Projective Cones for Generalized Dispersing Billiards

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Abstract

We construct Birkhoff cones for dispersing billiards, which are contracted by the action of the transfer operator. This construction permits the study of statistical properties not only of regular dispersing billiards but also of sequential billiards (the billiard changes at each collision in a prescribed manner), open billiards (the dynamics exits some region or dies when hitting some obstacle) and many other examples. In particular, we include applications to chaotic scattering and the random Lorentz gas.

1 Introduction

Billiards are a ubiquitous source of models in physics, in particular in statistical mechanics. The study of the ergodic properties of billiards is of paramount importance for such applications and also a source of innovative ideas in ergodic theory. In particular, starting at least with [Kry], it has become clear that a quantitive estimate of the speed of convergence to equilibrium is pivotal for this research program. The first strong result of this type dates back to Bunimovich, Sinai and Chernov [BSC] in 1990 but it relies on a Markov-partition-like technology that is not very well suited to producing optimal results. The next breakthrough is due to Lai-Sang Young [You98, You99] who put forward two techniques (towers and coupling) well suited to study the decay of correlations of a large class of systems, billiards included. The idea of coupling was subsequently refined by Dolgopyat [Do04a, Do04b, Do05] who introduced the notion of *standard pairs*, which have proved a formidable tool to study the statistical properties of dynamical systems in general and billiards in particular [C1, C2, CD, CZ]. See [CM, Chapter 7] for a detailed exposition of these ideas and related references.

In the meantime another powerful idea has appeared, following the seminal work of Ruelle [RS, Ru76] and Lasota-Yorke [LY], to study the spectral properties of the associated transfer operator acting on spaces of functions adapted to the dynamics. After some preliminary attempts [Fr86, Ru96, Ki99], the functional approach for hyperbolic systems was launched by the seminal paper [BKL], which was quickly followed and refined by a series of authors, including [B1, GL, BT, GL2]. Such an approach, when applicable, has provided the strongest results so far, see [B2] for a recent review. In particular, building on a preliminary result by Demers and Liverani [DL], it has

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been applied to billiards by Demers and collaborators [DZ1, DZ2, DZ3, D2, BD1, BD2]. This has led to manifold results, notably the proof of exponential decay of correlations for certain billiard flows [BDL].

Yet, lately there has been a growing interest in non-stationary systems, when the dynamical system changes with time. Since most systems of interest are not isolated, not even in first approximation, the possibility of a change to the system due to external factors clearly has physical relevance. Another important scenario in which non-stationarity appears is in dynamical systems in random media, e.g. [AL]. The functional approach as such seems not to be well suited to treat these situations since it is based on the study of an operator via spectral theory. In the non-stationary case a single operator is substituted by a product of different operators and spectral theory does not apply.

There exist several approaches that can be used to overcome this problem, notably:

1. consider random systems; in this case, especially in the annealed case, it is possible to recover an averaged transfer operator to which the theory applies. More recently, the idea has emerged to study quenched systems via infinite dimensional Oseledets theory, see e.g. [DFGV1, DFGV2] and references therein;

2. consider only very slowing changing systems that can be treated using the perturbation theory in [KL99, GL]. For example, see [DS], and references therein, for some recent work in this direction;

3. use the technology of standard pairs, which has the advantage of being very flexible and applicable to the non-stationary case [SYZ]. Note that the standard pair technology and the previous perturbation ideas can be profitably combined together, see [DeL1, DeL2, DLPV];

4. use the cone and Hilbert metric technology introduced in [L95a, L95b, LM].

The first two approaches, although effective, impose severe limitations on the class of nonstationary systems that can be studied. The second two approaches are more general and seem more or less equivalent. However, coupling arguments are often cumbersome to write in detail and usually provide weaker quantitative estimates compared to the cone method.

Therefore, in the present article we develop the the cone method and demonstrate that it can be successfully applied to billiards. Indeed, we introduce a relatively simple cone that is contracted by a large class of billiards. This implies that one can easily prove a loss of memory result for sequences of billiard maps. To show that the previous results have concrete applications we devote more than one third of this paper to developing applications to several physically relevant classes of models.

We emphasize that the present paper does not exhaust the possible applications of the present ideas. To have a more complete theory one should consider, to mention just a few, billiards with corner points, billiards with electric or magnetic fields, billiards with more general reflection laws, measures different from the SRB measure (that is transfer operators with generalized potentials as in [BD1, BD2]), etc. We believe that most of these cases can be treated by small modifications of the present theory; however, the precise implementation does require a non-negligible amount of work and hence exceeds the scope of the this presentation which aims only at introducing the basic ideas and producing a viable cone for dispersing billiards.

The plan of the paper is as follows. In Section 2 we introduce the type of billiards we will study and summarize our main results. In Section 3 we present some basic estimates (growth lemma) needed in the following and introduce one of our main characters, the transfer operator. In Section 4 we introduce our protagonist, the cone (see Section 4.3). Section 5 is devoted to showing that the cone so defined is invariant under the action of the transfer operators of the billiards in question. In Section 6 we show that in fact the cone is eventually strictly invariant (the image has finite diameter in the associated Hilbert metric) thanks to some mixing properties of the dynamics on a finite scale. The strict cone contration implies exponential mixing for a very large class of observables and densities as is explained in Section 7. Finally, Section 8 contains the announced applications, first to sequential systems with holes (open systems), then to chaotic scattering and finally to the Random Lorentz gas. Note that the last two applications are not fully satisfactory because it is necessary to introduce artificial boundaries for the theory to apply. This is due to the fact that the billiard dynamics takes some amount of time to strictly contract the cone and hence we enforce that the billiard does not change until this happens. Note however that this is not a conceptual limitation: it only means that to remove the artificial boundaries it is necessary to show that the needed finite size mixing properties hold also for a sequence of billiards and not only for a fixed one. This has nothing to do with the cone approach; it is a matter of billiard geometry and should be addressed independently.

2 Setting and Summary of Main Results

Let $\{B_i\}_{i=1}^K$ denote a finite number of pairwise disjoint convex sets in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We assume ∂B_i is a C^3 curve with strictly positive curvature. The billiard flow is defined by the motion of a point particle traveling at unit speed in $Q := \mathbb{T}^2 \setminus (\cup_i B_i)$ and reflecting elastically at collisions.

The associated billiard map T is the discrete-time collision map which maps a point on ∂Q to its next collision. Parameterizing ∂Q according to an arclength parameter r (oriented clockwise on each obstacle B_i) and denoting by φ the angle made by the post-collision velocity vector and the outward pointing normal to the boundary yields the cannonical coordinates for the phase space M of the billiard map. In these coordinates, $M = \cup_i(\partial B_i \times [-\pi/2, \pi/2])$.

For $x = (r, \varphi) \in M$, let $\tau(x)$ denote the time until the next collision for x under the flow. We assume that τ is bounded on M, i.e. the billiard has finite horizon. Thus since the scatterers are disjoint, there exist constants $\tau_{\min}, \tau_{\max} > 0$ such that $\tau_{\min} \le \tau(x) \le \tau_{\max} < \infty$ for all $x \in M$.

It is a standard fact that T preserves a smooth invariant probability measure, $\mu_{\text{SRB}} = c \cos\varphi \, dr \, d\varphi$, where c is the normalizing constant [CM].

As announced in the introduction, the main analytical tool developed in this paper is the construction of a convex cone of functions $\mathcal{C}_{c,A,L}(\delta)$, depending on parameters $\delta > 0$, $c, A, L > 1$, as defined in Section 4.3, that is contracted under the action of the transfer operator $\mathcal{L}f = f \circ T^{-1}$, defined in Section 3.3. This is summarized in the following theorem.

