any $\delta$.) The same is true of* (4.40)*–*(4.41)*. Finally, in* (4.42)*, we end up with $\exp(-\eta^{-\frac{\alpha-\beta}{24\varsigma}})$ plus $\eta^{-3/2}\exp(-\eta^{-\frac{1}{24\varsigma}})$, and this goes to 0 as $\eta$ goes to 0, for any $\varsigma > 0$ (in particular, there is no extra condition on $\varsigma$ from this estimate).*

4.5. **Spectral Gap for $\mathcal{L}_t$. Constructing $\mu_t$ (Proof of Theorem 4.1).** We harvest the results from the previous subsections to show Theorem 4.1 at the end of this section. Our first result follows from Proposition 4.6 and the exact growth for $Q_n(t)$ (Propositions 3.15 and 3.18).

**Proposition 4.12** (Quasi-compactness)**.** *Let $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$. Then we can choose parameters for $\mathcal{B}$ such that for any $t \in [t_0, t_1]$, the operator $\mathcal{L}_t$ acting on $\mathcal{B}$ is quasi-compact: its spectral radius is $e^{P_*(t)}$ and its essential spectral radius is at most $\sigma e^{P_*(t)}$, where*

$$\sigma := \max\{\Lambda^{-\beta+1/p}, \theta^{t-1/p}e^{-P_*(t)}, \Lambda^{-\gamma}\} < 1\,.$$

*Moreover, the peripheral spectrum of $\mathcal{L}_t$ contains no Jordan blocks.*

*Proof.* Since $t_0 > 0$ and $t_1 < t_*$, we can choose $p > 1$ such that $p > 2/t_0 \ge 2/t$ and (by Lemma 4.7) $\theta^{(t-1/p)}e^{-P_*(t)} < 1$ for any $t \in [t_0, t_1]$. Then (4.5) and Proposition 3.18(b) imply that the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}_w$ is at most $e^{P_*(t)}$. Combining (4.8) from Proposition 4.6 with Hennion's theorem and compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$ from Proposition 4.2, the essential spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is at most $\sigma e^{P_*(t)} < e^{P_*(t)}$. Hence the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is at most $e^{P_*(t)}$.

Next, notice that by Lemma 2.1 and our choice of $\delta_1$ in (3.47), we have for $W \in \mathcal{W}^s$ with $|W| \ge \delta_1/3$,

$$\begin{aligned}\int_W \mathcal{L}_t^n 1\, dm_W &= \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} \int_{W_i} |J^sT^n|^t\, dm_{W_i} \ge \sum_{W_i \in L_n^{\delta_1}(W)} \tfrac{1}{3}\delta_1 2^{-t} |J^sT^n|^t_{C^0(W_i)} \\ &\ge \tfrac{1}{4}\delta_1 2^{-t} \sum_{W_i \in \mathcal{G}_n^{\delta_1}(W)} |J^sT^n|^t_{C^0(W_i)} \ge \tfrac{1}{4}\delta_1 2^{-t} c_1 Q_n(t)\,,\end{aligned} \tag{4.43}$$

where for the final inequality we have applied Proposition 3.18(b). Then, since $Q_n(t) \ge e^{nP_*(t)}$ by the lower bound in Proposition 3.18(b), we conclude

$$\|\mathcal{L}_t^n\|_{\mathcal{B}} \ge \|\mathcal{L}_t^n 1\|_{\mathcal{B}}(\|1\|_{\mathcal{B}})^{-1} \ge (\|1\|_{\mathcal{B}})^{-1} C\delta_1 e^{P_*(t)n} \quad \Longrightarrow \quad \lim_{n\to\infty} \|\mathcal{L}_t^n\|_{\mathcal{B}}^{1/n} \ge e^{P_*(t)} .$$

Thus the spectral radius of $\mathcal{L}_t$ on $\mathcal{B}$ is in fact $e^{P_*(t)}$ and $\mathcal{L}_t$ is quasi-compact on $\mathcal{B}$.

Finally, to prove there are no Jordan blocks in the peripheral spectrum, assume to the contrary that there exist $f_0, f_1 \in \mathcal{B}$, $f_0 \neq 0$, and $\lambda \in \mathbb{C}$, $|\lambda| = e^{P_*(t)}$, such that $\mathcal{L}_t f_0 = \lambda f_0$ and $\mathcal{L}_t f_1 = \lambda f_1 + f_0$. Then $\mathcal{L}_t^n f_1 = \lambda^n f_1 + n\lambda^{n-1} f_0$, so that

$$n|f_0|_w \le e^{P_*(t)}|f_1|_w + e^{-(n-1)P_*(t)}|\mathcal{L}_t^n f_1|_w ,$$

and dividing by $n$, letting $n \to \infty$ and applying (4.5) and Proposition 3.18(b) yields $|f_0|_w = 0$. The injectivity of $\mathcal{B}_w$ into $\mathcal{B}$ given by Proposition 4.2 implies $f_0 = 0$ in $\mathcal{B}$, a contradiction. □

For $\varpi \in [0, 1)$, let $\mathbb{V}_\varpi$ denote the eigenspace of $\mathcal{L}_t$ on $\mathcal{B}$ corresponding to the eigenvalue $e^{P_*(t)}e^{2\pi i\varpi}$. Due to Proposition 4.12, we have the following decomposition of $\mathcal{L}_t$ on $\mathcal{B}$,

$$\mathcal{L}_t = \sum_{\varpi} e^{P_*(t)+2\pi i\varpi}\Pi_\varpi + R_t , \tag{4.44}$$

where the sum is over finitely many $\varpi$ due to the quasi-compactness of $\mathcal{L}_t$, and $\Pi_\varpi^2 = \Pi_\varpi$, $\Pi_\varpi\Pi_{\varpi'} = R_t\Pi_\varpi = \Pi_\varpi R_t = 0$ for $\varpi \neq \varpi'$ (mod $2\pi$), and the spectral radius of $R_t$ is strictly less than $e^{P_*(t)}$.

**Lemma 4.13.** *Define* $\nu_t = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-P_*(t)k}\mathcal{L}_t^k 1$.

a) *Then* $\nu_t \neq 0$ *is a nonnegative Radon measure, and* $e^{P_*(t)}$ *is in the spectrum of* $\mathcal{L}_t$.
b) *All elements of* $\mathbb{V} = \oplus_\varpi \mathbb{V}_\varpi$ *are complex measures, absolutely continuous with respect to* $\nu_t$.

Lemma 4.13 is standard, adapting what has been done in the SRB case. We give a proof for completeness.

*Proof.* (b) The lack of Jordan blocks enables us to define spectral projectors by

$$\Pi_\varpi : \mathcal{B} \to \mathbb{V}_\varpi , \quad \Pi_\varpi f = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-P_*(t)k-2\pi i\varpi k}\mathcal{L}_t^k f , \tag{4.45}$$

where convergence in the $\mathcal{B}$ norm is guaranteed by Propositions 4.6 and 3.18(b). Moreover, since $C^1(M)$ is dense in $\mathcal{B}$ and $\mathbb{V}_\varpi$ is finite-dimensional, for each $\nu \in \mathbb{V}_\varpi$, there exists $\bar f_\nu \in C^1(M)$ such that $\Pi_\varpi \bar f_\nu = \nu$.

Taking $\nu \in \mathbb{V}_\varpi$ and $\psi \in C^\alpha(M)$, we have

$$|\nu(\psi)| \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-P_*(t)k}|\mathcal{L}_t^k \bar f_\nu(\psi)| \le |\bar f_\nu|_\infty \Pi_0 1(|\psi|) \le |\bar f_\nu|_\infty |\psi|_\infty \Pi_0 1(1) .$$

The last two inequalities show respectively that $\nu$ is a complex Radon measure, and is absolutely continuous with respect to $\nu_t$, with density $f_\nu \in L^\infty(\nu_t)$. It may be that $f_\nu \neq \bar f_\nu$.

(a) Item (b) implies also that $\nu_t$ is a nonnegative Radon measure since $\bar f_{\nu_t} = 1$ and $\Pi_0$ is nonnegative. Also, if $\nu_t = 0$, then all elements of $\mathbb{V}_\varpi$ are 0, contradicting the fact that the spectral radius of $\mathcal{L}_t$ is $e^{P_*(t)}$. Thus $\nu_t \neq 0$ and $e^{P_*(t)}$ is in the spectrum of $\mathcal{L}_t$ since $\mathcal{L}_t\nu_t = e^{P_*(t)}\nu_t$. □

The dual operator $\mathcal{L}_t^*$ acting on $\mathcal{B}^*$ has the same spectrum as $\mathcal{L}_t$ on $\mathcal{B}$. Define

$$\tilde\nu_t := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-P_*(t)k}(\mathcal{L}_t^*)^k d\mu_{\mathrm{SRB}} , \tag{4.46}$$

which converges in the dual norm due to the absence of Jordan blocks. By the analogous arguments to item (b) of Lemma 4.13, the distribution $\tilde\nu_t \neq 0$ is a nonnegative Radon measure, and every other eigenvector corresponding to the peripheral spectrum is a Radon measure, absolutely continuous with respect to $\tilde\nu_t$, with bounded density.

With $\tilde\nu_t$, we will define our candidate $\mu_t$ for the equilibrium state in Proposition 4.15. For this (and in (6.23)), we shall use the following lemma (proved exactly as in [BD, Lemma 4.4]) which gives more precise information about the inclusion $\mathcal{B}_w \subset (C^1(M))^*$ in Proposition 4.2. Recalling (2.2), let $H^\alpha_{\mathcal{W}^s_{\mathbb{H}}}(\psi) = \sup_{W\in\mathcal{W}^s_{\mathbb{H}}} H^\alpha_W(\psi)$.

**Lemma 4.14.** *There exists $C > 0$ such that for all $f \in \mathcal{B}_w$ and $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$,*

$$|f(\psi)| \le C|f|_w\big(|\psi|_\infty + H^\alpha_{\mathcal{W}^s_{\mathbb{H}}}(\psi)\big)\,.$$

**Proposition 4.15** (Constructing $\mu_t$)**.** *For $\nu \in \mathcal{B}$ and $\tilde\nu \in \mathcal{B}^*$ we set $\langle \nu, \tilde\nu\rangle := \tilde\nu(\nu)$.*

a) *The measure $\tilde\nu_t \in \mathcal{B}^*$ is in fact an element of $\mathcal{B}^*_w$.*
b) *We have $\langle \nu_t, \tilde\nu_t\rangle \neq 0$, and the distribution $\mu_t$ defined for $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$ by*

$$\mu_t(\psi) := \frac{\langle \psi\nu_t, \tilde\nu_t\rangle}{\langle \nu_t, \tilde\nu_t\rangle}$$

*is a $T$-invariant probability measure.*

*Proof.* a) Let $g_n = n^{-1}\sum_{k=0}^{n-1} e^{-P_*(t)k}(\mathcal{L}^*_t)^k d\mu_{\mathrm{SRB}}$. By definition, $\|g_n - \tilde\nu_t\|_{\mathcal{B}^*} \to 0$ as $n\to\infty$. Thus for $f \in \mathcal{B}$, we have

$$|\langle f, \tilde\nu_t\rangle| \le |\langle f, \tilde\nu_t - g_n\rangle| + |\langle f, g_n\rangle| \le |\langle f, \tilde\nu_t - g_n\rangle| + C|f|_w\,,$$

where for the last inequality, we used the bound,

$$|\langle f, (\mathcal{L}^k_t)^* d\mu_{\mathrm{SRB}}\rangle| = |\langle \mathcal{L}^k_t f, d\mu_{\mathrm{SRB}}\rangle| \le C|\mathcal{L}^k_t f|_w \le C' e^{P_*(t)k}|f|_w\,,$$

by Lemma 4.14 and Proposition 4.6. Taking $n\to\infty$ yields the bound $|\langle f, \tilde\nu_t\rangle| \le C|f|_w$ for all $f\in\mathcal{B}$ and since $\mathcal{B}$ is dense in $\mathcal{B}_w$, the distribution $\tilde\nu_t$ extends to a bounded linear operator on $\mathcal{B}_w$, as required.

b) First we show the expression $\langle \psi\nu_t, \tilde\nu_t\rangle$ is well-defined for $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$. According to our convention, for $f \in C^1(M)$, we define for $n \ge 0$,

$$\langle f, \psi(\mathcal{L}^n_t)^* d\mu_{\mathrm{SRB}}\rangle = \int \mathcal{L}^n_t(f\psi)\, d\mu_{\mathrm{SRB}} \le CQ_n(t)|f|_w|\psi|_{C^\alpha(\mathcal{W}^s_{\mathbb{H}})}\,,$$

by the proof of Lemma 4.14. Thus $\psi(\mathcal{L}^n_t)^* d\mu_{\mathrm{SRB}}$ extends to a bounded linear functional on $\mathcal{B}_w$. Applying Proposition 3.18(b) and (4.46), we obtain

$$\psi\tilde\nu_t \in \mathcal{B}^*_w \ \text{ with } \ |\langle f, \psi\tilde\nu_t\rangle| \le C'|f|_w|\psi|_{C^\alpha(\mathcal{W}^s_{\mathbb{H}})},\ \ \forall f\in\mathcal{B}_w\,. \tag{4.47}$$

(We do not claim or need that $\psi f \in \mathcal{B}_w$, i.e. that $\psi f$ can be approached by a sequence of $C^1$ functions in the weak norm.) Thus $\langle \psi\nu_t, \tilde\nu_t\rangle := \langle \nu_t, \psi\tilde\nu_t\rangle$ is well-defined. Remark that the above argument also shows that $\mu_t(f\psi) = \langle f\nu_t, \psi\tilde\nu_t\rangle$ for all $f \in C^1(M)$, $\psi \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$.