Theorem 2.1. *Suppose c, A and L satisfy the conditions of Section 5.3, and that $\delta > 0$ satisfies (6.6) and (6.17). Then there exists $\chi < 1$ and $N_T \in \mathbb{N}$ such that if $n \ge N_T$, then $\mathcal{L}^n \mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$.*

In addition, for any $\chi \in \left(\max\{\frac{1}{2}, \frac{1}{L}, \frac{1}{\sqrt{A-1}}\}, 1\right)$, the cone $\mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$ has diameter at most $\log\left(\frac{(1+\chi)^2}{1-\chi)^2}\chi L\right) < \infty$ in $\mathcal{C}_{c,A,L}(\delta)$, provided $\delta > 0$ is chosen sufficiently small to satisfy (6.19).

The first statement of this theorem is proved in two steps: first, Proposition 5.1 shows that the parameters c and A contract due to the hyperbolicity of the map, subject to the constraints listed in Section 5.3; second, Theorem 6.10 proves the contraction of L using the mixing property of T. The second statement of Theorem 2.1 is proved by Proposition 6.11.

From this theorem follow the usual results on decay of correlations and convergence to equilibrium, starting from initial distributions of the form $f d\mu_{\text{SRB}}$ (Theorem 7.4) as well as distributions supported on individual stable curves (Theorem 7.3). These results and the necessary preliminaries are proved in Section 7.

Next, with this tool in hand, we are interested in studying the statistical properties of sequential billiards. This means that the obstacle configuration can change from one collision to the next, hence we have a sequence of phase spaces M_i and billiard maps $T_i : M_i \to M_{i+1}$. In addition, we want to include the case of *open billiards*, that is we allow the presence of holes in the system such that if the particle reaches one such hole, it exits the system and hence the dynamics is terminated. If H_i are the holes in the phase space M_i, and $\mathbb{1}_{H_i}$ denotes their indicator function, then our object of interest is

$$\int_{M_0} \psi \circ T_{n-1} \circ \cdots \circ T_0(x) \prod_{i=0}^{n-1} \mathbb{1}_{H_i}(T_{i-1} \circ \cdots \circ T_0(x)) f(x) \mu_{\text{SRB}}(dx) \tag{2.1}$$

which, depending on the way one interprets it, expresses the correlation between a measurement at time zero and a measurement at time n, or expresses the expectation of the observable ψ at time n, when the system, at time zero, was distributed according to the measure $f d\mu_{\text{SRB}}$.

A basic tool to study (2.1) are the Ruelle transfer operators $\mathcal{L}_i$ defined by $\mathcal{L}_i f = f \circ T_i^{-1}$. If we consider $\mathcal{L}_i$ as an operator from $L^2(M_{i+1}, \mu_{\text{SRB}}) \to L^2(M_i, \mu_{\text{SRB}})$, then this is nothing other than the adjoint of the Koopman operator T_i^* defined by $T_i^* = \psi \circ T_i$. If we define $\mathcal{L}_{i,H_i} f = \mathcal{L}_i(\mathbb{1}_{H_i} f)$, then we can rewrite (2.1) as

$$\int_{M_n} \psi(x) \mathcal{L}_{n-1,H_{n-1}} \cdots \mathcal{L}_{0,H_0} f(x) d\mu_{\text{SRB}}.$$

If the billiard tables and the holes do not change with time, then we have

$$\int_{M} \psi(x) \mathcal{L}_H^n f(x) d\mu_{\text{SRB}}.$$

and the expression is determined by the spectral properties of $\mathcal{L}_H$. Unfortunately, the spectral properties of such an operator, when acting on L^2 are very poor. The key step in this line of thought has been achieved in [DZ1, D1, D2] where the authors have constructed Banach spaces of distributions where the operators $\mathcal{L}_H$ are quasi compact, hence the needed information can be read from their spectra.

Unfortunately, in the sequential case, such as (2.1), spectral theory cannot be applied since the operators keep changing in time. As mentioned in the introduction, we will overcome this problem by proving that $\mathcal{L}_{i,H_i} \mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$ for some choice of c, A, L and δ with finite diameter measured in the Hilbert metric (see [L95a] for details). The strict cone contraction implies that the images of a bounded set of densities f, under the action of the transfer operators in (2.1) are contained in smaller and smaller sets, hence the loss of memory with respect to the original density.

This will allow us to prove theorems of the following type (see Theorem 8.10 for a precise formulation).

Theorem 2.2. *Under appropriate technical conditions, there exists $\vartheta < 1$ such that for each $\psi, f, g, h \in \mathcal{C}^1$, there exits $C > 0$ such that, for all $n \geq 0$,*

$$\left| \int_{M_n} \frac{\psi(x)}{Z_n(f)} \mathcal{L}_{n-1,H_{n-1}} \cdots \mathcal{L}_{0,H_0} f(x) d\mu_{SRB} - \int_{M_n} \frac{\psi(x)}{Z_n(g)} \mathcal{L}_{n-1,H_{n-1}} \cdots \mathcal{L}_{0,H_0} g(x) d\mu_{SRB} \right| \leq C \vartheta^n ,$$

where $Z_n(h) = \int_{M_n} \mathcal{L}_{n-1,H_{n-1}} \cdots \mathcal{L}_{0,H_0} h(x) d\mu_{SRB}$.

In turn this will allow us to apply our theory to relevant physical problems such as *chaotic scattering*, see Section 8.4, and a variant of the *random Lorentz gas*, see Section 8.5.

3 Hyperbolicity, Singularities and Transfer Operators

We start by recalling some fundamental properties of billiards that will be needed in the sequel.

3.1 Hyperbolicity and singularities

The map T is uniformly hyperbolic in the following sense. T has a family of invariant stable cones C^s, defined by

$$C^s(x) = \{(dr, d\varphi) \in \mathbb{R}^2 : -\mathcal{K}_{\max} - \tau_{\min}^{-1} \le d\varphi/dr \le -\mathcal{K}_{\min}\}, \quad \text{for } x \in M,$$

where $\mathcal{K}_{\min}$ and $\mathcal{K}_{\max}$ denote the minimum and maximum curvature of the boundaries of the scatterers, respectively. This family of cones is strictly invariant, $DT^{-1}C^s(x) \subset C^s(T^{-1}x)$, and T^{-1} enjoys uniform expansion of vectors in the stable cone: There exist $C_1 \in (0, 1]$ and $\Lambda > 1$ such that,

$$\|DT^{-n}(x)v\| \ge C_1 \Lambda^n \|v\|, \qquad \text{for all } v \in C^s(x). \tag{3.1}$$

T has a family of unstable cones C^u defined similarly, but with $\mathcal{K}_{\min} \le d\varphi/dr \le \mathcal{K}_{\max} + \tau_{\min}^{-1}$.

Near tangential collisions, $\|DT(x)v\| \approx \frac{\|v\|}{\cos\varphi(Tx)}$, for $v \in C^u(x)$. Due to this unbounded expansion, we define the standard homogeneity strips, following [BSC]. For some $k_0 \in \mathbb{N}$, to be chosen later in (3.5), we define

$$\mathbb{H}_{\pm k} = \{(r, \varphi) \in M : (k+1)^{-2} \le | \pm \tfrac{\pi}{2} - \varphi| \le k^{-2}\}, \text{ for all } k \ge k_0. \tag{3.2}$$

Set $\mathcal{S}_0 = \{(r, \varphi) \in M : \varphi = \pm\frac{\pi}{2}\}$. The singularity set for T^n is denoted by $\mathcal{S}_n = \cup_{i=0}^n T^{-i}\mathcal{S}_0$, for $n \in \mathbb{Z}$. On $M \setminus \mathcal{S}_n$, T^n is a C^2 diffeomorphism onto its image.