Next, suppose $\langle \nu_t, \tilde\nu_t\rangle = 0$. Then for any $f \in C^1(M)$, and $n \ge 1$, using (4.45),

$$\begin{aligned}\langle f, \tilde\nu_t\rangle &= \frac{1}{n}\sum_{k=0}^{n-1} e^{-P_*(t)k}\langle f, (\mathcal{L}^*_t)^k\tilde\nu_t\rangle = \frac{1}{n}\sum_{k=0}^{n-1} e^{-P_*(t)k}\langle \mathcal{L}^k_t f, \tilde\nu_t\rangle \\ &\xrightarrow[n\to\infty]{} \langle \Pi_0(f), \tilde\nu_t\rangle = c_t(f)\langle \nu_t, \tilde\nu_t\rangle = 0\,.\end{aligned} \tag{4.48}$$

By density of $C^1(M)$ in $\mathcal{B}$, this implies that $\tilde\nu_t = 0$ as an element of $\mathcal{B}^*$, a contradiction. Thus $\langle \nu_t, \tilde\nu_t\rangle \neq 0$, and indeed $c_t(f) = \frac{\langle f, \tilde\nu_t\rangle}{\langle \nu_t, \tilde\nu_t\rangle}$, so that $\mu_t$ is a well-defined element of $\mathcal{B}^* \subset (C^1(M))^*$. It is then easy to see that $\mu_t$ is a nonnegative distribution and thus a Radon measure. The fact that $\mu_t$ is $T$-invariant is an exercise, using that $\nu_t$ and $\tilde\nu_t$ are eigenvectors of $\mathcal{L}_t$ and $\mathcal{L}^*_t$. $\square$

Following [BD, Definition 7.5], we remark that elements of $\mathcal{B}$ and $\mathcal{B}_w$ can be viewed both as distributions on $M$, as well as families of leafwise distributions on stable manifolds. In particular, for $f \in C^1(M)$, $W \in \mathcal{W}^s$, the map defined by

$$\mathcal{D}_{W,f}(\psi) := \int_W f\,\psi\, dm_W\,, \quad \psi \in C^\alpha(W)\,,$$

can be viewed as a distribution of order $\alpha$ on $W$. Since $|\mathcal{D}_{W,f}(\psi)| \le |f|_w |\psi|_{C^\alpha(W)}$, the map $\mathcal{D}_{W,\cdot}$ can be extended to all $f \in \mathcal{B}_w$. We will use the notation $\int_W \psi\, f$ for this extension and call the associated family of distributions the leafwise distributions $(f, W)_{W\in\mathcal{W}^s}$ corresponding to $f$. If $f$ satisfies $\int \psi\, f \ge 0$ for all $f \ge 0$, then the leafwise distribution is a leafwise measure.

Next, we introduce the notation regarding the disintegration of the measure $\mu_{\mathrm{SRB}}$. We fix a foliation of $\mathcal{F} = \{W_\xi\}_{\xi\in\Xi} \subset \mathcal{W}^s_{\mathbb{H}}$ of maximal, homogeneous local stable manifolds belonging to $\mathcal{W}^s_{\mathbb{H}}$. The conditional measures are defined by $\mu^\xi_{\mathrm{SRB}} = |W_\xi|^{-1}\rho_\xi dm_{W_\xi}$, where $\rho_\xi$ satisfies [CM, Cor 5.30],

$$0 < c_\rho \le \inf_{\xi\in\Xi}\inf_{W_\xi} \rho_\xi \le \sup_{\xi\in\Xi} |\rho_\xi|_{C^\alpha(W_\xi)} \le C_\rho < \infty\,. \tag{4.49}$$

Denoting the factor measure on the index set $\Xi$ by $\hat\mu_{\mathrm{SRB}}$, we get an analogue of [BD, Lemma 7.7]:

**Lemma 4.16** ($\nu_t$ as a leafwise measure)**.** *Let $\nu_t^\xi$ and $\hat\nu_t$ denote the conditional measure on $W_\xi$ and the factor measure on $\Xi$, respectively, obtained by disintegrating $\nu_t$ on the foliation of stable manifolds $\mathcal{F}$. For all $\psi \in C^\alpha(M)$,*

$$\int_{W_\xi} \psi\, d\nu_t^\xi = \frac{\int_{W_\xi} \psi\,\rho_\xi\,\nu_t}{\int_{W_\xi}\rho_\xi\,\nu_t} \quad \forall \xi\in\Xi, \text{ and } \quad d\hat\nu_t(\xi) = |W_\xi|^{-1}\left(\int_{W_\xi}\rho_\xi\,\nu_t\right) d\hat\mu_{SRB}(\xi)\,.$$

*Moreover, viewed as a leafwise measure, $\nu_t(W) > 0$ for all $W \in \mathcal{W}^s_{\mathbb{H}}$.*

*Proof.* We begin by showing that $\nu_t(W) > 0$ for all $W \in \mathcal{W}^s_{\mathbb{H}}$. If $|W| \ge \delta_1/3$ (recalling that our choice of $\delta_1$ in (3.31) is uniform for $t \in [t_0, t_1]$), then the positivity follows immediately from the uniform lower bound (4.43). So assume $W \in \mathcal{W}^s_{\mathbb{H}}$ with $|W| < \delta_1/3$.

First, we claim that there exists $n_W = \mathcal{O}(\log|W|)$ such that at least one element of $\mathcal{G}^{\delta_1}_{n_W}(W)$ has length at least $\delta_1/3$. For any $n \ge 1$, if no elements $W_i \in \mathcal{G}^{\delta_1}_n(W)$ have length at least $\delta_1/3$, then $\mathcal{G}^{\delta_1}_n(W) = \mathcal{I}^{\delta_1}_n(W)$ so that by Lemma 3.3 with $\varsigma = 0$, $\sum_{W_i\in\mathcal{G}^{\delta_1}_n(W)} |J^sT^n|_{C^0(W_i)} \le C_0\theta^n$, while by (3.9), $\sum_{W_i\in\mathcal{G}^{\delta_1}_n(W)} |J^sT^n|_{C^0(W_i)} \ge C_1|W|\delta_1^{-1}$. This can continue only so long as

$$C_1|W|\delta_1^{-1} \le C_0\theta^n \implies n \le \frac{\log\left(\frac{C_1|W|}{C_0\delta_1}\right)}{\log\theta} =: n_W\,.$$

Next, letting $V \in \mathcal{G}^{\delta_1}_{n_W}(W)$ be such that $|V| \ge \delta_1/3$, we estimate as in (3.39), using the fact that since $V$ and $T^{n_W}V$ are both homogeneous, $|V| \le C|T^{n_W}V|^{\left(\frac{q+1}{2q+1}\right)^{n_W}}$ for some $C \ge 1$, so that

$$|J^sT^{n_W}|^t_{C^0(V)} \ge e^{-tC_d}\frac{|T^{n_W}V|^t}{|V|^t} \ge e^{-tC_d}(\delta_1/(3C))^{t\left(\frac{2q+1}{q+1}\right)^{n_W}}\,.$$

Finally, recalling our choice of $n_1$ from (3.47), and using the fact that $|V| \ge \delta_1/3$, we estimate,[35]

$$
\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} e^{-P_*(t)k}\mathcal{L}_t^k 1\, dm_W &\ge e^{-P_*(t)n_W}\frac{1}{n}\sum_{k=n_1+n_W}^{n-1} e^{-P_*(t)(k-n_W)}\int_V \mathcal{L}_t^{k-n_W}1\,|J^sT^{n_W}|^t\, dm_V \\
&\ge e^{-P_*(t)n_W} e^{-tC_d}\big(\tfrac{\delta_1}{3C}\big)^{t\left(\frac{2q+1}{q+1}\right)^{n_W}}\frac{1}{n}\sum_{k=n_1+n_W}^{n-1} e^{-P_*(t)(k-n_W)}\int_V \mathcal{L}_t^{k-n_W}1\, dm_V \\
&\ge e^{-P_*(t)n_W} e^{-tC_d}\big(\tfrac{\delta_1}{3C}\big)^{t\left(\frac{2q+1}{q+1}\right)^{n_W}}\,\tfrac{n-1-n_1-n_W}{n}\,\tfrac{1}{4}\delta_1 2^{-t}c_1\,,
\end{aligned}
\tag{4.50}
$$

where in the last line we have applied (4.43) and Proposition 3.18(b).

These lower bounds depend only on $|W|$ and carry over to $\nu_t(W)$ since they are uniform in $n$.

With the lower bounds established, the remainder of the proof follows precisely as in [BD, Lemma 7.7], disintegrating the measure $\left(\frac{1}{n}\sum_{k=0}^{n-1} e^{-kP_*(t)}\mathcal{L}_t^k 1\right) d\mu_{\mathrm{SRB}}$ on the foliation of stable manifolds $\mathcal{F}$, using that convergence in $\mathcal{B}$ to $\nu_t$ implies convergence of the integral on each $W_\xi \in \mathcal{F}$. The lower bounds on $\nu_t(W)$ imply that the ratio $\dfrac{\int_{W_\xi}\psi\,\rho_\xi\,\nu_t}{\int_{W_\xi}\rho_\xi\,\nu_t}$ is well-defined for each $W_\xi \in \mathcal{F}$. $\square$

In view of (4.52) in the proof of Lemma 4.17 below (and also (6.25)), it is convenient to define $\mathcal{L}_t$ acting explicitly on distributions. For any point $x \in M$ that has a stable manifold of zero length, we define $W^s(x) = \{x\}$, and extend $\mathcal{W}^s$ to a larger collection $\widetilde{\mathcal{W}}^s$ including these singletons. For $\alpha \le 1$, let

$$
C^\alpha(\widetilde{\mathcal{W}}^s) := \{\psi \text{ bounded and measurable } \mid |\psi|_{C^\alpha(\widetilde{\mathcal{W}}^s)} := \sup_{W\in\widetilde{\mathcal{W}}^s}|\psi|_{C^\alpha(W)} < \infty\}\,.
$$

Let $C^\alpha_{\cos}(\widetilde{\mathcal{W}}^s)$ denote the set of measurable functions $\psi$ such that $\psi\cos\varphi \in C^\alpha(\widetilde{\mathcal{W}}^s)$. It follows from the uniform hyperbolicity of $T$ that if $\psi \in C^\alpha(\widetilde{\mathcal{W}}^s)$, then $\psi\circ T \in C^\alpha(\widetilde{\mathcal{W}}^s)$ (see (4.11)). Also, as in the proof of Lemma 2.2, by [CM, eq. (5.14)], we have $J^sT(x) \approx \cos\varphi(x)$ for $x \in M'$. We extend $J^sT$ to all $x \in M$ by defining it to be 1 on $M\setminus M'$. Then using (2.3), we have $\psi\circ T/|J^sT|^{1-t} \in C^\alpha_{\cos}(\widetilde{\mathcal{W}}^s)$ whenever $\psi \in C^\alpha(\widetilde{\mathcal{W}}^s)$ and $\alpha \le 1/(q+1)$. Using these facts, for a distribution $\mu \in (C^\alpha_{\cos}(\widetilde{\mathcal{W}}^s))^*$, define $\mathcal{L}_t : (C^\alpha_{\cos}(\widetilde{\mathcal{W}}^s))^* \to (C^\alpha(\widetilde{\mathcal{W}}^s))^*$ by

$$
\mathcal{L}_t\mu(\psi) = \mu\left(\frac{\psi\circ T}{|J^sT|^{1-t}}\right)\,,\quad \text{for all } \psi \in C^\alpha(\widetilde{\mathcal{W}}^s).
\tag{4.51}
$$

To reconcile this definition with (1.11), for $f \in C^\alpha(\mathcal{W}^s)$, we identify $f$ with the measure $f d\mu_{\mathrm{SRB}}$. Such a measure belongs to $(C^\alpha_{\cos}(\widetilde{\mathcal{W}}^s))^*$ since $1/\cos\varphi \in L^1(\mu_{\mathrm{SRB}})$. With this convention, the measure $\mathcal{L}_t f$ has density with respect to $\mu_{\mathrm{SRB}}$ given by (1.11). Finally, note that $\mathcal{B} \subset (C^\alpha_{\cos}(\widetilde{\mathcal{W}}^s))^*$, due to Lemma 4.14 and Remark 4.5.

We are finally ready to prove that $\mathcal{L}_t$ enjoys a spectral gap, using Lemma 4.16 (which exploited that $\mu_{\mathrm{SRB}}$ has smooth *stable* conditional densities, a very nongeneric property in the setting of hyperbolic dynamics).

**Lemma 4.17** (Spectral Gap)**.** *$\mathcal{L}_t$ has a spectral gap on $\mathcal{B}$, i.e., $e^{P_*(t)}$ is a simple eigenvalue and all other eigenvalues of $\mathcal{L}_t$ have modulus strictly less than $e^{P_*(t)}$.*

*Proof. Step 1: the spectrum of $e^{-P_*(t)}\mathcal{L}_t$ consists of finitely many cyclic groups;* in particular, each $\varpi$ in (4.44) is rational. To prove this, suppose $\nu \in \mathbb{V}_\varpi$, $\nu \neq 0$, and $\psi \in C^\alpha(M)$. Then by

[35] In fact, estimating more carefully for $t \le 1$, one can obtain the more precise lower bound $C'\delta_1^{1-t}|W|^{C''P_*(t)}|W|^t$ for some $C', C'' > 0$, but we will not need this here.

Lemma 4.13(b) and viewing $\nu$ as a distribution in the sense of (4.51)

$$
\begin{aligned}
\int_M \psi\, f_\nu\, d\nu_t &= \nu(\psi) = e^{-P_*(t)-2\pi i\varpi}\mathcal{L}_t\nu(\psi) = e^{-P_*(t)-2\pi i\varpi}\nu\left(\frac{\psi\circ T}{|J^sT|^{1-t}}\right) \\
&= e^{-P_*(t)-2\pi i\varpi}\nu_t\left(f_\nu\frac{\psi\circ T}{|J^sT|^{1-t}}\right) = e^{-P_*(t)-2\pi i\varpi}\mathcal{L}_t\nu_t\left(\psi\, f_\nu\circ T^{-1}\right) \\
&= e^{-2\pi\varpi}\nu_t\left(\psi\, f_\nu\circ T^{-1}\right),
\end{aligned}
\tag{4.52}
$$

so that $f_\nu\circ T^{-1} = e^{2\pi i\varpi}f_\nu$, $\nu_t$-almost everywhere.

Defining $\nu_{k,t} = (f_\nu)^k\nu_t$, for $k\in\mathbb{N}$, we claim that $e^{P_*(t)+2\pi i\varpi k}$ belongs to the spectrum of $\mathcal{L}_t$ and $\nu_{k,t}\in\mathbb{V}_{\varpi k}$. The claim completes the proof of Step 1 since the peripheral spectrum is finite, forcing $\varpi k = 0 \pmod 1$ for some $k\geq 1$, so that $\varpi$ must be rational.

To prove the claim, set $f_\nu = 0$ outside the support of $\nu_t$, and define the measure $\langle f_\nu\nu_t, \cdot\,\tilde\nu_t\rangle = \langle \nu, \cdot\,\tilde\nu_t\rangle$. We claim that this measure is not identically zero. If it were, then for any $\psi\in\mathcal{B}^*$, making the dual argument to (4.48),

$$
\langle\nu,\psi\rangle = \langle\Pi_\varpi\nu,\psi\rangle = \langle\nu,\Pi^*_\varpi\psi\rangle = \langle\nu,\tilde f_\varpi\tilde\nu_t\rangle\tilde c_\varpi(\psi) = 0\,,
$$

where we have used that every eigenvector corresponding to the peripheral spectrum of $\mathcal{L}_t^*$ is absolutely continuous with respect to $\tilde\nu_t$, i.e. $\tilde\nu_\varpi = \tilde f_\varpi\tilde\nu_t$, as explained after (4.46). Thus $\nu = 0$, a contradiction.