In order to achieve bounded distortion, we will consider the boundaries of the homogeneity strips as an extended singularity set for T. To this end, define $\mathcal{S}_0^{\mathbb{H}} = \mathcal{S}_0 \cup (\cup_{k \ge k_0}(\partial\mathbb{H}_k \cup \partial\mathbb{H}_{-k}))$, and $\mathcal{S}_n^{\mathbb{H}} = \cup_{i=0}^n T^{-i}\mathcal{S}_0^{\mathbb{H}}$, for $n \in \mathbb{Z}$.

We call a curve $W \subset M$ a stable curve if for each $x \in W$, the tangent vector to W at x belongs to C^s. A stable curve is called homogeneous if it lies in one homogeneity strip or outside their union. Denote by $\mathcal{W}^s$ the set of homogeneous stable curves with length at most δ_0 (defined by (3.5)) and with curvature at most $\bar{B}$. We may choose $\bar{B}$ sufficiently large that $T^{-1}\mathcal{W}^s \subset \mathcal{W}^s$, up to subdividing the curves of length larger than δ_0.

Similarly, we define an analogous set of homogeneous unstable curves by $\mathcal{W}^u$.

We have the following distortion bound for homogeneous stable curves. Suppose $W \in \mathcal{W}^s$ is such that $T^iW \in \mathcal{W}^s$ for $i = 0, \dots n$. There exists $C_d > 0$, independent of W and n, such that for all $x, y \in W$,

$$|\log J_W T^n(x) - \log J_W T^n(y)| \le C_d d(x, y)^{1/3}, \tag{3.3}$$

where $J_W T^n$ is the (stable) Jacobian of T^n along W and $d(\cdot, \cdot)$ denotes arclength on W with respect to the metric $dr^2 + d\varphi^2$.

Similar bounds hold for stable Jacobians lying on the same unstable curve. Suppose $V_1, V_2 \in \mathcal{W}^s$ are such that $T^iV_1, T^iV_2 \in \mathcal{W}^s$ for $0 \le i \le n$, in particular they are not cut by any singularity, and there exists a foliation of unstable curves $\{\ell_x\}_{x \in V_1} \subset \mathcal{W}^u$ creating a one-to-one correspondence between V_1 and V_2 and such that $\{T^n\ell_x\}_{x \in V_1} \subset \mathcal{W}^u$ creates a one-to-one correspondence between T^nV_1 and T^nV_2. For $x \in V_1$, define $\bar{x} = \ell_x \cap V_2$. Then there exists $C_d > 0$, independent of V_1, V_2, n and x, such that,

$$|\log J_{V_1} T^n(x) - \log J_{V_2} T^n(\bar{x})| \le C_d(d(x, \bar{x})^{1/3} + \phi(x, \bar{x})), \tag{3.4}$$

where $\phi(x, \bar{x})$ denotes the angle between the tangent vectors to V_1 and V_2 at x and $\bar{x}$, respectively. For simplicity, we use the same symbol C_d to represent the distortion constants in (3.3) and (3.4). The proofs for these distortion bounds in this form can be found in [DZ1, Appendix A] (see also [CM, Section 5.8]).

3.2 Growth lemma

The control on complexity for the billiard is given by the following one-step expansion condition due to Chernov. Recalling (3.2), for k_0 sufficiently large, there exist $\delta_0 > 0$ and $\theta_0 < 1$ such that

$$\sup_{\substack{W \in \mathcal{W}^s \\ |W| \le \delta_0}} \sum_{V_i} |J_{V_i} T|_* = \theta_0, \tag{3.5}$$

where V_i are the homogeneous components of $T^{-1}W$ and $|J_{V_i} T|_*$ is the supremum of the Jacobian of T along V_i in an adapted metric [CM, Lemma 5.56].

In Section 6.1, we will find it convenient to increase the contraction by replacing T with a higher iterate T^n and choosing δ_0 sufficiently small so that (3.5) holds for $T_* = T^n$ with constant θ_0^n. This is possible since if W is a stable curve, then $|T^{-1}W| \le C|W|^{1/2}$ [CM, Exercise 4.50], so we may choose δ_0 so small that no connected component of $T^{-k}(W)$ is longer than δ_0 for $k = 0, \dots, n$. Since no artificial subdivisions are necessary, we apply (3.5) inductively in k to obtain the desired contraction.

Choose $\bar{n}$ and fix $\delta_0 \in (0, 1)$ such that $\theta_1 := \theta_0^{\bar{n}}$ satisfies

$$3C_0 \frac{\theta_1}{1 - \theta_1} \le \frac{1}{4} \qquad \text{and} \qquad \sup_{\substack{W \in \mathcal{W}^s \\ |W| \le \delta_0}} \sum_{V_i} |J_{V_i} T^{\bar{n}}|_* \le \theta_1, \tag{3.6}$$

where V_i are the homogeneous components of $T^{-\bar{n}}W$. Note that if we shrink δ_0 further, then (3.6) will continue to hold for the same value of $\bar{n}$.

We shall work with the map $T_* = T^{\bar{n}}$ throughout the following. To simplify notation we will call T_* again T as no confusion can arise.

The following growth lemma is contained in [DZ1, Lemmas 3.1, 3.2], but we include the proof of item (b) here for convenience and to draw out the explicit dependence on the constants.

For $W \in \mathcal{W}^s$, we denote by $\mathcal{G}_n(W)$ the homogeneous components of $T^{-n}W$, where we have subdivided the elements of $T^{-n}W$ longer that δ_0 into elements with length between δ_0 and $\delta_0/2$ so that $\mathcal{G}_n(W) \subset \mathcal{W}^s$. We call $\mathcal{G}_n(W)$ the nth generation of W. Let $\mathcal{I}_n(W)$ denote the set of curves $W_i \in \mathcal{G}_n(W)$ such that $T^j(W_i)$ is not contained in an element of $\mathcal{G}_{n-j}(W)$ having length at least $\delta_0/3$ for any $j = 0, \dots n$

Lemma 3.1. *There exists $\bar{C}_0 > 0$ such that for all $W \in \mathcal{W}^s$ and $n \ge 0$,*

a) $\displaystyle\sum_{W_i \in \mathcal{I}_n(W)} |J_{W_i} T^n|_{C^0(W_i)} \le C_0 \theta_1^n;$

b) $\displaystyle\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i} T^n|_{C^0(W_i)} \le \bar{C}_0 \delta_0^{-1} |W| + C_0 \theta_1^n.$

Proof. Item (a) follows by induction on n from (3.6) and the constant C_0 comes from translating from the adapted metric to the Euclidean metric at the last step (the two metrics are uniformly equivalent; see [DZ1, Lemma 3.1]). We focus on proving item (b).

For $W \in \mathcal{W}^s$, let $L_k(W) \subset \mathcal{G}_k(W)$ denote those elements of $\mathcal{G}_k(W)$ having length at least $\delta_0/3$. For $k \le n$ and $W_i \in \mathcal{G}_n(W)$, we say that $V_j \in L_k(W)$ is the most recent long ancestor of W_i if $k \le n$ is the largest time that $T^{n-k}W_i$ is contained in an element of $L_k(W)$. Then by definition, $W_i \in \mathcal{I}_{n-k}(V_j)$. Note that if $W_i \in L_n(W)$, then $k = n$ and $W_i = V_j$. Now we estimate,

$$\begin{aligned}\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i}T^n|_{C^0(W_i)} &\le \sum_{k=1}^{n} \sum_{V_j \in L_k(W)} \sum_{W_i \in \mathcal{I}_{n-k}(V_j)} |J_{W_i}T^{n-k}|_{C^0(W_i)} |J_{V_j}T^k|_{C^0(V_j)} \\ &\quad + \sum_{W_i \in \mathcal{I}_n(W)} |J_{W_i}T^n|_{C^0(W_i)} \\ &\le \sum_{k=1}^{n} \sum_{V_j \in L_k(W)} C_0 \theta_1^{n-k} e^{C_d \delta_0^{1/3}} \frac{|T^k V_j|}{|V_j|} + C_0 \theta_1^n ,\end{aligned}$$

where we have used item (a) of the lemma to sum over $W_i \in \mathcal{I}_{n-k}(W)$ and (3.3) to replace $|J_{V_j}T^k|_{C^0(V_j)}$ with $\frac{|T^k V_j|}{|V_j|}$. Now since $\cup_{V_j \in L_k(W)} T^k V_j \subset W$, and $|V_j| \ge \delta_0/3$, we have

$$\sum_{W_i \in \mathcal{G}_n(W)} |J_{W_i}T^n|_{C^0(W_i)} \le \sum_{k=1}^{n} C_0 \theta_1^{n-k} 3 \delta_0^{-1} |W| e^{C_d \delta_0^{1/3}} + C_0 \theta_1^n ,$$

which proves the lemma with $\bar{C}_0 := \frac{3C_0}{1-\theta_1} e^{C_d \delta_0^{1/3}}$. □

Remark 3.2. *It is not necessary to work with $T = T^{\bar{n}}$ in Lemma 3.1. It follows equally well from (3.5) with θ_1 replaced by θ_0. Moreover, if $|W| \ge \delta_0/3$, then all pieces $W_i \in \mathcal{G}_n(W)$ have a long ancestor and can be included in the sum over k; in this case, the second term on the right side of item (b) is not needed, and the value of $\bar{C}_0$ remains unchanged.*

3.3 Transfer operator

We define the transfer operator $\mathcal{L}$ associated with T acting on scales of spaces of distributions as in [DZ1]. We denote by $T^{-n}\mathcal{W}^s$ the set of curves $W \in \mathcal{W}^s$ such that $T^i W \in \mathcal{W}^s$ for all $i = 0, \dots n$. For $\alpha \le 1/3$, let $C^\alpha(T^{-n}\mathcal{W}^s)$ denote the set of complex valued functions on M that are Hölder continuous on elements of $T^{-n}\mathcal{W}^s$. Then for $\psi \in C^\alpha(\mathcal{W}^s)$, we have $\psi \circ T^n \in C^\alpha(T^{-n}\mathcal{W}^s)$ (see Lemma 5.2(a)). Define

$$\mathcal{L}^n \mu(\psi) = \mu(\psi \circ T^n), \text{ for } \mu \in (C^\alpha(T^{-n}\mathcal{W}^s))^* .$$

This defines $\mathcal{L} : (C^\alpha(T^{-n}\mathcal{W}^s))^* \to (C^\alpha(T^{-n+1}\mathcal{W}^s))^*$ for any $n \ge 1$. See [DZ1] for details.

Recall that T preserves the smooth invariant measure $\mu_{\text{SRB}} = c \cos\varphi dr d\varphi$, where c is the normalizing constant. When $d\mu = f d\mu_{\text{SRB}}$ is a measure absolutely continuous with respect to μ_{SRB}, we identify μ with its density f. With this identification, the transfer operator acting on densities has the following familiar expression,

$$\mathcal{L}f = f \circ T^{-1}.$$

We choose this identification of functions in order to simplify our later work: using the reference measure μ_{SRB}, the Jacobian of the transformation is 1, making $\mathcal{L}$ simpler to work with.

4 Cones and Distributions

Given a closed,[1] convex cone $\mathcal{C}$ satisfying $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we define an order relation by $f \preceq g$ if and only if $g - f \in \mathcal{C} \cup \{0\}$. We can then define a projective metric by

$$\begin{aligned} \bar{\alpha}(f,g) &= \sup\{\lambda \in \mathbb{R}^+ : \lambda f \preceq g\} \\ \bar{\beta}(f,g) &= \inf\{\mu \in \mathbb{R}^+ : g \preceq \mu f\} \\ \rho(f,g) &= \log\left(\frac{\bar{\beta}(f,g)}{\bar{\alpha}(f,g)}\right). \end{aligned} \tag{4.1}$$

4.1 A cone of test functions

For $W \in \mathcal{W}^s$, $\alpha \in (0,1]$ and $a \in \mathbb{R}^+$, define a cone of test functions by

$$\mathcal{D}_{a,\alpha}(W) = \left\{\psi \in C^0(W) : \psi > 0, \frac{\psi(x)}{\psi(y)} \le e^{ad(x,y)^\alpha}\right\},$$

where $d(\cdot,\cdot)$ is the arclength distance along W.

The Hilbert metric associated with this cone and defined by (4.1) depends on the constant a and the exponent α determining the regularity of the functions. For each such choice, the Hilbert metric has the following convenient representation.

Lemma 4.1 ([L95a, Lemma 2.2])**.** *Choose $\alpha \in (0,1]$. For $\psi_1, \psi_2 \in \mathcal{D}_{a,\alpha}(W)$, the corresponding metric $\rho_{W,a,\alpha}(\cdot,\cdot)$ is given by*

$$\rho_{W,a,\alpha}(\psi_1,\psi_2) = \log\left[\sup_{x,y,u,v\in W} \frac{e^{ad(x,y)^\alpha}\psi_1(x) - \psi_1(y)}{e^{ad(x,y)^\alpha}\psi_2(x) - \psi_2(y)} \cdot \frac{e^{ad(u,v)^\alpha}\psi_2(u) - \psi_2(v)}{e^{ad(u,v)^\alpha}\psi_1(u) - \psi_1(v)}\right].$$

A corollary of this lemma is that $\mathcal{D}_{a,\alpha}(W)$ has finite diameter in $\mathcal{D}_{a,\beta}(W)$ if $\beta < \alpha$ and $|W| < 1$.

The next two lemmas are simple consequences of the regularity of functions in $\mathcal{D}_{a,\alpha}(W)$ for $W \in \mathcal{W}^s$. We denote by m_W the measure induced by arclength along W.

Lemma 4.2. *For any $\alpha \in (0,1]$ and $W \in \mathcal{W}^s$ with $|W| \in [\delta, 2\delta]$, any $\psi \in \mathcal{D}_{a,\alpha}(W)$ and $x \in W$, we have*

$$\frac{\delta\psi(x)}{\int_W \psi\, dm_W} \le \frac{|W|\psi(x)}{\int_W \psi dm_W} \le e^{a|W|^\alpha}.$$

Proof. The estimate is immediate since $\inf_{y\in W}\psi(y) \ge \psi(x)e^{-a|W|^\alpha}$. □

Lemma 4.3. *Given $\alpha \in (0,1]$, $W \in \mathcal{W}^s$, $\psi_1, \psi_2 \in \mathcal{D}_{a,\alpha(W)}$ and $x, y \in W$,*

$$e^{-\rho_{W,a,\alpha}(\psi_1,\psi_2)} \le \frac{\psi_1(x)\psi_2(y)}{\psi_2(x)\psi_1(y)} \le e^{\rho_{W,a,\alpha}(\psi_1,\psi_2)}$$

Proof. According to (4.1), we must have,

$$\psi_2(x) - \bar{\alpha}\psi_1(x) \ge 0 \quad \forall x \in W \qquad \text{and} \qquad \psi_2(y) - \bar{\beta}\psi_1(y) \le 0 \quad \forall y \in W.$$

This in turn implies that

$$\rho_{W,a,\alpha}(\psi_1,\psi_2) = \log\frac{\bar{\beta}(\psi_1,\psi_2)}{\bar{\alpha}(\psi_1,\psi_2)} \ge \log\left[\frac{\psi_1(x)\psi_2(y)}{\psi_2(x)\psi_1(y)}\right] \qquad \forall x,y \in W.$$

□

[1] Closed here means that for all $f, g \in \mathcal{C}$ and sequence $\{\alpha_n\} \subset \mathbb{R}$ such that $\lim_{n\to\infty}\alpha_n = \alpha$ and $g + \alpha_n f \in \mathcal{C}$ for all $n \in \mathbb{N}$ we have $g + \alpha f \in \mathcal{C} \cup \{0\}$.