Since $\langle f_\nu\nu_t,\cdot\,\tilde\nu_t\rangle$ is not identically zero, it follows that $\langle (f_\nu)^k\nu_t,\cdot\,\tilde\nu_t\rangle$ is not identically zero. Thus there exists $\psi\in C^\alpha(M)$ such that $\langle (f_\nu)^k\nu_t,\psi\tilde\nu_t\rangle\neq 0$.

For $\varepsilon>0$, choose $g\in C^1(M)$ such that $\mu_t(|g-(f_\nu)^k|)<\varepsilon$. Note that $g\nu_t\in\mathcal{B}$ by [DZ2, Lemma 5.3]. We will show that $\Pi_{\varpi k}(g\nu_t)\neq 0$. For $\psi\in C^\alpha(M)$ and each $j\geq 0$,

$$
\begin{aligned}
e^{-P_*(t)j-2\pi i\varpi kj}\langle\mathcal{L}_t^j(g\nu_t),\psi\tilde\nu_t\rangle &= e^{-P_*(t)j-2\pi i\varpi kj}\langle g\nu_t,\psi\circ T^j(\mathcal{L}_t^*)^j\tilde\nu_t\rangle \\
&= e^{-2\pi i\varpi kj}\langle\nu_t,\tilde\nu_t\rangle\,\mu_t(g\,\psi\circ T^j)\,,
\end{aligned}
$$

where we have used $(\mathcal{L}_t^*)^j\tilde\nu_t = e^{P_*(t)j}\tilde\nu_t$. Also, due to the invariance of $\mu_t$,

$$
\langle (f_\nu)^k\nu_t,\psi\tilde\nu_t\rangle = e^{-2\pi i\varpi kj}\langle (f_\nu)^k\circ T^{-j}\,\nu_t,\psi\tilde\nu_t\rangle = e^{-2\pi i\varpi kj}\langle\nu_t,\tilde\nu_t\rangle\,\mu_t((f_\nu)^k\,\psi\circ T^j)\,.
$$

Putting these two expressions together, we estimate,

$$
\begin{aligned}
&\left|\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}e^{-P_*(t)j-2\pi i\varpi kj}\langle\mathcal{L}_t^j(g\nu_t),\psi\tilde\nu_t\rangle - \langle (f_\nu)^k\nu_t,\psi\tilde\nu_t\rangle\right| \\
&\leq \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\langle\nu_t,\tilde\nu_t\rangle\,\mu_t(|g-(f_\nu)^k|)|\psi|_\infty \leq \varepsilon\langle\nu_t,\tilde\nu_t\rangle|\psi|_\infty\,.
\end{aligned}
$$

Since $\langle (f_\nu)^k\nu_t,\cdot\,\tilde\nu_t\rangle\neq 0$ and $\varepsilon>0$ was arbitrary, this estimate shows that (i) $\Pi_{\varpi k}(g\nu_t)\neq 0$, so that $\mathbb{V}_{\varpi k}$ is not empty, and (ii) $\nu_{k,t} = (f_\nu)^k\nu_t$ can be approximated by elements of $\mathbb{V}_{\varpi k}$, and so must belong to $\mathbb{V}_{\varpi k}$, as claimed.

*Step 2: $\mathcal{L}_t$ has a spectral gap.* It suffices to show that the ergodicity of $(T,\mu_{\mathrm{SRB}})$ implies that the positive eigenvalue $e^{P_*(t)}$ is simple. For then applying Step 1, suppose $\nu\in\mathbb{V}_\varpi$ for $\varpi = a/b$. Then both $\mathcal{L}_t^b\nu = e^{P_*(t)b}\nu$ and $\mathcal{L}_t^b\nu_t = e^{P_*(t)b}\nu_t$, so that $\mathcal{L}_t^b$ has eigenvalue $e^{P_*(t)b}$ of multiplicity 2, and this is also its spectral radius, contradicting the fact that $(T^b,\mu_{\mathrm{SRB}})$ is also ergodic.

Now, suppose $\nu\in\mathbb{V}_0$. By Lemma 4.13(b), there exists $f_\nu\in L^\infty(\nu_t)$ such that $d\nu = f_\nu d\nu_t$. We will show that $f_\nu$ is $\nu_t$-a.e. a constant.

By (4.52) $f_\nu \circ T = f_\nu$, $\nu_t$-a.e. so that setting

$$S_n f_\nu = \sum_{j=0}^{n-1} f_\nu \circ T^j \,,$$

we see that $\frac{1}{n} S_n f_\nu = f_\nu$ for all $n \geq 1$. Thus $f_\nu$ is constant on stable manifolds. Next, since the factor measure $\hat{\nu}_t$ is equivalent to $\hat{\mu}_{\mathrm{SRB}}$ on the index set $\Xi$ by Lemma 4.16, it follows that $f_\nu = f_\nu \circ T$ on $\hat{\mu}_{\mathrm{SRB}}$-a.e. $W_\xi \in \mathcal{F}$. So $f_\nu = f_\nu \circ T$, $\mu_{\mathrm{SRB}}$-a.e. Since $\mu_{\mathrm{SRB}}$ is ergodic, $f_\nu$ is constant $\mu_{\mathrm{SRB}}$-a.e. But since $f_\nu$ is constant on each stable manifold $W_\xi \in \mathcal{F}$, it follows that there exists $c > 0$ such that $f_\nu = c$ for $\hat{\mu}_{\mathrm{SRB}}$-a.e. $\xi \in \Xi$, and once again using the equivalence of $\hat{\mu}_{\mathrm{SRB}}$ and $\hat{\nu}_t$, we conclude that $f_\nu$ is constant $\nu_t$-a.e. □

*Proof of Theorem 4.1.* All claims except the last sentence of the theorem follow from Propositions 4.12 and 4.15, and Lemma 4.17. Exponential decay of correlations for $C^\alpha$ functions with rate $\upsilon$ satisfying (4.4) follows from the classical spectral decomposition

$$\mathcal{L}_t^k f = e^{kP_*(t)}[c_t(f) \cdot \nu_t + \mathcal{R}_t^k(f)] \,, \text{ where } \exists C < \infty \text{ s. t. } \|\mathcal{R}_t^k f\| < C\upsilon^k \|f\| \,, \forall k \geq 0 \,, \forall f \in \mathcal{B} \,,$$

and $c_t(f) = \frac{\langle f, \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle}$. Indeed, by [DZ2, Lemma 5.3] for $\psi \in C^\alpha(M)$,

$$\psi \circ T^{-j} f \in \mathcal{B} \quad \text{and} \quad \|\psi \circ T^{-j} f\|_{\mathcal{B}} \leq C_j |\psi|_{C^\alpha} \|f\|_{\mathcal{B}} \quad \text{for all} \quad j \geq 1 \,, \tag{4.53}$$

we find for $f_1, f_2 \in C^\alpha(M)$ (using (4.53) with $j = k$),

$$\begin{aligned}\int (f_1 \circ T^k) f_2 d\mu_t &= \frac{\langle (f_1 \cdot f_2 \circ T^{-k} \nu_t, \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} = e^{-kP_*(t)} \frac{\langle f_1 \mathcal{L}_t^k (f_2 \nu_t), \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} \\ &= c_t(f_2 \nu_t) \frac{\langle f_1 \nu_t, \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} + \frac{\langle f_1 \mathcal{R}_t^k (f_2 \nu_t), \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} = \int f_1 d\mu_t \int f_2 d\mu_t + \frac{\langle f_1 \mathcal{R}_t^k (f_2 \nu_t), \tilde{\nu}_t \rangle}{\langle \nu_t, \tilde{\nu}_t \rangle} \,,\end{aligned}$$

and we have, using again (4.53) (with $j = 0$),

$$\left| \langle f_1 \mathcal{R}_t^k (f_2 \nu_t), \tilde{\nu}_t \rangle \right| \leq |f_1|_{C^\alpha} \|\mathcal{R}_t^k (f_2 \nu_t)\| \leq C |f_1|_{C^\alpha} \upsilon^k \|f_2 \nu_t\| \leq C |f_1|_{C^\alpha} |f_2|_{C^\alpha} \upsilon^k \,.$$

Exponential mixing for Hölder functions of exponent smaller than $\alpha$ then follows from mollification (a lower exponent may worsen the rate of mixing). Finally, mixing is obtained by a standard argument: Since $\mu$ is a Borel probability measure and $M$ is a compact metric space (and thus a normal topological space), any $f \in L^2(\mu)$ can be approximated by a sequence of continuous functions in the $L^2(\mu)$ norm, using Urysohn functions. So, by Cauchy–Schwartz, we may reduce to proving mixing for continuous test functions. Clearly, Lipschitz functions form a subalgebra of the Banach algebra of continuous functions, the constant function $\equiv 1$ is Lipschitz, and for any $x \neq y$ in $M$ there exists a Lipschitz function $\tilde{f}$ with $\tilde{f}(x) \neq \tilde{f}(y)$. Since $M$ is a compact metric space the Stone–Weierstrass theorem implies that any continuous function on $M$ can be approached in the supremum norm by a sequence of Lipschitz functions on $M$. To conclude, use Cauchy–Schwartz. □

## 5. Final Properties of $\mu_t$ (Proof of Theorem 1.1 and Theorem 2.4)

In this section we show Proposition 5.1, Corollary 5.3, Proposition 5.5, Lemma 5.6, and Proposition 5.7, which, together with Theorem 4.1, give Theorem 1.1.

### 5.1. Measuring Neighbourhoods of Singularity Sets – $\mu_t$ is $T$-adapted.

In this section, we show Proposition 5.1, which gives in particular that $\mu_t$ is $T$-adapted. For any $\varepsilon > 0$ and any $A \subset M$, we set $\mathcal{N}_\varepsilon(A) = \{x \in M \mid d(x, A) < \epsilon\}$. The proof will be based on controlling the measure of small neighbourhoods of singularity sets.

**Proposition 5.1.** *Let $\mu_t$ be given by Theorem 4.1 for $t \in [t_0, t_1]$, with $p > 2$ the norm parameter.*

a) *For any $C^1$ curve $S$ uniformly transverse to the stable cone, there exists $C > 0$ such that $\mu_t(\mathcal{N}_\varepsilon(S)) \le C\varepsilon^{1/p}$ for all $\varepsilon > 0$.*
b) *The measure $\mu_t$ has no atoms. We have $\mu_t(\mathcal{S}_n) = 0$ for any $n \in \mathbb{Z}$, and $\mu_t(W) = 0$ for any local stable or unstable manifold $W$.*
c) *The measure $\mu_t$ is adapted, i.e., $\int |\log d(x, \mathcal{S}_{\pm 1})| \, d\mu_t < \infty$.*
d) *For any $p' > 2p$, $\mu_t$-almost every $x$ and each $n \in \mathbb{Z}$, there exists $C > 0$ such that*

$$d(T^j x, \mathcal{S}_n) \ge C j^{-p'}, \forall j \ge 0 . \tag{5.1}$$

e) *$\mu_t$-almost every $x \in M$ has stable and unstable manifolds of positive lengths.*

*Proof.* We proceed as in [BD, Corollary 7.4]. The key fact is that for any $n \in \mathbb{N}$ there exists $C_n < \infty$ such that for all $\varepsilon > 0$

$$\mu_t(\mathcal{N}_\epsilon(\mathcal{S}_{-n})) < C_n \varepsilon^{1/p}, \quad \mu_t(\mathcal{N}_\epsilon(\mathcal{S}_n)) < C_n \varepsilon^{1/(2p)} . \tag{5.2}$$

Denoting by $1_{n,\varepsilon}$ the indicator function of the set $\mathcal{N}_\varepsilon(\mathcal{S}_{-n})$, Proposition 4.15(a) implies

$$\mu_t(\mathcal{N}_\varepsilon(\mathcal{S}_{-n})) = \langle 1_{n,\varepsilon} \nu_t, \tilde{\nu}_t \rangle \le C |1_{n,\varepsilon} \nu_t|_w ,$$

for $n \ge 0$. The bound $|1_{n,\varepsilon} f|_w \le A_n \|f\|_s |\varepsilon|^{1/p}$ for all $f \in \mathcal{B}$ follows exactly as the proof of [BD, Lemma 7.3], replacing the logarithmic modulus of continuity $|\log \varepsilon|^{-\gamma}$ in the strong stable norm there by our Hölder modulus of continuity $\varepsilon^{1/p}$, and using the fact that $\mathcal{S}_{-n}$ is uniformly transverse to the stable cone. This proves the first inequality in (5.2). The second follows from the invariance of $\mu_t$, together with the fact that $T(\mathcal{N}_\varepsilon(\mathcal{S}_n)) \subset \mathcal{N}_{C\varepsilon^{1/2}}(\mathcal{S}_{-n})$.

Claim a) of the proposition follows from the proof of (5.2), since the only property required of $\mathcal{S}_{-n}$ is that it comprises finitely many smooth curves uniformly transverse to the stable cone. The bound (5.2) applied to arbitrary stable curves immediately implies that $\mu_t$ has no atoms, and that $\mu_t(\mathcal{S}_n) = 0$ for any $n \in \mathbb{Z}$. Next, if we had $\mu_t(W) > 0$ for a local stable manifold, then $\mu_t(T^n W) > 0$ for all $n > 0$. Since $\mu_t$ is a probability measure and $T^n$ is continuous on stable manifolds, $\cup_{n \ge 0} T^n W$ must be the union of finitely many smooth curves. Since $|T^n W| \to 0$, there is a subsequence $(n_j)$ such that $\cap_{j \ge 0} T^{n_j} W = \{x\}$. Thus $\mu_t(\{x\}) > 0$, a contradiction. For an unstable manifold $W$, use the fact that $T^{-n}$ is continuous on $W$. So we have established b).