4.2 Distances between curves and functions

Due to the global stable cones for the map T, we may consider stable curves $W \in \mathcal{W}^s$ as graphs of C^2 functions over an interval I_W in the r-coordinate:

$$W = \{G_W(r) = (r, \varphi_W(r)) : r \in I_W\}.$$

Using this representation, we define a notion of distance between $W^1, W^2 \in \mathcal{W}^s$ by

$$d_{\mathcal{W}^s}(W^1, W^2) = |\varphi_{W^1} - \varphi_{W^2}|_{C^1(I_{W^1} \cap I_{W^2})} + |I_{W^1} \triangle I_{W^2}|, \tag{4.2}$$

if W^1 and W^2 lie in the same homogeneity strip and $|I_{W^1} \cap I_{W^2}| > 0$; otherwise, we set $d_{\mathcal{W}^s}(W^1, W^2) = \infty$. Note that $d_{\mathcal{W}^s}$ is not a metric, but this is irrelevant for our purposes.

We will also find it necessary to compare between test functions on two different stable curves. Given $W^1, W^2 \in \mathcal{W}^s$ with $d_{\mathcal{W}^s}(W^1, W^2) < \infty$, and $\psi_i \in \mathcal{D}_{a,\beta}(W_i)$, define

$$d_*(\psi_1, \psi_2) = |\psi_1 \circ G_{W^1} \|G'_{W^1}\| - \psi_2 \circ G_{W^2} \|G'_{W^2}\| \,|_{C^\beta(I_{W^1} \cap I_{W^2})}, \tag{4.3}$$

to be the (Hölder) distance between ψ_1 and ψ_2, where $\|G'_W\| = \sqrt{1 + (d\varphi_W/dr)^2}$.

Also, by the bound $\bar{B}$ on the curvature of elements of $\mathcal{W}^s$, there exists $B_* > 0$ such that

$$B_* = \sup_{W \in \mathcal{W}^s} |\varphi''_W|_{C^0(W)} < \infty \,. \tag{4.4}$$

Remark 4.4. *Note that if $I_{W^1} = I_{W^2}$ and $d_*(\psi_1, \psi_2) = 0$, then*

$$\int_{W^1} \psi_1 \, dm_{W_1} = \int_{W^2} \psi_2 \, dm_{W_2}.$$

4.3 Definition of the cone

In order to define a cone of functions adapted to our dynamics, we will fix the following exponents, $\alpha, \beta, \gamma, q > 0$ and constant $a > 1$ large enough. Choose $q \in (0, 1/2)$, $\beta < \alpha \le 1/3$ and finally $\gamma \le \min\{\alpha - \beta, q\}$.

For a length scale $\delta \le \delta_0/3$, define $\mathcal{W}^s_-(\delta)$ to be those curves in $\mathcal{W}^s$ with length $|W| \le 2\delta$ and $\mathcal{W}^s(\delta)$ to be those curves in $\mathcal{W}^s$ with length $|W| \in [\delta, 2\delta]$.

Let $\mathcal{A}$ denote the set of functions on M whose restriction to each $W \in \mathcal{W}^s$ is integrable with respect to the arclength measure dm_W. For $f \in \mathcal{A}$ define,

$$|||f|||_+ = \sup_{\substack{W \in \mathcal{W}^s(\delta) \\ \psi \in \mathcal{D}_{a,\beta}(W)}} \frac{\int_W f\psi \, dm_W}{\int_W \psi \, dm_W}, \qquad |||f|||_- = \inf_{\substack{W \in \mathcal{W}^s(\delta) \\ \psi \in \mathcal{D}_{a,\beta}(W)}} \frac{\int_W f\psi \, dm_W}{\int_W \psi \, dm_W}, \tag{4.5}$$

Note that if $f \in \mathcal{A}$, it must be that $|||f|||_- < \infty$.

Denote the average value of ψ on W by $\fint_W \psi \, dm_W = \frac{1}{|W|} \int_W \psi \, dm_W$. Since all of our integrals in this section and the next will be taken with respect to the arclength dm_W, to keep our notation concise, we will drop the measure from our integral notation in what follows.

Now for $a, c, A, L > 1$, and $\delta \in (0, \delta_0/3]$, define the cone

$$\mathcal{C}_{c,A,L}(\delta) = \Big\{ f \in \mathcal{A} : \qquad |||f|||_+ \le L|||f|||_- ; \tag{4.6}$$

$$\sup_{W \in \mathcal{W}^s_-(\delta)} \sup_{\psi \in \mathcal{D}_{a,\beta}(W)} |W|^{-q} \frac{|\int_W f\psi|}{\fint_W \psi} \le A\delta^{1-q}|||f|||_- ; \tag{4.7}$$

$$\forall\, W^1, W^2 \in \mathcal{W}^s_-(\delta) : d_{\mathcal{W}^s}(W^1, W^2) \le \delta, \forall \psi_i \in \mathcal{D}_{a,\alpha}(W_i) : d_*(\psi_1, \psi_2) = 0,$$

$$\left| \frac{\int_{W^1} f\psi_1}{\fint_{W^1} \psi_1} - \frac{\int_{W^2} f\psi_2}{\fint_{W^2} \psi_2} \right| \le d_{\mathcal{W}^s}(W^1, W^2)^\gamma\, \delta^{1-\gamma} cA|||f|||_- \Big\}. \tag{4.8}$$

We write the constants c, A, L explicitly as subscripts in our notation for the cone since these will be the parameters which are contracted by the dynamics.

By contrast, the exponents α, β, γ, q are fixed and will not be altered by the dynamics, while the constant a, which will be chosen in Lemma 5.2, will not appear directly in the contraction constant of the cone.

For convenience, we will require that δ_0 is sufficiently small that

$$e^{2a\delta_0^\beta} \le 2\,. \tag{4.9}$$

This will imply similar bounds in terms of δ since $\delta \le \delta_0/3$.

Remark 4.5. *As will become clear from our estimates in Sections 5 and 6, in order to prove that the parameters contract, we will need to choose A large compared to L, and c large compared to A. This yields the compatible set of restrictions, $1 < L < A < c$.*

By contrast, the exponents are fixed by the regularity properties of the map: $\alpha \le 1/3$ due to (3.3), and $\beta < \alpha$ so that $\mathcal{D}_{a,\beta}(W)$ has finite diameter in $\mathcal{D}_{a,\alpha}(W)$, while $\gamma \le \alpha - \beta$ is convenient to obtain the required contraction in Lemma 5.5. See Section 5.3 for all the conditions the constants must satisfy for Proposition 5.1. Several further conditions are specified in Theorem 6.10 to prove the strict contraction of the cone.

Remark 4.6. *Note that condition (4.6) implies $(L-1)|||f|||_- \ge |||f|||_+ - |||f|||_- \ge 0$, hence for all $W \in \mathcal{W}^s(\delta), \psi \in \mathcal{D}_{a,\beta}(W)$,*

$$\int_W f\psi\, dm_W \ge |||f|||_- \int_W \psi\, dm_W \ge 0. \tag{4.10}$$

In addition, condition (4.7) implies

$$A|||f|||_- \ge \sup_{W \in \mathcal{W}^s_-(\delta)} \sup_{\psi \in \mathcal{D}_{a,\beta}(W)} \delta^{q-1}|W|^{1-q} \frac{|\int_W f\psi|}{\int_W \psi} \ge |||f|||_+.$$

However condition (4.6) is not vacuous since we assume $A > L$.

We will need the following lemma in Section 6.2.

Lemma 4.7. *For all $f \in \mathcal{C}_{c,A,L}(\delta)$, $W \in \mathcal{W}^s(\delta)$ and all $\psi_1, \psi_2 \in \mathcal{D}_{a,\beta}(W)$,*

$$\left| \frac{\int_W f\psi_1}{\fint_W \psi_1} - \frac{\int_W f\psi_2}{\fint_W \psi_2} \right| \le 2\delta L \rho_{W,a,\beta}(\psi_1, \psi_2)|||f|||_-\,.$$

Proof. Let $f \in \mathcal{C}_{c,A,L}(\delta)$, $W \in \mathcal{W}^s(\delta)$ and $\psi_1, \psi_2 \in \mathcal{D}_{a,\beta}(W)$. For each $\lambda, \mu > 0$ such that $\lambda\psi_1 \preceq \psi_2 \preceq \mu\psi_1$, hence also $\lambda\psi_1 \le \psi_2 \le \mu\psi_1$, we have

$$\frac{\int_W f\psi_2}{\fint_W \psi_2} = \frac{\lambda \int_W f\psi_1 + \int_W f(\psi_2 - \lambda\psi_1)}{\fint_W \psi_2} \ge \frac{\lambda \int_W f\psi_1}{\mu \fint_W \psi_1},$$

where we have dropped the second term above due to (4.10) since $\psi_2 - \lambda\psi_1 \in \mathcal{D}_{a,\beta}(W)$. Taking the sup on λ and the inf on μ, and recalling (4.1), yields

$$\frac{\int_W f\psi_1}{\fint_W \psi_1} - \frac{\int_W f\psi_2}{\fint_W \psi_2} \le \frac{\int_W f\psi_1}{\fint_W \psi_1}(1 - e^{-\rho_{W,a,\beta}(\psi_1,\psi_2)}) \le \rho_{W,a,\beta}(\psi_1,\psi_2)\frac{\int_W f\psi_1}{\fint_W \psi_1}.$$

Then, since $|W| \ge \delta$, we use (4.6) to estimate,

$$\frac{\int_W f\psi_1}{\fint_W \psi_1} \le |W| |\!|\!| f |\!|\!|_+ \le 2\delta L |\!|\!| f |\!|\!|_-.$$

Reversing the roles of ψ_1 and ψ_2 completes the proof of the lemma. □

5 Cone Estimates

In this section, we will prove the following proposition. Let $n_0 \ge 1$ be such that $AC_0\theta_1^{n_0} \le 1/16$.

Proposition 5.1. *If the conditions on δ, n_0, a, c, A, L specified in Section 5.3 are satisfied, then there exists $\chi < 1$ such that for all $n \ge n_0$,*

$$\mathcal{L}^n \mathcal{C}_{c,A,L}(\delta) \subseteq \mathcal{C}_{\chi c, \chi A, 3L}(\delta).$$

Before proving Proposition 5.1 we need some facts concerning the behaviour of the test functions under the dynamics.

5.1 Contraction of test functions

For $W \in \mathcal{W}^s$, $\psi \in \mathcal{D}_{a,\beta}(W)$, and $W_i \in \mathcal{G}_n(W)$, define

$$\widehat{T}^n_{W_i}\psi = \widehat{T}^n_i\psi := \psi \circ T^n J_{W_i} T^n.$$

The following lemma is a consequence of the hyperbolicity of T.

Lemma 5.2. *Let $n \ge 0$ be such that $C_1^{-1}\Lambda^{-\beta n} < 1$, where $C_1 \le 1$ is from (3.1), and fix $a > (1 - C_1^{-1}\Lambda^{-\beta n})^{-1} C_d \delta_0^{1/3-\beta}$. For each $\beta \in (0, 1/3]$, there exist $\sigma, \bar{\xi} < 1$ such that for all $W \in \mathcal{W}^s$ and $W_i \in \mathcal{G}_n(W)$,*

a) $\widehat{T}^n_i(\mathcal{D}_{a,\beta}(W)) \subset \mathcal{D}_{\sigma a,\beta}(W_i)$;

b) $\rho_{W_i,a,\beta}(\widehat{T}^n_i\psi_1, \widehat{T}^n_i\psi_2) \le \bar{\xi}\rho_{W,a,\beta}(\psi_1,\psi_2)$ for all $\psi_1,\psi_2 \in \mathcal{D}_{a,\beta}(W)$.

Proof. (a) We need to measure the log-Hölder norm of $\widehat{T}^n_i\psi$ for $\psi \in \mathcal{D}_{a,\beta}(W)$. For $x, y \in W_i$, we estimate,

$$\frac{\widehat{T}^n_i\psi(x)}{\widehat{T}^n_i\psi(y)} = \frac{\psi(T^n x) J_{W_i}T^n(x)}{\psi(T^n y) J_{W_i}T^n(y)} \le e^{ad(T^n x, T^n y)^\beta + C_d d(x,y)^{1/3}} \le e^{(aC_1^{-\beta}\Lambda^{-\beta n} + C_d\delta_0^{1/3-\beta}) d(x,y)^\beta},$$

where we have used (3.1) and (3.3) as well as the fact that $\beta \le 1/3$. This proves the first statement of the lemma since $aC_1^{-1}\Lambda^{-\beta n} + C_d\delta_0^{1/3-\beta} < a$.
(b) Using Lemma 4.1, if $\psi_1, \psi_2 \in \mathcal{D}_{\sigma a,\beta}(W_i)$, then,

$$\begin{aligned}\rho_{W_i,a,\beta}(\psi_1,\psi_2) &= \log\left[\sup_{x,y,u,v\in W_i} \frac{e^{ad(x,y)^\beta}\psi_1(x) - \psi_1(y)}{e^{ad(x,y)^\beta}\psi_2(x) - \psi_2(y)} \cdot \frac{e^{ad(u,v)^\beta}\psi_2(u) - \psi_2(v)}{e^{ad(u,v)^\beta}\psi_1(u) - \psi_1(v)}\right] \\ &\le \log\left[\sup_{x,y,u,v\in W} \frac{e^{(a+\sigma a)d(x,y)^\beta} - 1}{e^{(a-\sigma a)d(x,y)^\beta} - 1}\frac{e^{(a+\sigma a)d(u,v)^\beta} - 1}{e^{(a-\sigma a)d(u,v)^\beta} - 1}\frac{\psi_1(y)\psi_2(v)}{\psi_2(y)\psi_1(u)}\right] \\ &\le \log\left[\frac{(a+\sigma a)^2}{(a-\sigma a)^2}e^{2a(1+\sigma)\delta_0^\beta}e^{2a\delta_0^\beta}\right] =: K.\end{aligned} \tag{5.1}$$

Thus the diameter of $\mathcal{D}_{\sigma a,\beta}(W_i)$ is finite in $\mathcal{D}_{a,\beta}(W_i)$. Part (b) of the lemma then follows from [L95a, Theorem 1.1], with $\bar{\xi} = \tanh(K/4) < 1$. □

Corollary 5.3. *Let n_1 denote the least positive integer satisfying $C_1^{-1}\Lambda^{-\beta n} < 1$ and $aC_1^{-1}\Lambda^{-\beta n_1} + C_d\delta_0^{1/3-\beta} < a$. Define $\xi = \bar{\xi}^{\frac{1}{2n_1}} < 1$. Then for $W \in \mathcal{W}^s$, $n \ge n_1$ and $W_i \in \mathcal{G}_n(W)$,*

$$\rho_{W_i,a,\beta}(\widehat{T}_i^n\psi_1 \widehat{T}_i^n\psi_2) \le \xi^n \rho_{W,a,\beta}(\psi_1,\psi_2) \qquad \text{for all } \psi_1,\psi_2 \in \mathcal{D}_{a,\beta}(W).$$

Proof. The proof follows immediately from Lemma 5.2 once we decompose $n = kn_1 + r$, where $r \in [0, n_1)$ and write

$$\widehat{T}^n_{W_i}\psi = \widehat{T}^{n_1+r}_{W_i} \circ \widehat{T}^{n_1}_{T^{n_1+r}W_i} \circ \widehat{T}^{n_1}_{T^{2n_1+r}W_i} \circ \cdots \circ \widehat{T}^{n_1}_{T^{(k-1)n_1+r}W_i}\psi.$$

Each of the operators $\widehat{T}^{n_1}_{T^{jn_1+r}W_i}$ satisfies Lemma 5.2 with the same σ and $\bar{\xi}$. The corollary then follows using the observation that $\bar{\xi}^{\lfloor n/n_1\rfloor} \le \xi^n$, $\forall n \ge n_1$. □

It is important for what follows that the contractive factor $\bar{\xi} < 1$ is explicitly given in terms of the diameter K, which depends only on a and σ, and not on δ. While n_1 depends on the parameter choice β, it also is independent of δ.