To show c), choose $p' > 2p$. Then by (5.2)

$$\begin{aligned} \int_{M \setminus \mathcal{N}_1(\mathcal{S}_1)} |\log d(x, \mathcal{S}_1)| \, d\mu_t &= \sum_{j \ge 1} \int_{\mathcal{N}_{j^{-p'}}(\mathcal{S}_1) \setminus \mathcal{N}_{(j+1)^{-p'}}(\mathcal{S}_1)} |\log d(x, \mathcal{S}_1)| \, d\mu_t \\ &\le p' \sum_{j \ge 1} \log(j+1) \cdot \mu_t(\mathcal{N}_{j^{-p'}}(\mathcal{S}_1)) \le p' C_1 \sum_{j \ge 1} \log(j+1) \cdot j^{-p'/(2p)} < \infty. \end{aligned}$$

A similar estimate holds for $\int |\log d(x, \mathcal{S}_{-1})| \, d\mu_t$.

Next, fix $\eta > 0$, $p' > 2p$ and $n \in \mathbb{Z}_+$. Since both sums

$$\sum_{j \ge 1} \mu_t(\mathcal{N}_{\eta j^{-p'}}(\mathcal{S}_{-n})) \le \tilde{C} C_{-n} \eta^{\frac{1}{p}} \sum_{j \ge 1} j^{-\frac{p'}{p}} , \quad \sum_{j \ge 1} \mu_t(\mathcal{N}_{\eta j^{-p'}}(\mathcal{S}_n)) \le \tilde{C} C_n \eta^{\frac{1}{2p}} \sum_{j \ge 1} j^{-\frac{p'}{2p}} , \tag{5.3}$$

are finite, the Borel–Cantelli Lemma implies that $\mu_t$-almost every $x \in M$ visits $\mathcal{N}_{\eta j^{-p'}}(\mathcal{S}_n)$ only finitely many times. This gives (5.1) and thus claim d). Finally, the existence of nontrivial stable and unstable manifolds claimed in e) follows from the Borel–Cantelli estimate (5.3) by a standard argument, choosing $p' > 2p$ and $\eta \ge 1$ such that $\Lambda^j > \eta^{-1} j^{p'}$ for all $j$ (see [CM, Sect. 4.12]). $\square$

5.2. $\mu_t$ **is an Equilibrium State. Variational Principle for** $P_*(t)$**.** For $\epsilon > 0$, $x \in M$, and $n \ge 1$ denote by $B_n(x, \epsilon)$ the dynamical (Bowen) ball for $T^{-1}$:

$$B_n(x, \epsilon) = \{ y \in M \mid d(T^{-j}(y), T^{-j}(x)) \le \epsilon , \ \forall \, 0 \le j \le n \} . \tag{5.4}$$

**Proposition 5.2** (Upper Bounds on the Measure of Dynamical Balls). *Let $t_0 \in (0,1)$ and $t_1 \in (1, t_*)$. There exists $A < \infty$ such that for all small enough $\epsilon > 0$, all $x \in M$, and all $n \ge 1$, the measure $\mu_t$ constructed in Theorem 4.1 for $t \in [t_0, t_1]$ satisfies*

$$\mu_t(\overline{B_n(x,\epsilon)}) \le A e^{-nP_*(t) + t\sum_{k=1}^n \log J^sT(T^{-k}(x))} . \tag{5.5}$$

**Corollary 5.3** (Equilibrium State for $-t\log J^u$. Variational principle for $P_*(t)$.). *The measure $\mu_t$ constructed in Theorem 4.1 for $t \in (0, t_*)$ satisfies $P_{\mu_t}(-t\log J^uT) = P_*(t) = P(t)$.*

*Proof of Corollary 5.3.* By definition we have $P_{\mu_t}(T) \le P(t)$, and Proposition 2.3 gives $P(t) \le P_*(t)$, so it is enough to show $P_{\mu_t}(T) \ge P_*(t)$. We follow [BD, Cor. 7.17]. Since $\int |\log d(x, \mathcal{S}_{\pm1})|\, d\mu_t < \infty$ by Proposition 5.1, and $\mu_t$ is ergodic, we may apply [DWY, Prop. 3.1] (a slight generalization of the Brin–Katok local theorem [BK], using [M, Lemma 2], continuity of the map is not used) to $T^{-1}$. This gives that for $\mu_t$-almost every $x \in M$,

$$\lim_{\epsilon\to0}\liminf_{n\to\infty} -\tfrac{1}{n}\log\mu_t(B_n(x,\epsilon)) = \lim_{\epsilon\to0}\limsup_{n\to\infty} -\tfrac{1}{n}\log\mu_t(B_n(x,\epsilon)) = h_{\mu_t}(T^{-1}) = h_{\mu_t}(T) .$$

Using (5.5) it follows that for any $\epsilon$ sufficiently small,

$$\limsup_{n\to\infty} -\tfrac{1}{n}\log\mu_t(B_n(x,\epsilon)) \ge P_*(t) - \lim_{n\to\infty}\tfrac{t}{n}\sum_{k=1}^n \log J^sT(T^{-k}(x)) \ge P_*(t) - t\int_M \log J^sT\, d\mu_t ,$$

for all $\mu_t$-typical $x$. Thus applying (1.6), we get $P_{\mu_t}(-t\log J^uT) \ge P_*(t)$. □

*Proof of Proposition 5.2.* For $x \in M$ and $n \ge 0$, let $1^B_{n,\epsilon}$ denote the indicator function of $B_n(x,\epsilon)$. Since $\nu_t$ is attained as the (averaged) limit of $e^{-nP_*(t)}\mathcal{L}_t^n 1$ in the weak (and strong) norm and since we have $\int_W (\mathcal{L}_t^n 1)\,\psi dm_W \ge 0$ whenever $\psi \ge 0$, it follows that, viewing $\nu_t$ as a leafwise distribution,

$$\int_W \psi\,\nu_t \ge 0, \quad \text{for all } \psi \ge 0. \tag{5.6}$$

Then the inequality $|\int_W \psi\,\nu_t| \le \int_W |\psi|\,\nu_t$ implies that the supremum in the weak norm can be obtained by restricting to $\psi \ge 0$. In addition, for each $n \ge 0$,

$$\begin{aligned} \int_W \psi\,\mathcal{L}_t^n\nu_t &= \lim_k e^{-kP_*(t)}\int_W \psi\,\mathcal{L}_t^n(\mathcal{L}_t^k 1)\,dm_W \\ &= \lim_k e^{-kP_*(t)}\int_{T^{-n}W} \psi\circ T^n\,\mathcal{L}_t^k 1\,|J^sT^n|^t\,dm_{T^{-1}W} = \int_{T^{-n}W}\psi\circ T^n\,|J^sT^n|^t\,\nu_t , \end{aligned} \tag{5.7}$$

for each $W \in \mathcal{W}^s$ and $\psi \in C^\beta(W)$.

Let $W \in \mathcal{W}^s$ be a curve intersecting $B_n(x,\epsilon)$, and let $\psi \in C^\alpha(W)$ satisfy $\psi \ge 0$ and $|\psi|_{C^\alpha(W)} \le 1$. Then, since $\mathcal{L}_t\nu_t = e^{P_*(t)}\nu_t$, we have

$$\int_W \psi\,1^B_{n,\epsilon}\,\nu_t = \int_W \psi\,1^B_{n,\epsilon}\,e^{-nP_*(t)}\mathcal{L}_t^n\nu_t = e^{-nP_*(t)}\sum_{W_i\in\mathcal{G}_n(W)}\int_{W_i}(\psi\circ T^n)\,(1^B_{n,\epsilon}\circ T^n)|J^sT^n|^t\,\nu_t . \tag{5.8}$$

In the proof of [BD, Prop. 7.12] we showed that $1^B_{n,\varepsilon}f \in \mathcal{B}_w$ (and $\mathcal{B}$) for each $f \in \mathcal{B}$ and $n \ge 0$. In the proof of [BD, Lemma 3.4], we found (using our strong notion of finite horizon) $\tilde\varepsilon > 0$ such that there if $x, y$ lie in different elements of $\mathcal{M}_0^n$, then $\max_{0\le i\le n} d(T^ix, T^iy) \ge \tilde\varepsilon$. Since $B_n(x,\epsilon)$ is defined with respect to $T^{-1}$, we will use the time reversal counterpart of this property: If $\epsilon < \tilde\varepsilon$, we conclude that $B_n(x,\epsilon)$ is contained in a single component of $\mathcal{M}_{-n}^0$, i.e., $B_n(x,\epsilon)\cap\mathcal{S}_{-n} = \emptyset$, so that $T^{-n}$ is a diffeomorphism of $B_n(x,\epsilon)$ onto its image. Note that $T^{-n}(B_n(x,\epsilon))$ is contained in a single component of $\mathcal{M}_0^n$, denoted $A_{n,\epsilon}$. Thus, $W_i \cap A_{n,\epsilon} = W_i$ for each $W_i \in \mathcal{G}_n(W)$. By (5.6),

$$\int_{W_i}(\psi\circ T^n)\,1_{T^{-n}(B_n(x,\epsilon))}|J^sT^n|^t\,\nu_t \le \int_{W_i}(\psi\circ T^n)|J^sT^n|^t\,\nu_t .$$

In the proof of [BD, Prop. 7.12] we observed that there are at most two $W_i \in \mathcal{G}_n(W)$ having nonempty intersection with $T^{-n}(B_n(x,\epsilon))$. Using these facts together with (4.11) and (4.12) (which implies $||J^sT^n|^t|_{C^\alpha(W_i)} \le C||J^sT^n|^t|_{C^0(W_i)}$), we sum over $W_i' \in \mathcal{G}_n(W)$ such that $W_i' \cap T^{-n}(B_n(x,\epsilon)) \neq 0$, to obtain

$$\int_W \psi\, 1^B_{n,\epsilon}\, \nu_t \le e^{-nP_*(t)} \sum_i \int_{W_i'} (\psi \circ T^n)\, |J^sT^n|^t\, \nu_t \le 2Ce^{-nP_*(t)+t\sum_{k=0}^{n-1}\log J^sT(T^{k-n}(x)))} |\nu_t|_w\,,$$

where we also used the distortion bounds from Lemma 2.2 to switch to $J^sT^n(T^{-n}x)$ since $T^{-n}x$ may not belong to $W_i'$. This yields $|1^B_{n,\varepsilon}\nu_t|_w \le 2Ce^{-nP_*(t)+t\log J^sT^n(T^{-n}x)}|\nu_t|_w$. Applying Proposition 4.15(a) gives (5.5). □

5.3. **Definition of $h_*$. Sparse Recurrence. Proof that** $\lim_{t\downarrow 0} P(t) = h_*$**.** In [BD, Lemma 3.3] we showed that the limit below exists

$$h_* := \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{M}_0^n\,.$$

The number $h_*$ generalises topological entropy, in particular, $P(0) \le h_*$ [BD, Theorem 2.3].

Using $h_*$, we can state the sparse recurrence condition:

**Definition 5.4** (Sparse Recurrence to Singularities)**.** *For $\varphi < \pi/2$ and $n \in \mathbb{N}$, define $s_0(\varphi, n) \in (0,1]$ to be the smallest number such that any orbit of length $n$ has at most $s_0 n$ collisions whose angles with the normal are larger than $\varphi$ in absolute value. We say that $T$ satisfies the sparse recurrence condition if there exist $\varphi_0 < \pi/2$ and $n_0 \in \mathbb{N}$ such that $h_* > s_0(\varphi_0, n_0)\log 2$.*

We refer to [BD, §2.4] for a discussion of the sparse recurrence condition. We proved in [BD] that sparse recurrence implies $P(0) = h_*$. The following proposition connects $h_*$ to $P_*(t)$ for $t > 0$, despite the use of different partitions, $\mathcal{M}_0^n$ and $\mathcal{M}_0^{n,\mathbb{H}}$.

**Proposition 5.5.** *If $T$ satisfies sparse recurrence then* $\lim_{t\downarrow 0} P_*(t) = \lim_{t\downarrow 0} P(t) = h_*$.

Assuming the sparse recurrence condition [BD, Theorem 2.4] we have $P(0) = h_*$. So in this case the function $P(t)$ is continuous on $[0, t_*)$. In the general case, we cannot exclude $P(0) < h_*$ even if we can show $\lim_{t\downarrow 0} P(t) = P(0)$.

*Proof.* Recall that $P(t) = P_*(t)$ for $t \in (0, t_*)$ (using Proposition 2.3 and Corollary 5.3).

Showing[36] $\lim_{t\downarrow 0} P(t) \le h_*$ does not require the sparse recurrence condition: Any invariant probability measure $\mu$ satisfies $\int_M \log J^uT\, d\mu \ge \log\Lambda$ due to (1.2). Also, $h_\mu(T) \le h_*$ by [BD, Theorem 2.3]. Thus for $t > 0$, we have $P(t) \le h_* - t\log\Lambda$, so that, $\lim_{t\downarrow 0} P(t) \le h_*$.

To prove the upper bound, assume the sparse recurrence condition and let $\mu_0$ denote the measure of maximal entropy for $T$ constructed in [BD, Theorem 2.4] (called $\mu_*$ in that paper). Since $\mu_0$ is T-adapted [BD, Theorem 2.6], the Jacobian $J^uT$ is defined $\mu_0$-almost everywhere and $\int \log J^uT\, d\mu_0 = \chi^+_{\mu_0} < \infty$. Thus for $t > 0$,

$$P(t) \ge P_{\mu_0}(-t\log J^uT) = h_{\mu_0} - t\int_M \log J^uT\, d\mu_0 = h_* - t\chi^+_{\mu_0}\,,$$

and $\lim_{t\downarrow 0} P(t) \ge h_*$. □

5.4. **Full Support of $\mu_t$.** It follows from Lemma 4.16 that the measure $\nu_t$ is fully supported on $M$. In this section, we will prove the analogous property for $\mu_t$ combining mixing of the SRB measure and a direct use of Cantor rectangles, bypassing the absolute continuity argument which was used in [BD, Section 7.3] to show full support of the measure of maximal entropy there. Recall the definition of maximal Cantor rectangle $R = R(D)$ comprising the intersection of all homogeneous stable and unstable manifolds completely crossing a solid rectangle $D$ as described in the proof of Proposition 3.14. The boundary of the solid rectangle $D$ comprises two stable and unstable manifolds which also belong to $R$. Let $\Xi_R \subset \mathcal{W}^s$ denote the family of stable manifolds corresponding to $R$ (i.e. the set of homogeneous stable manifolds that completely cross $D$).

[36] Note that the limit exists since $P(t) = P_*(t)$ is monotonic.

**Lemma 5.6.** *For any maximal Cantor rectangle $R$, if $\mu_{SRB}(\cup_{W\in\Xi_R}W) > 0$ then we also have $\mu_t(\cup_{W\in\Xi_R}W) > 0$. Consequently, for any nonempty open set $O \subset M$, we have $\mu_t(O) > 0$.*

*Proof.* Let $\psi \in C^1(M)$ such that $\psi \geq 0$ and $\psi \equiv 1$ on $\cup_{W\in\Xi_R}W$. Due to the spectral decomposition of $\mathcal{L}_t^*$, setting $c = \langle \nu_t, \tilde\nu_t\rangle^{-1}$, we have

$$\mu_t(\psi) = c \lim_{n\to\infty} e^{-nP_*(t)}\langle \psi\nu_t, (\mathcal{L}_t^*)^n d\mu_{\mathrm{SRB}}\rangle = c \lim_{n\to\infty} e^{-nP_*(t)}\langle \mathcal{L}_t^n(\psi\nu_t), d\mu_{\mathrm{SRB}}\rangle\,. \tag{5.9}$$

Then, using the disintegration of $\mu_{\mathrm{SRB}}$, introduced before Lemma 4.16, into conditional measures on a fixed foliation $\mathcal{F} = \{W_\xi\}_{\xi\in\Xi}$ of stable manifolds, and a transverse measure $\hat\mu_{\mathrm{SRB}}$ on the index set $\Xi$, and recalling (5.7), we estimate for $n \geq 0$,

$$\begin{aligned}\langle \mathcal{L}_t^n(\psi\nu_t), d\mu_{\mathrm{SRB}}\rangle &= \int_\Xi |W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi) \int_{W_\xi} \mathcal{L}_t^n(\psi\nu_t)\,\rho_\xi \\ &= \int_\Xi |W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi) \sum_{W_i\in\mathcal{G}_n(W_\xi)} \int_{W_i} \psi\nu_t |J^sT^n|^t \rho_\xi\circ T^n \\ &\geq C\int_\Xi |W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi) \sum_{W_i\in\mathcal{G}_n(W_\xi)} |J^sT^n|^t_{C^0(W_i)} \int_{W_i}\psi\nu_t\,,\end{aligned}$$

where in the last line we have used (4.49), bounded distortion for $J^sT$ and the positivity of $\nu_t$. Next, note that if $W_i \in \mathcal{G}_n(W_\xi)$ properly crosses[37] $R$, then using again the positivity of $\nu_t$, we have

$$\int_{W_i} \psi\,\nu_t \geq \zeta(\ell_R)\,, \tag{5.10}$$

where $\ell_R$ is the minimum length of a stable manifold in $\Xi_R$ and $\zeta$ is a function depending only on $t$ (uniform in $[t_0,t_1]$) and $\delta_1(t_0,t_1)$ from (3.47) via (4.50). Thus letting $\mathcal{G}_n^R(W_\xi)$ denote those elements of $\mathcal{G}_n(W)$ that properly cross $R$, we have

$$\langle \mathcal{L}_t^n(\psi\nu_t), d\mu_{\mathrm{SRB}}\rangle \geq C'\zeta(\ell_R)\int_\Xi |W_\xi|^{-1} d\hat\mu_{\mathrm{SRB}}(\xi) \sum_{W_i\in\mathcal{G}_n^R(W_\xi)} |J^sT^n|^t_{C^0(W_i)}\,. \tag{5.11}$$

As in the proof of Proposition 3.14, by [CM, Lemma 7.87], we choose a finite number of locally maximal homogeneous Cantor rectangles $\mathcal{R}(\delta_1) = \{R_1,\dots,R_k\}$ such that there exists $n_* = n_*(\delta_1, R)$ such that $T^{-n_*}(D(R_i))$ contains a homogeneous connected component that properly crosses $R$ for all $i = 1,\dots,k$. Therefore, if $V \in \mathcal{W}^s$ has $|V| \geq \delta_1/3$, then at least one element of $\mathcal{G}_{n_*}(V)$ properly crosses $R$. Thus, if $|W_\xi| \geq \delta_1/3$ and $n - n_* \geq n_1$, then using (3.31), and letting $\delta_1'$ denote the minimum length of a stable manifold belonging to any of the $R_i$,

$$\begin{aligned}\sum_{W_i\in\mathcal{G}_n^R(W_\xi)} |J^sT^n|^t_{C^0(W_\xi)} &\geq e^{-tC_d} \sum_{W_j\in L^{\delta_1}_{n-n_*}(W_\xi)} |J^sT^{n-n_*}|^t_{C^0(W_j)} |J^sT^{n_*}|^t_{C^0(W_i)} \\ &\geq \tfrac34 e^{-tC_d} C(\delta_1')^{t\left(\frac{2q+1}{q+1}\right)^{n_*}} \sum_{W_j\in\mathcal{G}^{\delta_1}_{n-n_*}(W_\xi)} |J^sT^{n-n_*}|^t_{C^0(W_j)} \\ &\geq \tfrac34 e^{-tC_d} C(\delta_1')^{t\left(\frac{2q+1}{q+1}\right)^{n_*}} c_1 e^{(n-n_*)P_*(t)}\,,\end{aligned} \tag{5.12}$$

where in the second line we have estimated $J^sT^{n_*}$ from below on $W_i$ as in (3.39) using the fact that $|W_i| \geq \delta_1'$, and in the third line we have applied Propositions 3.14 and 3.15.

Substituting (5.12) into (5.11) and letting $\Xi^{\delta_1}$ denote those elements $W_\xi \in \mathcal{F}$ with $|W_\xi| \geq \delta_1/3$,

$$e^{-nP_*(t)}\langle \mathcal{L}_t^n(\psi\nu_t), d\mu_{\mathrm{SRB}}\rangle \geq C''\zeta(\ell_R)\delta_1(\delta_1')^{t\left(\frac{2q+1}{q+1}\right)^{n_*}} e^{-n_*P_*(t)}\hat\mu_{\mathrm{SRB}}(\Xi^{\delta_1})\,.$$

[37]See the proof of Proposition 3.14 for the definition of proper crossing.

Since this lower bound is independent of $n$, by (5.9) we have $\mu_t(\psi) > 0$, and since this holds for all $\psi \in C^1(M)$ with $\psi \equiv 1$ on $\cup_{W \in \Xi_R} W$, the first statement of the lemma is proved. Then the second statement of the lemma follows from the fact that any nonempty open set $O \subset M$ has a locally maximal Cantor set $R$ such that $D(R) \subset O$ and $\mu_{\mathrm{SRB}}(R) > 0$. □

5.5. **Uniqueness of Equilibrium State. (Strong) Variational Principle for $P_*(t,g)$.** In this section, we prove the following uniqueness result:

**Proposition 5.7.** *For any $0 < t < t_*$, the measure $\mu_t$ from Theorem 4.1 is the unique equilibrium state for $-t \log J^u T$.*

The proof of the proposition will give a more general statement (shown at the end of this section):

**Theorem 5.8** (Strong Variational Principle for $P_*(t,g)$)**.** *For any $[t_0, t_1] \subset (0, t_*)$ there exists $\upsilon_0 > 0$ such that for any $C^1$ function $g : M \to \mathbb{R}$ with $|g|_{C^1} \le \upsilon_0$ we have*

$$P_*(t,g) = P(t,g) = \max\{h_\mu + \int (-t \log J^u T + g)\, d\mu : \mu \text{ a } T\text{-invariant probability measure } \}\,,$$

*and the equilibrium state for $-t \log J^u + g$ is unique.*

(We restrict to $C^1$ functions $g$ for simplicity. The result also holds Hölder $g$ of suitable exponent.)

Fix $0 < t_0 < t_1 < t_*$. For $\phi \in C^1(M)$, $t \in [t_0, t_1]$, and $\upsilon \in \mathbb{R}$, define the transfer operator $\mathcal{L}_{t,\upsilon} = \mathcal{L}_{t,\upsilon,\phi}$ by

$$\mathcal{L}_{t,\upsilon} f = \frac{f \circ T^{-1}}{|J^s T|^{1-t} \circ T^{-1}} e^{\upsilon \phi \circ T^{-1}}\,, \quad \text{for all } f \in C^1(M).$$

Since $\mathcal{L}_{t,\upsilon} f = e^{\upsilon \phi \circ T^{-1}} \mathcal{L}_t f$ and the discontinuities of $\phi \circ T^{-1}$ are uniformly transverse to the stable cone, [DZ2, Lemma 5.3] implies that $\mathcal{L}_{t,\upsilon} f \in \mathcal{B}$ (with $\mathcal{B} = \mathcal{B}(t_0, t_1)$ the space for $\mathcal{L}_t$) and $\|\mathcal{L}_{t,\upsilon} f\|_{\mathcal{B}} \le C \|f\|_{\mathcal{B}} |e^{\upsilon \phi}|_{C^1}$, so that $\mathcal{L}_{t,\upsilon}$ defines a bounded linear operator on $\mathcal{B}$. By [DZ1, Lemma 6.1] the map $\upsilon \mapsto \mathcal{L}_{t,\upsilon}$ is analytic. Thus since $\mathcal{L}_t = \mathcal{L}_{t,0}$ has a spectral gap, so does $\mathcal{L}_{t,\upsilon}$ for $|\upsilon|$ sufficiently small, and the leading eigenvalue $\lambda_{t,\upsilon}$ varies analytically in $\upsilon$ [Ka, VII, Thm 1.8, II.1.8]; moreover, $\lambda_{t,0} = e^{P(t)}$ and, with $\mu_t$ from Theorem 4.1, we have [Ka, II.2.1, (2.1), (2.33)]

$$\frac{d}{d\upsilon} \lambda_{t,\upsilon} \Big|_{\upsilon=0} = e^{P(t)} \int \phi \, d\mu_t\,, \, \forall t \in [t_0, t_1]\,. \tag{5.13}$$

Recalling the definition $P(t, \upsilon\phi)$ in (1.8), the following result will give Proposition 5.7:

**Proposition 5.9.** *Fix $0 < t_0 < t_1 < t_*$. For $\phi \in C^1(M)$, $t \in [t_0, t_1]$, and $\upsilon \in \mathbb{R}$, with $|\upsilon|$ sufficiently small, the spectral radius of $\mathcal{L}_{t,\upsilon}$ on $\mathcal{B}(t_0, t_1)$ is $\lambda_{t,\upsilon\phi} = e^{P(t,\upsilon\phi)}$.*

*Proof of Proposition 5.7.* We use tangent measures, inspired by the proof of [Br, Theorem 16]: If $\mu$ is an equilibrium state for $-t \log J^u T$ then $\mu$ is a $C^1$-tangent measure at $t$ (see e.g.[38] [W, Theorem 9.14]) in the sense that,

$$P(t, \phi) \ge P(t, 0) + \int \phi \, d\mu \qquad \text{for all } \phi \in C^1(M). \tag{5.14}$$

Thus, Proposition 5.9 together with (5.13) imply that,

$$\int \phi \, d\mu_t = \lim_{\upsilon \downarrow 0} \frac{P(t, \upsilon\phi) - P(t, 0)}{\upsilon} \ge \int \phi \, d\mu \quad \text{and}$$
$$\int \phi \, d\mu_t = \lim_{\upsilon \uparrow 0} \frac{P(t, \upsilon\phi) - P(t, 0)}{\upsilon} \le \int \phi \, d\mu\,.$$

Thus $\int \phi \, d\mu_t = \int \phi \, d\mu$ for all $\phi \in C^1(M)$. Since $M$ is a compact metric space, $C^1(M)$ is dense in $C^0(M)$ and so $\mu = \mu_t$ showing the uniqueness claim in the proposition. □

[38] The standard definitions use $C^0$ rather than $C^1$ in (5.14) For our purposes, $C^1$ will suffice.

*Proof of Proposition 5.9.* Let $|\upsilon|$ be small enough such that $g := \upsilon\phi$ satisfies (3.6), (3.30), (3.46) and their analogues for the number of interpolations needed to reach $t_1 < t_*$ in Section 3.6. The constants $n_1$, $n_2$, $\delta_1$, $\delta_2$, $C_2$, $c_0$, $c_1$, $c_2$, and $C_\kappa$ from Section 3 then hold for all $g = \upsilon'\phi$ with $|\upsilon'| < |\upsilon|$ and all $t \in [t_0, t_1]$. In particular the constants $c_1(\upsilon') > 0$ from Proposition 3.14 and Proposition 3.18(a) and $c_2(\upsilon') > 0$ in Proposition 3.15 and Proposition 3.18(b) are uniform in $|\upsilon'| < |\upsilon|$ and $t \in [t_0, t_1]$.

*Step 1. The Spectral Radius $\lambda_{t,\upsilon}$ of $\mathcal{L}_{t,\upsilon}$ on $\mathcal{B}$ is $e^{P_*(t,\upsilon\phi)}$.* Possibly reducing $|\upsilon|$ further, $\mathcal{L}_{t,\upsilon}$ has a spectral gap on $\mathcal{B}$, as observed above. The upper bound on $\lambda_{t,\upsilon} \le e^{P_*(t,\upsilon\phi)}$ can thus be proved as in Proposition 4.12, once we know that the spectral radius of $\mathcal{L}_{t,\upsilon}$ on $\mathcal{B}_w$ is at most $e^{P_*(t,\upsilon\phi)}$. For this, by the upper bound in Proposition 3.18(b), it suffices to find $C < \infty$ such that

$$|\mathcal{L}^n_{t,\upsilon} f|_w \le C Q_n(t, \upsilon\phi)|f|_w\,,\ \forall f \in C^1\,. \tag{5.15}$$

To prove (5.15), note that due to (2.9), we have for $W \in \mathcal{W}^s$ and $W_i \in \mathcal{G}_n(W)$,

$$|e^{\upsilon S_n\phi}|_{C^\alpha(W_i)} \le (1 + C_*|\nabla\phi|_{C^0} \cdot \delta_0^{1-\alpha})|e^{\upsilon S_n\phi}|_{C^0(W_i)}\,, \tag{5.16}$$

then, for $W \in \mathcal{W}^s$ and $\psi \in C^\alpha(W)$ with $|\psi|_{C^\alpha(W)} \le 1$, we follow (4.10) and apply (4.11), (4.12), Lemma 3.4, and (5.16) to write,

$$\begin{aligned}\int_W \mathcal{L}^n_{t,\upsilon} f\,\psi\,dm_W &\le \sum_{W_i\in\mathcal{G}_n(W)} |f|_w|\psi\circ T^n|_{C^\alpha(W)}||J^sT^n|^t e^{\upsilon S_n\phi}|_{C^\alpha(W_i)}\\ &\le |f|_w C_1^{-1}(1+2^tC_d)(1+C_*|\nabla\phi|_{C^0}) \sum_{W_i\in\mathcal{G}_n(W)} |J^sT^n|^t_{C^0(W_i)}|e^{\upsilon S_n\phi}|_{C^0(W_i)} \le C|f|_w Q_n(t,\upsilon\phi)\,.\end{aligned}$$

The lower bound $\lambda_{t,\upsilon} \ge e^{P_*(t,\upsilon\phi)}$ on the spectral radius follows as in the proof of Proposition 4.12.