In what follows, we require $n_0 \ge n_1$ by definition, so that Lemma 5.2 and Corollary 5.3 will hold for all $n \ge n_0$.

5.2 Proof of Proposition 5.1

This section is devoted to the proof of Proposition 5.1.

5.2.1 Preliminary estimate on L

Denote by $Sh_n(W;\delta)$ the elements of $\mathcal{G}_n(W)$ of length less than δ and by $Lo_n(W;\delta)$ the elements of $\mathcal{G}_n(W)$ of length at least δ.

Lemma 5.4. *Fix $\delta \in (0, \delta_0/3)$ so that $4A\delta\delta_0^{-1}\bar{C}_0 \le 1/4$, then, for all $f \in \mathcal{C}_{c,A,L}(\delta)$ and $n \ge n_0$,*

$$|||\mathcal{L}^n f|||_+ \le \tfrac{3}{2}|||f|||_+ \quad \text{and} \quad |||\mathcal{L}^n f|||_- \ge \tfrac{1}{2}|||f|||_-.$$

Proof. Let $W \in \mathcal{W}^s(\delta)$, $\psi \in \mathcal{D}_{a,\beta}(W)$. Then,

$$\int_W \mathcal{L}^n f \, \psi = \sum_{W_i \in Lo_n(W;\delta)} \int_{W_i} f \, \psi \circ T^n \, J_{W_i} T^n + \sum_{W_i \in Sh_n(W;\delta)} \int_{W_i} f \, \psi \circ T^n \, J_{W_i} T^n. \tag{5.2}$$

Now since $\psi \circ T^n J_{W_i} T^n \in \mathcal{D}_{a,\beta}(W_i)$ by Lemma 5.2, we subdivide elements $W_i \in Lo_n(W;\delta)$ into curves U_ℓ having length between δ and 2δ and use the definition of $|||f|||_+$ on each such curve to estimate,

$$\int_{W_i} f \, \psi \circ T^n \, J_{W_i} T^n \le \sum_\ell |||f|||_+ \int_{U_\ell} \psi \circ T^n J_{V_j} T^n = |||f|||_+ \int_{T^n W_i} \psi \, .$$

To estimate the short pieces, we apply (4.7) and use Lemma 3.1-(b) since $Sh_n(W;\delta) \subset \mathcal{G}_n(W)$.

$$\begin{aligned} \sum_{W_i \in Sh_n(W;\delta)} \int_{W_i} f \, \psi \circ T^n \, J_{W_i} T^n &\le \sum_{W_i \in Sh_n(W;\delta)} |||f|||_- A |W_i|^q \delta^{1-q} \fint_{W_i} \psi \circ T^n \, J_{W_i} T^n \\ &\le \delta A |||f|||_- e^{a(2\delta)^\beta} \fint_W \psi \, (\bar{C}_0 \delta_0^{-1} |W| + C_0 \theta_1^n). \end{aligned}$$

Putting these estimates together in (5.2) and using that $|W| \ge \delta$, we obtain,

$$\begin{aligned} \int_W \mathcal{L}^n f \, \psi &\le \sum_{W_i \in Lo_n(W;\delta)} |||f|||_+ \int_{T^n W_i} \psi + A |||f|||_- e^{a(2\delta)^\beta} \int_W \psi \, (\bar{C}_0 \delta \delta_0^{-1} + C_0 \theta_1^n) \\ &\le |||f|||_+ \int_W \psi \, \left(1 + A e^{a(2\delta)^\beta} (\delta \delta_0^{-1} \bar{C}_0 + C_0 \theta_1^n)\right), \end{aligned}$$

where we have used Lemma 4.2. Now (4.9) implies $e^{a(2\delta)^\beta} \le 2$, and our choices of n_0 and δ imply $2A \max\{\bar{C}_0 \delta \delta_0^{-1}, C_0 \theta_1^{n_0}\} \le 1/4$, which yields the required estimate on $|||\mathcal{L}^n f|||_+$ for all $n \ge n_0$.

For the bound on $|||\mathcal{L}^n f|||_-$, we perform a similar estimate, except noting that for $W_i \in Lo_n(W;\delta)$,

$$\int_{W_i} f \psi \circ T^n J_{W_i} T^n \ge |||f|||_- \int_{T^n W_i} \psi,$$

we follow (5.2) to estimate,

$$\begin{aligned} \int_W \mathcal{L}^n f \, \psi &\ge \sum_{W_i \in Lo_n(W;\delta)} |||f|||_- \int_{T^n W_i} \psi - A |||f|||_- e^{a(2\delta)^\beta} \int_W \psi \, (\bar{C}_0 \delta \delta_0^{-1} + C_0 \theta_1^n) \\ &\ge |||f|||_- \int_W \psi \, \left(1 - 2A e^{a(2\delta)^\beta} (\delta \delta_0^{-1} \bar{C}_0 + C_0 \theta_1^n)\right). \end{aligned}$$

Again using our choice of n_0 and δ, we have $4AC_0\theta_1^n \le 1/4$ and $4A\delta\delta_0^{-1}\bar{C}_0 \le 1/4$, which yields $|||\mathcal{L}^n f|||_- \ge \frac{1}{2} |||f|||_-$. □

In particular the above implies the estimate: for all $n \ge n_0$,

$$\frac{|||\mathcal{L}^n f|||_+}{|||\mathcal{L}^n f|||_-} \le 3 \frac{|||f|||_+}{|||f|||_-} \le 3L. \tag{5.3}$$

5.2.2 Contraction of the parameter A

We prove that the parameter A contracts in (4.7). Choose $f \in \mathcal{C}_{c,A,L}(\delta)$. Let $W \in \mathcal{W}^s$ with $|W| \le 2\delta$, $\psi \in \mathcal{D}_{a,\beta}(W)$ and $x \in W$. From now on, we will refer to $Lo_n(W;\delta)$ and $Sh_n(W;\delta)$ as simply $Lo_n(W)$ and $Sh_n(W)$. We follow (5.2) to write

$$\begin{aligned}\left|\int_W \mathcal{L}^n f\,\psi\right| &\le \sum_{W_i\in Lo_n(W)} \int_{W_i} f\,\psi\circ T^n\,J_{W_i}T^n + \sum_{W_i\in Sh_n(W)} \left|\int_{W_i} f\,\psi\circ T^n\,J_{W_i}T^n\right| \\ &\le \sum_{W_i\in Lo_n(W)} |\!|\!| f|\!|\!|_+ \int_{W_i} \psi\circ T^n\,J_{W_i}T^n + \sum_{W_i\in Sh_n(W)} A\delta^{1-q}|W_i|^q |\!|\!| f|\!|\!|_- \fint_{W_i} \psi\circ T^n\,J_{W_i}T^n \\ &\le \sum_{W_i\in Lo_n(W)} |\!|\!| f|\!|\!|_- L\int_{T^nW_i} \psi + A\delta^{1-q}|W|^q |\!|\!| f|\!|\!|_- |\psi|_{C^0} \sum_{W_i\in Sh_n(W)} \frac{|W_i|^q}{|W|^q}\frac{|T^nW_i|}{|W_i|},\end{aligned}$$

where in the second line we have used (4.7) for the sum on short pieces. Since $|W| \le 2\delta$, the first sum above is bounded by

$$|\!|\!| f|\!|\!|_- L|W| \fint_W \psi \le \|f\|\!|_- 2L\delta^{1-q}|W|^q \fint_W \psi\,.$$