*Step 2. $P_*(t,\upsilon\phi) = P(-t\log J^uT + \upsilon\phi)$.* Denoting by $\nu_{t,\upsilon}$ the eigenmeasure associated to $e^{P_*(t,\upsilon\phi)}$ and by $\tilde\nu_{t,\upsilon}$ the eigenmeasure of the dual operator $\mathcal{L}^*_{t,\upsilon}$, defined as in Lemma 4.13 and (4.46), we construct an invariant probability measure $\mu_{t,\upsilon}$ as in Proposition 4.15.

We claim the following analogue of Proposition 5.2: There exists $A < \infty$ such that for all sufficiently small $|\upsilon|$, all $\epsilon > 0$ sufficiently small, all $x \in M$ and $n \ge 1$,

$$\mu_{t,\upsilon}(B_n(x,\epsilon)) \le A e^{-nP_*(t,\upsilon\phi) + t\log J^sT^n(T^{-n}x) + \upsilon S_n\phi(T^{-n}x)}\,, \tag{5.17}$$

where $B_n(x,\epsilon)$ is the Bowen ball defined in (5.4). Using (5.17), the proof of Corollary 5.3 yields that $P_{\mu_{t,\upsilon}}(-t\log J^uT + \upsilon\phi) \ge P_*(t,\upsilon\phi)$, and this, together with Proposition 2.3 yields $P(-t\log J^uT + \upsilon\phi) = P_*(t,\upsilon\phi)$. By Step 1, this ends the proof of Proposition 5.9. (In addition, we have established that $\mu_{t,\upsilon}$ is an equilibrium state for $-t\log J^uT + \upsilon\phi$.)

Finally, (5.17) follows easily from the proof of Proposition 5.2. The property in (5.6) extends to $\nu_{t,\upsilon}$ due to its definition as a limit of $e^{-nP_*(t,\upsilon\phi)}\mathcal{L}^n_{t,\upsilon}1$. The analogue of (5.7) holds for the same reason, so that the modification of (5.8) yields,

$$\int_W \psi\,1^B_{n\epsilon}\,\nu_{t,\upsilon} = e^{-nP_*(t,\upsilon\phi)} \sum_{W_i\in\mathcal{G}_n(W)} \int_{W_i} (\psi\circ T^n)(1^B_{n,\epsilon}\circ T^n)|J^sT^n|^t e^{\upsilon S_n\phi}\,\nu_{t,\upsilon}\,,$$

where $1^B_{n,\epsilon}$ denotes the indicator function of $B_n(x,\epsilon)$. The subsequent estimates in the proof of Proposition 5.2 go through with the obvious changes, so that

$$|1^B_{n,\epsilon}\nu_{t,\upsilon}|_w \le C' e^{-nP_*(t,\upsilon\phi) + t\log J^sT^n(T^{-n}x) + \upsilon S_n\phi(T^{-n}x)}\,,$$

where the only additional factor needed is the distortion constant $C_*|\nabla\phi|_{C^0}$ from (2.9). Applying the analogue of Proposition 4.15(a) completes the proof of (5.17). □

*Proof of Theorem 5.8.* The upper bound $P(t,g) \le P_*(t,g)$ is the content of Proposition 2.3. Taking $\phi = g$, the equilibrium state for $-t\log J^u + g$ is $\mu_{t,1}$ constructed in Step 2 of the proof of Proposition 5.9. The proof of uniqueness can be obtained by a straightforward adaptation of the argument proving uniqueness of the equilibrium state for $-t\log J^u$, up to taking small enough $|g|_{C^1}$. □

## 6. Derivatives of $P(t)$ and Strict Convexity (Proof of Theorem 1.2)

This section contains the proof of Theorem 1.2 and Corollaries 1.4 and 1.5.

The maximal eigenvalue of $\mathcal{L}_t^n$ is $\exp(nP(t))$. Showing that $nP(t)$ is analytic for some integer $n \ge 1$ is equivalent to showing that $P(t)$ is analytic. Recall the one-step expansion factor $\theta^{-1} > 1$ from Lemma 3.1. In the remainder of this section[39]:

$$\text{Fix } n_0 \ge 1 \text{ such that } |J^s T^{n_0}| < C_0 \theta^{n_0} \le \frac{1}{2}\,, \text{ and set } \mathcal{T} := T^{n_0}\,.$$

By standard results on analytic perturbations of simple isolated eigenvalues [Ka], analyticity of $P(t) = P_*(t)$ will be an immediate consequence of the following result:

**Proposition 6.1** (Analyticity of $t \mapsto \mathcal{L}_t^{n_0}$)**.** *Fix $0 < t_0 < t_1 < t_*$. Then the map $t \mapsto \mathcal{L}_t^{n_0}$ is analytic from $(t_0, t_1)$ to the space of bounded operators from $\mathcal{B}$ to $\mathcal{B}$, with*

$$\partial_t^j \mathcal{L}_t^{n_0}(f)|_{t=w} = \mathcal{L}_w^{n_0}\big((\log J^s\mathcal{T})^j f\big)\,,\ \forall j \ge 1\,,\ \forall w \in (t_0,t_1)\,,\ \forall f \in \mathcal{B}\,. \tag{6.1}$$

*Proof.* We claim that it suffices to prove that, for any $0 < t_0 < t_1 < t_*$, we have

$$\text{there exists } C < \infty \text{ such that } \|\mathcal{L}_w^{n_0}\big((\log J^s\mathcal{T})^j f\big)\|_{\mathcal{B}} \le j(Cj)^j\,\|f\|_{\mathcal{B}}\,,\ \forall f \in \mathcal{B}\,, \tag{6.2}$$

for all $w \in (t_0,t_1)$ and all $j \ge 0$. (The bound (6.2) is the content of Proposition 6.3.)

Indeed, by the Stirling formula, (6.2) implies

$$\frac{\|\mathcal{L}_w^{n_0}(f(\log J^s\mathcal{T})^j)\|_{\mathcal{B}}}{j!} \le jC^j C^j_{Stirling}\|f\|_{\mathcal{B}}\,. \tag{6.3}$$

Now, for $w \in (t_0,t_1)$ and $t \in \mathbb{C}$, first write

$$\mathcal{L}_t^{n_0} f = \frac{f \circ \mathcal{T}^{-1}}{|J^s\mathcal{T}|^{1-w} \circ \mathcal{T}^{-1}} e^{(w-t)\log J^s\mathcal{T}\circ\mathcal{T}^{-1}} = \frac{f \circ \mathcal{T}^{-1}}{|J^s\mathcal{T}|^{1-w} \circ \mathcal{T}^{-1}} \sum_{j=0}^{\infty} \frac{(w-t)^j}{j!} (\log J^s\mathcal{T})^j \circ \mathcal{T}^{-1}$$

where (6.3), with $w = 1$ and $f \equiv 1$, gives that the series converges in norm for $|w-t| < (CC_{Stirling})^{-1}$. Then note that

$$\frac{f \circ \mathcal{T}^{-1}}{|J^s\mathcal{T}|^{1-w} \circ \mathcal{T}^{-1}} \sum_{j=0}^{\infty} \frac{(w-t)^j}{j!} (\log J^s\mathcal{T})^j \circ \mathcal{T}^{-1} = \sum_{j=0}^{\infty} \frac{(w-t)^j}{j!} \mathcal{L}_w^{n_0}(f(\log J^s\mathcal{T})^j)\,, \tag{6.4}$$

where the sum commutes with $\mathcal{L}_w^{n_0}$ due to (6.3), with $w$ and $f$, so that this series also converges in norm for $|w-t| < (CC_{Stirling})^{-1}$. The radius of convergence is independent of $w \in (t_0,t_1)$, giving the claimed analyticity there. The power series representation (6.4) immediately implies (6.1). □

The key to the analyticity result in this section is the following elementary lemma which extends the distortion estimate Lemma 2.1 to expressions of the type $(\log|J^s\mathcal{T}|)^j |J^s\mathcal{T}|^t$:

**Lemma 6.2** (Distortion for $\exp(\Psi)(\Psi)^j$)**.** *Fix $I \subset \mathbb{R}$ a compact interval and let $\Psi : I \to \mathbb{R}_-$. Then, for any $\upsilon > 0$, there exists $C_\upsilon < \infty$ such that*

$$|\exp(\upsilon\Psi)|\Psi|^j|_{C^0(I)} \le (C_\upsilon j)^j\,,\ \forall j \ge 1\,. \tag{6.5}$$

[39] The value 1/2 below is for convenience, giving the number $-\log 2$ in Lemma 6.2; what is important is $C_0\theta^n < 1$.

*In addition, if* $|\sup \Psi| = \inf |\Psi| \geq \log 2$ *and there exist* $\alpha \in (0,1)$ *and* $C_\Psi < \infty$ *such that*[40]

$$|\Psi(x) - \Psi(y)| \leq C_\Psi |x-y|^\alpha \,,\ \forall x, y \in I \,, \tag{6.6}$$

*then*

$$|\log|\Psi(x)| - \log|\Psi(y)|| \leq 4C_\Psi|x-y|^\alpha \,, \quad \forall x, y \in I \,, \tag{6.7}$$

*and, for any* $t > 0$*,*

$$|\exp(t\Psi)|\Psi|^j|_{C^\alpha(I)} \leq (1 + eC_\Psi(4j+t))|\exp(t\Psi)|\Psi|^j|_{C^0(I)} \,, \quad \forall j \geq 0 \,. \tag{6.8}$$

(The lemma will be applied to $\Psi = \log|J^s\mathcal{T}|$, with $\alpha \leq 1/(q+1)$, and $I$ an interval giving an arc length parametrisation of a weakly homogeneous stable manifold.)

*Proof.* The proof of (6.5) is a straightforward exercise in calculus (with $C_\upsilon = (e \cdot \upsilon)^{-1}$): It suffices to show that $\sup_{X \in [0,1]} |\log X|^j X^\upsilon \leq (\frac{j}{e \cdot \upsilon})^j$.

Next, for any $x, y \in I$, the Mean Value Theorem applied to the logarithm yields, for some $Z$ between $|\Psi(x)|$ and $|\Psi(y)|$,

$$|\log|\Psi(x)| - \log|\Psi(y)|| \leq \frac{1}{Z}|\Psi(x) - \Psi(y)| \leq \frac{C_\Psi|x-y|^\alpha}{\log 2} \leq 4C_\Psi|x-y|^\alpha \,. \tag{6.9}$$

From (6.9) we get (6.7) and also, for any $x, y \in I$,

$$\begin{aligned}\left|\log \frac{\exp(t\Psi(x))|\Psi(x)|^j}{\exp(t\Psi(y))|\Psi(y)|^j}\right| &\leq j|\log|\Psi(x)| - \log|\Psi(y)|| + t|\Psi(x) - \Psi(y)| \\ &\leq (j4C_\Psi + tC_\Psi)|x-y|^\alpha \,.\end{aligned}$$

This implies

$$\exp(-C_\Psi(4j+t)|x-y|^\alpha) \leq \frac{\exp(t\Psi(x))|\Psi(x)|^j}{\exp(t\Psi(y))|\Psi(y)|^j} \leq \exp(C_\Psi(4j+t)|x-y|^\alpha) \,. \tag{6.10}$$

For $|x-y|^\alpha < (4jC_\Psi + tC_\Psi)^{-1}$ (other pairs $(x,y)$ are trivial to handle), (6.10) implies

$$\left|1 - \frac{\exp(t\Psi(x))|\Psi(x)|^j}{\exp(t\Psi(y))|\Psi(y)|^j}\right| \leq eC_\Psi(4j+t)|x-y|^\alpha \,.$$

Multiplying both sides above by $\exp(t\Psi(y))|\Psi(y)|^j \leq |\exp(t\Psi)|\Psi|^j|_{C^0(I)}$, proves (6.8). □

Recalling that $\mathcal{T} = T^{n_0}$ for fixed $n_0$, we define, for all integers $j \geq 0$,

$$\mathcal{M}_t^{(j)} f := \mathcal{L}_t^{n_0}((\log|J^s\mathcal{T}|)^j f) \,, \tag{6.11}$$

acting on measurable functions. We first prove (6.2):

**Proposition 6.3.** *For any* $0 < t_0 < t_1 < t_*$*, there exists* $C < \infty$ *such that*

$$\|\mathcal{M}_t^{(j)} f\|_\mathcal{B} \leq C^j\, j^{j+1}\, \|f\|_\mathcal{B} \,,\ \forall j \geq 1 \,, \forall f \in \mathcal{B} \,, \forall t \in [t_0, t_1] \,. \tag{6.12}$$

**Remark 6.4.** *A modification of the proof of Lemma 4.3 shows that for any* $f \in C^1(M)$*,* $\mathcal{M}_t^{(j)} f$ *can be approximated by* $C^1(M)$ *functions in the* $\mathcal{B}$ *norm, using the fact that Lemma 4.10 holds for the function* $(\log|J^s\mathcal{T}|)^j|J^s\mathcal{T}|^t$ *by Lemma 6.2. By density of* $C^1(M)$ *in* $\mathcal{B}$*, this, together with Proposition 6.3, implies* $\mathcal{M}_t^{(j)} f \in \mathcal{B}$ *for all* $f \in \mathcal{B}$ *and* $j \geq 0$*.*

[40]The bound (6.6) is equivalent to $\exp(-C_\Psi|x-y|^\alpha) \leq \exp(\Psi(x))/\exp(\Psi(y)) \leq \exp(C_\Psi|x-y|^\alpha)$ or, for small enough $|x-y|$, to $|1 - \exp(\Psi(x))/\exp(\Psi(y))| \leq C_{d,\Psi}|x-y|^\alpha$.

*Proof of Proposition 6.3.* It is enough to consider $f \in C^1(M)$. We first bound the stable norm. Fix $W \in \mathcal{W}^s$, and $\psi \in C^\beta(W)$ such that $|\psi|_{C^\beta(W)} \le |W|^{-1/p}$. For $t > 0$, we have

$$\begin{aligned}\int_W \mathcal{M}_t^{(j)} f\, \psi\, dm_W &= \sum_{W_i \in \mathcal{G}_{n_0}(W)} \int_{W_i} f(\psi \circ \mathcal{T}) |J^s\mathcal{T}|^t (\log |J^s\mathcal{T}|)^j \, dm_{W_i} \\ &\le \sum_{W_i \in \mathcal{G}_{n_0}(W)} \|f\|_s |\psi \circ \mathcal{T}|_{C^\beta(W)} \frac{|W_i|^{1/p}}{|W|^{1/p}} ||J^s\mathcal{T}|^t (\log |J^s\mathcal{T}|)^j|_{C^\beta(W)} \,.\end{aligned} \tag{6.13}$$

On the one hand, we have seen in §4.3.1 that $|\psi \circ \mathcal{T}|_{C^\beta(W_i)} \le \tilde C |\psi|_{C^\beta(W)}$. On the other hand, recalling that $\sup_{W,W_i} |J^s\mathcal{T}|_{C^0(W_i)} < 1$, and using (6.5) from Lemma 6.2, for any $\upsilon > 0$, there exists $C_\upsilon$ such that for any $W_i \in \mathcal{G}_n(W)$, all $t \in [t_0, t_1]$, and all $j \ge 1$,

$$\sup_{W_i} (|\log |J^s\mathcal{T}|^j| |J^s\mathcal{T}|^t) \le (jC_\upsilon)^j \sup_{W_i} |J^s\mathcal{T}|^{t-\upsilon} \,. \tag{6.14}$$

Therefore, since $\beta < \alpha$, choosing[41] $\upsilon < t_0/2 - 1/p$ and applying Lemma 6.2 , we deduce from (6.13) and (6.14) that for all $j \ge 1$ and $f \in C^1$, takig $C_d' = 1 + eC_\Psi(4+t)$ from (6.8),

$$\begin{aligned}\int_W \mathcal{M}_t^{(j)} f\, \psi\, dm_W &\le C_\upsilon^j j^j \, jC_d' \sum_{W_i \in \mathcal{G}_{n_0}(W)} \|f\|_s \frac{|W_i|^{1/p}}{|W|^{1/p}} |J^s\mathcal{T}|^{t-\upsilon}_{C^0(W_i)} \\ &\le C_d' C_2[0]\, j (jC_\upsilon)^j \|f\|_s Q_{n_0}(t - \upsilon - 1/p) \,,\end{aligned}$$

where $C_\Psi = C_d$ by (2.3), and we used Lemma 3.4 with $\varsigma = 1/p$. Taking the suprema over $\psi \in C^\beta(W)$ with $|\psi|_{C^\beta(W)} \le |W|^{-1/p}$ and $W \in \mathcal{W}^s$ yields $C_s < \infty$ such that

$$\|\mathcal{M}_t^{(j)}(f)\|_s \le j \frac{(jC_s)^j}{2} \|f\|_s \,, \ \forall j \ge 1 \,, \ \forall f \in C^1 \,, \forall t \in [t_0, t_1] \,. \tag{6.15}$$

For the unstable norm, let $\varepsilon < \varepsilon_0$ and let $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) \le \varepsilon$. For $\ell = 1, 2$, we partition $\mathcal{T}^{-1} W^\ell$ into matched pieces $U_k^\ell$ and unmatched pieces $V_i^\ell$ as in §4.3.3, and we find, for any $\psi_\ell \in C^\alpha(W^\ell)$ with $|\psi_\ell|_{C^\alpha(W^\ell)} \le 1$ and $d(\psi_1, \psi_2) = 0$,

$$\begin{aligned}&\left| \int_{W^1} \mathcal{M}_t^{(j)}(f)\, \psi_1 - \int_{W^2} \mathcal{M}_t^{(j)}(f)\, \psi_2 \right| \\ &\le \sum_k \left| \int_{U_k^1} f\, (\psi_1 \circ \mathcal{T})\, |J^s\mathcal{T}|^t (\log |J^s\mathcal{T}|)^j - \int_{U_k^2} f\, (\psi_2 \circ \mathcal{T})\, |J^s\mathcal{T}|^t (\log |J^s\mathcal{T}|)^j \right| \\ &\qquad + \sum_{\ell, i} \left| \int_{V_i^\ell} f\, (\psi_\ell \circ \mathcal{T})\, |J^s\mathcal{T}|^t (\log |J^s\mathcal{T}|)^j \right| \,.\end{aligned} \tag{6.16}$$

For the unmatched pieces, adapting (4.19), by using Lemma 6.2 combined with Lemma 2.1 and (6.14), we find, for $\ell = 1, 2$ (choosing again $\upsilon < t_0/2 - 1/p$ so that $t - \upsilon - 1/p > t_0/2$),

$$\begin{aligned}&\sum_{\ell, i} \left| \int_{V_i^\ell} f\, (\psi_\ell \circ \mathcal{T})\, |J^s\mathcal{T}|^t (\log |J^s\mathcal{T}|)^j \right| \\ &\le \|f\|_s C_1^{-1} jC_d' (jC_\upsilon)^j \sum_{\ell, i} |\mathcal{T} V_i^\ell|^{1/p} ||J^s\mathcal{T}|^{t - \upsilon - 1/p}|_{C^0(V_i^\ell)} \\ &\le \|f\|_s 4C_2[0] C_1^{-1} jC_d' (jC_\upsilon)^j \varepsilon^{1/p} Q_{n_0}(t - \upsilon - 1/p) \,, \forall t \in [t_0, t_1] \,, \forall j \ge 1 \,,\end{aligned} \tag{6.17}$$

using Lemma 3.4 with $\varsigma = 0$. Next, we consider matched pieces. Recalling (4.17), we define

$$(\widetilde{\log} J^s\mathcal{T})^j(x) := (\log J^s\mathcal{T})^j \circ G_{U_k^2} \circ G_{U_k^1}^{-1}(x) \,, \ \forall x \in U_k^1 \,, \ ,\forall j \ge 1 \,.$$

[41] This is always possible since $p > q + 1$ and $t_0 q \ge 4$ from Definition 3.2.

Now, using $\widetilde{\psi}_2$ and $\widetilde{J}^s\mathcal{T} = \widetilde{J}^sT^{n_0}$ as defined above (4.22), and injecting Lemma 6.2 in the proof of Sublemma 4.8(b) gives, for $\upsilon$ as above,

$$
\begin{aligned}
&|(\psi_1 \circ \mathcal{T})(\log J^s\mathcal{T})^j |J^s\mathcal{T}|^t - \widetilde{\psi}_2(\widetilde{\log} J^s\mathcal{T})^j \widetilde{J}^s\mathcal{T}|^t|_{C^\beta(U_k^1)} \\
&\qquad \le C(jC_\upsilon)^j jC'_d 2^t |J^sT^{n_0}|^{t-\upsilon}_{C^0(U_k^1)} \varepsilon^{\alpha-\beta}\,, \ \forall k\,,\ \forall j\,, \forall t \in [t_0,t_1]\,. \qquad (6.18)
\end{aligned}
$$

Then we split

$$
\begin{aligned}
&\left| \int_{U_k^1} f\,(\psi_1 \circ \mathcal{T})\,(\log J^s\mathcal{T})^j |J^s\mathcal{T}|^t - \int_{U_k^2} f\,(\psi_2 \circ \mathcal{T})\,(\log J^s\mathcal{T})^j |J^s\mathcal{T}|^t \right| \\
&\le \left| \int_{U_k^1} f\, \Big( (\psi_1 \circ \mathcal{T})\,(\log J^s\mathcal{T})^j |J^s\mathcal{T}|^t - \widetilde{\psi}_2(\widetilde{\log} J^s\mathcal{T})^j |\widetilde{J}^s\mathcal{T}|^t \Big) \right| \qquad (6.19) \\
&\quad + \left| \int_{U_k^1} f\,\widetilde{\psi}_2(\widetilde{\log} J^s\mathcal{T})^j |\widetilde{J}^s\mathcal{T}|^t - \int_{U_k^2} f\,(\psi_2 \circ \mathcal{T})\,(\log J^s\mathcal{T})^j |J^s\mathcal{T}|^t \right|. \qquad (6.20)
\end{aligned}
$$

We estimate (6.19) for all $t \in [t_0,t_1]$ and $j \ge 1$ using (6.18),

$$
\left| \int_{U_k^1} f\, \Big( (\psi_1 \circ \mathcal{T})(\log J^s\mathcal{T})^j\, |J^s\mathcal{T}|^t - \widetilde{\psi}_2(\widetilde{J}^s\mathcal{T})^t \Big) \right| \le \|f\|_s \delta_0^{1/p} jC'_d (C_\upsilon j)^j 2^t |J^s\mathcal{T}|^{t-\upsilon}_{C^0(I_k)} \varepsilon^{\alpha-\beta}\,.
$$

Then, noting that $d(\psi_1 \circ \mathcal{T}\,(\log J^s\mathcal{T})^j |J^s\mathcal{T}|^t, \widetilde{\psi}_2(\widetilde{\log} J^s\mathcal{T})^j |\widetilde{J}^s\mathcal{T}|^t) = 0$ by definition, and that the $C^\alpha$ norms of both test functions are bounded by $C(jC_\upsilon)^j jC'_d |J^s\mathcal{T}|^{t-\upsilon}_{C^0(I_k)}$, using Lemma 6.2, we estimate (6.20) for all $f \in C^1$ and $t \in [t_0,t_1]$ as follows

$$
\begin{aligned}
\left| \int_{U_k^1} f\,(\psi_1 \circ \mathcal{T})\,|J^s\mathcal{T}|^t - \widetilde{\psi}_2(\widetilde{\log} J^s\mathcal{T})^j (\widetilde{J}^s\mathcal{T})^t) \right| &\le \|f\|_u d_{\mathcal{W}^s}(U_k^1, U_k^2)^\gamma jC'_d (jC_\upsilon)^j C |J^s\mathcal{T}|^{t-\upsilon}_{C^0(I_k)} \\
&\le C'(jC_\upsilon)^j jC'_d \|f\|_u n_0{}^\gamma \Lambda^{-n_0\gamma} \varepsilon^\gamma |J^s\mathcal{T}|^{t-\upsilon}_{C^0(I_k)}\,,
\end{aligned}
$$

where we used Lemma 4.8(a) in the second inequality. Putting these estimates into (6.19), combining with (6.17) in (6.16), and summing over $k$ gives, for all $t \in [t_0,t_1]$ and $j \ge 1$,

$$
\begin{aligned}
&\left| \int_{W_1} \mathcal{M}_t^{(j)} f\,\psi_1 - \int_{W_2} \mathcal{M}_t^{(j)} f\,\psi_2 \right| \\
&\quad \le j(j\bar{C})^j \left( \|f\|_u n_0{}^\gamma \Lambda^{-n_0\gamma} \varepsilon^\gamma Q_{n_0}(t-\upsilon) + \|f\|_s (\varepsilon^{1/p} Q_{n_0}(t-1/p-\upsilon) + \varepsilon^{\alpha-\beta} Q_{n_0}(t-\upsilon)) \right).
\end{aligned}
$$

Finally, since $\alpha - \beta \le \gamma$ and $1/p \le \gamma$, while $n_0$ is fixed, we have found $C_u < \infty$ such that

$$
\|\mathcal{M}_t^{(j)} f\|_u \le j\frac{(jC_u)^j}{2}(\|f\|_s + \|f\|_u)\,,\ \forall f \in C^1\,,\ \forall t \in [t_0,t_1]\,, \forall j \ge 1\,.
$$

With (6.15), taking $C = \max\{C_s, C_u\}$, this concludes the proof of Proposition 6.3. □

*Proof of Theorem 1.2.* Since $\exp(n_0 P(t)) > 0$ is a simple isolated eigenvalue of $\mathcal{L}_t^{n_0}$, analyticity of $\exp(n_0 P(t))$ is an immediate consequence of Proposition 6.1 and [Ka, VII, Theorem 1.8, II.1.8]. Since $\inf_{[t_0,t_1]} \exp(n_0 P(t)) > 0$, the function $P(t)$ is also analytic. The formulas

$$
n_0 P'(t) \exp(n_0 P(t))\,, \qquad n_0 P''(t) \exp(n_0 P(t)) + n_0{}^2 P'(t)^2 \exp(n_0 P(t))
$$

can be read off [Ka, II.2.2, (2.1), (2.33) p.79] (taking $m = 1$ there). It is then easy to extract the claimed formula (1.12) for $P'(t)$. In order to[42] establish (1.13) for $P''(t)$, use (1.12), and note that, recalling $\chi_t = P'(t) = \int \log J^sT\,d\mu_t$,

$$
\sum_{k \ge 0} \left[ \int (\log |J^sT| \circ T^k) \log |J^sT|\,d\mu_t - \chi_t^2 \right]
$$

[42]Formulas (1.12)–(1.13) are classical in smooth hyperbolic dynamics, see [Ru, Chap. 5, ex. 5b] for $P''(t)$.

$$= \langle \log |J^sT| (1 - e^{-P_*(t)} \mathcal{L}_t)^{-1} \Big( (\log |J^sT| - \chi_t) \nu_t \Big), \tilde\nu_t \rangle \, .$$

If there exists $f \in L^2(\mu_t)$ such that $\log |J^sT| - \chi_t = f - f \circ T$ then it is easy to see that $P''(t) = 0$. For the converse statement, we will use a martingale CLT result à la Gordin (see e.g. Viana [BDV, Theorem E.11]) as in [DRZ]: Let $\mathcal{A}_0$ be the sigma-algebra generated by the ($\mu_t$-mod 0) partition of $M$ into maximal connected, strongly homogeneous local stable manifolds for $T$ (this partition is measurable since it has a countable generator, see e.g. [CM, §5.1]). Then $\mathcal{A}_n = T^{-n}\mathcal{A}_0$, for $n \in \mathbb{Z}$, is a decreasing sequence of sigma algebras. Therefore, if $P''(t) = 0$, to obtain $f \in L^2(\mu_t)$ such that $\log |J^sT| = \chi_t + f - f \circ T$ from Gordin's Theorem ([BDV, Theorem E.11] or [DRZ, Theorem 5.1]), we only need to check the following two conditions:

$$\sum_{n=0}^{\infty} \| \log |J^sT| - E((\log |J^sT| - \chi_t)|\mathcal{A}_{-n})\|_{L^2(\mu_t)} < \infty \, , \tag{6.21}$$

$$\sum_{n=0}^{\infty} \| E((\log |J^sT| - \chi_t)|\mathcal{A}_{n})\|_{L^2(\mu_t)} < \infty \, . \tag{6.22}$$

We first discuss (6.21). If $n \geq 0$, then the elements of $\mathcal{A}_{-n}$ are of the form $T^n(V^s(x))$ where $V^s(x)$ is the maximal connected, strongly homogeneous stable manifold of (almost every) $x$. From Lemma 6.2, the function $\log |J^sT|$ is (Hölder) continuous on $T^n(V^s(x))$ for any $n \geq 1$, so, letting $\mathcal{A}_{-n}(x)$ be the element of $\mathcal{A}_{-n}$ containing $x$, we have

$$E(\log |J^sT| |\mathcal{A}_{-n})(x) = \log |J^sT|(y) \, ,$$

for some $y \in \mathcal{A}_{-n}(x)$. Thus (see the proof of [DRZ, (5.3)]), (6.7) with $\alpha = 1/(q+1)$ gives

$$\begin{aligned} &\| \log |J^sT| - E((\log |J^sT| - \chi_t)|\mathcal{A}_{-n})\|_{L^2(\mu_t)} \\ &\quad \leq \| \log |J^sT| - E((\log |J^sT| - \chi_t)|\mathcal{A}_{-n})\|_{L^\infty(\mu_t)} \leq C \Lambda^{-n/(q+1)} \, , \forall n \geq 1 \, , \; \forall t \in [t_0, t_1] \, . \end{aligned}$$

(The length of any element $T^n(V^s(x))$ in $\mathcal{A}_{-n}$ is bounded by $C_0 \Lambda^{-n}$.) This proves (6.21).

To establish (6.22), we also adapt the argument in [DRZ], starting from

$$\begin{aligned} &\sum_{n=0}^{\infty} \| E((\log |J^sT| - \chi_t)|\mathcal{A}_{n})\|_{L^2(\mu_t)} \\ &\qquad = \sum_{n=0}^{\infty} \sup \Big\{ \int (\log |J^sT| - \chi_t) \cdot (\psi \circ T^n) \, d\mu_t \mid \psi \in L^2(\mathcal{A}_0, \mu_t) \text{ with } \|\psi\|_{L^2(\mu_t) = 1} \Big\} \, . \end{aligned}$$

The key new ingredient is the fact that,[43] since $\nu_t = e^{-n_0 P(t)} \mathcal{L}_t^{n_0}(\nu_t)$, with $\nu_t \in \mathcal{B}$, we get from Proposition 6.3 and Remark 6.4 that

$$(\log |J^sT^{n_0}| \circ T^{-n_0}) \nu_t = e^{-n_0 P(t)} \mathcal{M}_t^{(1)}(\nu_t) \in \mathcal{B} \subset \mathcal{B}_w \, . \tag{6.23}$$

By definition, any $\mathcal{A}_0$-measurable function $\psi$ is constant on each curve in $\mathcal{A}_0$. If in addition $\psi$ is bounded, then for any $k \geq 0$, $\psi \circ T^k \in C^\alpha(\mathcal{W}^s_{\mathbb{H}})$ and $|\psi \circ T^k|_{C^\alpha(\mathcal{W}^s_{\mathbb{H}})} = |\psi|_{C^0(\mathcal{W}^s_{\mathbb{H}})} =: |\psi|_\infty$. Thus by Lemma 4.14 and (4.47),

$$\psi \circ T^k \tilde\nu_t \in \mathcal{B}_w^* \text{ and } |\langle f, \psi \circ T^k \tilde\nu_t \rangle| \leq C' |f|_w |\psi|_\infty \, , \forall f \in \mathcal{B}_w \, . \tag{6.24}$$

Then, recalling that $\mu_t(f) = \langle f\nu_t, \tilde\nu_t \rangle / \langle \nu_t, \tilde\nu_t \rangle$ for suitable $f$, following [DRZ], we write for $n \geq n_0$, and any bounded $\mathcal{A}_0$-measurable function $\psi$,

$$\int (\log |J^sT| - \chi_t) \cdot (\psi \circ T^n) \, d\mu_t = \frac{1}{n_0} \int (\log |J^sT^{n_0}| \circ T^{-n_0} - n_0 \chi_t) \cdot (\psi \circ T^{n-n_0}) \, d\mu_t$$

[43]Since $f \equiv 1 \in \mathcal{B}$, Remark 6.4 also implies $\log |J^sT^{n_0}| \circ T^{-n_0} = \log |J^s\mathcal{T}| \circ \mathcal{T}^{-1} = \mathcal{M}_1^{(1)}(1) \in \mathcal{B}$.

$$= \frac{1}{n_0}\langle (\log |J^sT^{n_0}| \circ T^{-n_0} - n_0\chi_t)\nu_t\,,\,(\psi \circ T^{n-n_0})\tilde\nu_t\rangle / \langle \nu_t, \tilde\nu_t\rangle$$

$$= \frac{1}{n_0}\langle e^{(n-n_0)P(t)}\mathcal{L}_t^{n-n_0}\big((\log |J^sT^{n_0}| \circ T^{-n_0} - n_0\chi_t)\nu_t\big)\,,\,\psi\tilde\nu_t\rangle / \langle \nu_t, \tilde\nu_t\rangle\,. \tag{6.25}$$

(The expressions in the first line are well defined and coincide because $(\log |J^sT| - \chi_t) \in L^1(d\mu_t)$ and $\psi$ is bounded. The expression in the second line is well defined by (6.23) and (6.24). Therefore, the second equality holds due to the definition of $\mu_t$ in Proposition 4.15(b). The last equality is clear.) Clearly $\tilde\nu_t\big(\nu_t(\log |J^sT^{n_0}| \circ T^{-n_0})\big) = n_0\chi_t$, so that Corollary 1.4 and (6.23) give constants $\rho < 1$ and $C_1', C_2' < \infty$ such that for all $n \ge n_0$ and all $t \in [t_0, t_1]$

$$\begin{aligned} |e^{(n-n_0)P(t)}\mathcal{L}_t^{n-n_0}&\big(\nu_t(\log |J^sT^{n_0}| \circ T^{-n_0} - n_0\chi_t)\big)|_w \\ &\le \|e^{(n-n_0)P(t)}\mathcal{L}_t^{n-n_0}\big(\nu_t(\log |J^sT^{n_0}| \circ T^{-n_0}) - n_0\chi_t)\big)\|_{\mathcal{B}} \\ &\le C_1'\rho^{n-n_0}\|\nu_t(\log |J^sT^{n_0}| \circ T^{-n_0})\|_{\mathcal{B}} \le C_2'\rho^n\,. \end{aligned} \tag{6.26}$$

Next, (6.25) together with the bounds (6.24) and (6.26), gives $C < \infty$ such that, for any bounded function $\psi$ which is $\mathcal{A}_0$-measurable,

$$|\int (\log |J^sT| - \chi_t) \cdot (\psi \circ T^n)\, d\mu_t| \le C\rho^n|\psi|_{L^\infty(M)}\,,\ \forall n \ge 1\,,\ \forall t \in [t_0, t_1]\,. \tag{6.27}$$

Then the proof of Lemma 5.1(c) (the $T$-adapted property of $\mu_t$) not only implies that $\log |J^sT|(x) \le C\log(d(x, \mathcal{S}_1))$ is in $L^1(d\mu_t)$ but also in $L^\ell(d\mu_t)$ for all $\ell \ge 1$ (use that $\sum_{j\ge 1}(\log(j+1))^\ell j^{-p'/(2p)} < \infty$ for all $\ell$ if $p' > 2p$). It follows that [DRZ, Lemma 5.2] holds for $\bar s = (\log |J^sT|) - \chi_t$, bootstrapping the $L^\infty$ bound (6.27) to the required $L^2$ control (6.22). The only change required in the proof (since the observable $\bar s$ in [DRZ, Lemma 5.2] is bounded while ours is not), is to replace the second term on the right-hand side of [DRZ, eq. (5.7)] by the Hölder bound $(\int |\bar s|^3 d\mu_t)^{1/3}(\int |\psi - \psi_L|^{3/2} d\mu_t)^{2/3}$ (where $\psi_L(x) = \psi(x)$ if $|\psi(x)| \le L$ and $\psi_L(x) = 0$ otherwise). Then using the fact that $\psi \in L^2(\mu_t)$,

$$\int |\psi - \psi_L|^{3/2} d\mu_t = \int 1_{|\psi|>L} \cdot |\psi|^{3/2}\, d\mu_t \le L^{-1/2}|\psi|^2_{L^2}\,,$$

using the Markov bound $\mu(|\psi| > L) \le L^{-2}|\psi|^2_{L^2}$. Setting $L = \rho^{-3n/4}$ instead of $L = \rho^{-n/2}$ in [DRZ, eq. (5.8)] completes the proof of [DRZ, Lemma 5.2] with modified rate $\rho^{n/4}$ for our observable $\bar s$. This verifies (6.22) and concludes the proof that $\log |J^sT|$ is cohomologous in $L^2(\mu_t)$ to the constant $\chi_t < 0$ if $P''(t) = 0$.

Finally, $P''(t) \ge 0$ implies that $P'(t)$ is increasing so that $\int \log J^uT\, d\mu_t = -P'(t)$ is decreasing, while $h_{\mu_t} = P(t) - tP'(t)$ is decreasing since $P(t)$ and $-tP'(t)$ are decreasing. □

*Proof of Corollary 1.4.* For a compact subinterval $I$ of $(0, t_*)$ the bound $\sigma(t)$ in Proposition 4.12 satisfies $\sigma_I := \sup_{t\in I}\sigma(t) < 1$. If each $\mathcal{L}_t$, for $t \in I$, has its spectrum on $\mathcal{B}$ contained in $e^{P_*(t)} \cup \{|z| < \sigma_I \cdot e^{P_*(t)}\}$, the corollary follows. Otherwise, use Proposition 6.1 and continuity [Ka, §IV.3.5] of any (finite) set of eigenvalues of finite multiplicities of bounded operators. □

*Proof of Corollary 1.5.* The corollary follows from (6.21) and (6.22), using Gordin's Theorem ([BDV, Theorem E.11] or [DRZ, Theorem 5.1]). □

## References


[Ba] V. Baladi, *Dynamical Zeta Functions and Dynamical Determinants for Hyperbolic Maps, a Functional Approach,* Springer Ergebnisse, **68** (2018).

[BD] V. Baladi and M.F. Demers, *On the measure of maximal entropy for finite horizon Sinai billiard maps,* J. Amer. Math. Soc. **33** (2020) 381–449.

[BDL] V. Baladi, M.F. Demers, and C. Liverani, *Exponential decay of correlations for finite horizon Sinai billiard flows,* Inventiones Math. **211** (2018) 39–177.

[BDT] P. Bálint, J. De Simoi, and I.P. Tóth, *A proof of Theorem 5.67 in "Chaotic Billiards" by Chernov and Markarian,* preprint, `http://www.math.utoronto.ca/jacopods/pdf/kolya-patch.pdf`

[BDV] C. Bonatti, L.J. Díaz, and M. Viana, *Dynamics Beyond Uniform Hyperbolicity. A global geometric and probabilistic perspective,* Encycl. Math. Sciences, **102,** Springer: Berlin (2005).

[Bo2] R. Bowen, *Topological entropy for non-compact sets,* Trans. Amer. Math. Soc. **49** (1973) 125–136.

[BK] M. Brin and A. Katok, *On local entropy,* Geometric Dynamics (Rio de Janeiro, 1981), Lecture Notes Math. **1007,** Springer: Berlin (1983) 30–38.

[Br] H. Bruin, *Notes on Thermodynamic Formalism*, `https://www.mat.univie.ac.at/~bruin/TF.pdf`

[CWZ] J. Chen, F. Wang, and H.-K. Zhang, *Markov partition and thermodynamic formalism for hyperbolic systems with singularities*, arXiv:1709.00527v2 (April 2019).

[CM] N.I. Chernov and R. Markarian, *Chaotic Billiards,* Math. Surveys and Monographs **127**, Amer. Math. Soc. (2006).

[D] M.F. Demers, *Uniqueness and exponential mixing for the measure of maximal entropy for piecewise hyperbolic maps,* arXiv (2020), to appear Discrete Contin. Dynam. Sys., doi: 10.3934/dcds.2020217

[DRZ] M.F. Demers, L. Rey-Bellet, and H.-K. Zhang, *Fluctuation of the entropy production for the Lorentz gas under small external forces,* Comm. Math. Phys. **363** (2018) 699–740.

[DWY] M.F. Demers, P. Wright, and L.-S. Young, *Entropy, Lyapunov exponents and escape rates in open systems,* Ergod. Th. Dynam. Sys. **32** (2012) 1270–1301.

[DZ1] M.F. Demers and H.-K. Zhang, *Spectral analysis of the transfer operator for the Lorentz gas*, J. Modern Dynam. **5** (2011) 665–709.

[DZ2] M.F. Demers and H.-K. Zhang, *A functional analytic approach to perturbations of the Lorentz gas*, Comm. Math. Phys. **324** (2013) 767-830.

[DZ3] M.F. Demers and H.-K. Zhang, *Spectral analysis of hyperbolic systems with singularities,* Nonlinearity **27** (2014) 379–433.

[GL] S. Gouëzel and C. Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties,* J. Diff. Geom. **79** (2008) 433–477.

[Ka] T. Kato, *Perturbation Theory for Linear Operators*, 2nd edition (Grundlehren der mathematischen Wissenchaften, 132). Springer: Berlin, 1984.

[KH] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems,* Cambridge University Press (1995).

[KS] A. Katok and J.M. Strelcyn, *Invariant Manifolds, Entropy and Billiards. Smooth Maps with Singularities*, Lecture Notes Math. **1222,** Springer: Berlin (1986).

[LM] Y. Lima and C. Matheus, *Symbolic dynamics for non-uniformly hyperbolic surface maps with discontinuities,* Ann. Sci. Éc. Norm. Supér. **51** (2018) 1–38.

[M] R. Mañé, *A proof of Pesin's formula,* Ergodic Th. Dynam. Sys. **1** (1981) 95–102.

[Pa] W. Parry, *Generators and strong generators in ergodic theory,* Bull. A.M.S. **72** (1966) 294–296.

[Ru] D. Ruelle, *Thermodynamic Formalism. The Mathematical Structures of Classical Equilibrium Statistical Mechanics,* Addison-Wesley Publishing Co., Reading, Mass. (1978).

[W] P. Walters, *An Introduction to Ergodic Theory,* Grad. Texts Math. **79,** Springer (1982).

[Y] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity,* Ann. of Math. **147** (1998) 585–650.

Laboratoire de Probabilités, Statistique et Modélisation (LPSM), CNRS, Sorbonne Université, Université de Paris, 4, Place Jussieu, 75005 Paris, France

*Email address*: `baladi@lpsm.paris`

Department of Mathematics, Fairfield University, Fairfield CT 06824, USA

*Email address*: `mdemers@fairfield.edu`