For the sum on short pieces, we use Lemmas 3.1-(b) and a Hölder inequality to estimate

$$\begin{aligned}\sum_{W_i\in Sh_n(W)} \frac{|W_i|^q}{|W|^q}\frac{|T^nW_i|}{|W_i|} &\le \left(\sum_{W_i\in Sh_n(W)} \frac{|T^nW_i|}{|W|}\right)^q \left(\sum_{W_i\in Sh_n(W)} |J_{W_i}T^n|_{C^0(W_i)}\right)^{1-q} \\ &\le (\bar{C}_0\delta_0^{-1}|W| + C_0\theta_1^n)^{1-q}\end{aligned}$$

Combining these two estimates with Lemma 4.2 yields,

$$\frac{|\int_W \mathcal{L}^n f\,\psi|}{\fint_W \psi} \le A\delta^{1-q}|W|^q |\!|\!| f|\!|\!|_- \left(2LA^{-1} + e^{a(2\delta)^\beta}(\bar{C}_0\delta_0^{-1}|W| + C_0\theta_1^n)^{1-q}\right). \tag{5.4}$$

This contracts the parameter A if $2LA^{-1} + e^{a(2\delta)^\beta}(2\bar{C}_0\delta\delta_0^{-1} + C_0\theta_1^n)^{1-q} < 1$, which we can achieve if $e^{a(2\delta)^\beta} \le 2$,

$$A > 4L, \quad \text{and} \quad (2\bar{C}_0\delta\delta_0^{-1} + C_0\theta_1^{n_0})^{1-q} < 1/4\,. \tag{5.5}$$

Remark that since $L \ge 1$, we have $A > 4$, and so according to the assumption of Lemma 5.4, $2\bar{C}_0\delta\delta_0^{-1} \le 1/32$. Moreover, $C_0\theta_1^{n_0} \le 1/64$ by choice of n_0, and since $1-q \ge 1/2$, the second condition in (5.5) is always satisfied under the assumption of Lemma 5.4.

5.2.3 Contraction of the parameter c

Finally, we verify the contraction of c via (4.8). Let $f \in \mathcal{C}_{c,A,L}(\delta)$ and $W^1, W^2 \in \mathcal{W}^s$ with $|W^k| \le 2\delta$ and $d_{\mathcal{W}^s}(W^1,W^2) \le \delta^2$. Take $\psi_k \in \mathcal{D}_{a,\alpha}(W^k)$ with $d_*(\psi_1,\psi_2) = 0$.

Without loss of generality we can assume $|W^2| \ge |W^1|$ and $\fint_{W_1} \psi_1 = 1$. Next, note that cone condition (4.7) implies (see section 5.2.2)

$$\left|\frac{\int_{W^1} \mathcal{L}^n f\psi_1}{\fint_{W^1}\psi_1} - \frac{\int_{W^2} \mathcal{L}^n f\psi_2}{\fint_{W^2}\psi_2}\right| \le 2A\delta^{1-q}|W^2|^q |\!|\!| \mathcal{L}^n f|\!|\!|_-$$

It follows that the contraction of the parameter c is trivial for $|W^2|^q \le \delta^{q-\gamma}\frac{d_{\mathcal{W}^s}(W^1,W^2)^\gamma c}{4}$. Thus it suffices to consider the case

$$|W^2|^q \ge \delta^{q-\gamma}\frac{d_{\mathcal{W}^s}(W^1,W^2)^\gamma c}{4}\,. \tag{5.6}$$

We claim that (5.6) implies that $I_{W_1}\cap I_{W_2}\neq\emptyset$. Define $C_s := \sqrt{1+(\mathcal{K}_{\max}+\tau_{\min}^{-1})^2}$ to be the maximum absolute value of the slopes of curves in the stable cone defined in (3.1). If $I_{W^1}\cap I_{W^2}=\emptyset$, then $d_{\mathcal{W}^s}(W^1,W^2)=|I_{W^1}\triangle I_{W^2}|=|I_{W^1}|+|I_{W^2}|$, so it must be that $|I_{W^2}|\le\delta$. Yet (5.6) implies that

$$C_s|I_{W^2}|\ge|W^2|\ge\delta^{(q-\gamma)/q}d_{\mathcal{W}^s}(W^1,W^2)^{\gamma/q}(\tfrac{c}{4})^{1/q}\ge\delta^{(q-\gamma)/q}|I_{W^2}|^{\gamma/q}(\tfrac{c}{4})^{1/q},$$

so we obtain the contradiction $|I_{W^2}|\ge\delta\cdot 2^{1/(q-\gamma)}$ provided

$$q>\gamma\,;\; c\ge 8C_s^q\,. \tag{5.7}$$

Next, for any two manifolds $U^i\in\mathcal{W}^s_-(\delta)$ defined on the intervals I_i with $J=I_1\cap I_2\neq\emptyset$, by the distance definition (4.2) we have,

$$\begin{aligned}\big|\,|U^1|-|U^2|\,\big| &\le \int_J(\|G_1'\|-\|G_2'\|)dr+\sum_{i=1}^{1}\int_{I_i\setminus J}\|G_i'\|dr\\ &\le\int_J\|G_2'-G_1'\|dr+C_s|I_1\Delta I_2|\le(|U^1|+C_s)d_{\mathcal{W}^s}(U^1,U^2).\end{aligned}\tag{5.8}$$

Since $\fint_{W_1}\psi_1=1$, we have $|\psi_1|_\infty\le e^{a(2\delta)^\alpha}$. On the other hand, since $I_{W^1}\cap I_{W^2}\neq\emptyset$ and $d_*(\psi_1,\psi_2)=0$, there must exist $r\in I_{W^1}\cap I_{W^2}$ such that $\psi_1\circ G_{W^1}(r)\|G'_{W^1}(r)\|=\psi_2\circ G_{W^2}(r)\|G'_{W^2}(r)\|$. Thus since,

$$\begin{aligned}\frac{\|G'_{W^1}(r)\|}{\|G'_{W^2}(r)\|}&=\sqrt{\frac{1+(\varphi'_{W^1}(r))^2}{1+(\varphi'_{W^2}(r))^2}}=\sqrt{1+\frac{(\varphi'_{W^1}(r)-\varphi'_{W^2}(r))(2\varphi'_{W^2}(r)+(\varphi'_{W^1}(r)-\varphi'_{W^2}(r)))}{1+(\varphi'_{W^2}(r))^2}}\\ &\le\sqrt{1+d_{\mathcal{W}^s}(W^1,W^2)(2+d_{\mathcal{W}^s}(W^1,W^2))}\le\sqrt{1+3\delta}\le 2\,,\end{aligned}$$

where we use $\delta<1$, we estimate,

$$|\psi_2|_\infty\le 2e^{a(2\delta)^\alpha}|\psi_1|_\infty\le 2e^{2a(2\delta)^\alpha}\,. \tag{5.9}$$

Then recalling Remark 4.4, it follows that

$$\left|\int_{W^1}\psi_1-\int_{W^2}\psi_2\right|\;\le\;e^{a(2\delta)^\alpha}C_s|I_{W^1}\setminus I_{W^2}|+e^{2a(2\delta)^\alpha}2C_s|I_{W^2}\setminus I_{W^1}|\le 2C_se^{2a(2\delta)^\alpha}d_{\mathcal{W}^s}(W^1,W^2).$$

Putting this together with (5.8) and using $\int_{W_1}\psi_1=|W_1|$, we estimate,

$$\begin{aligned}\left||W^2|-\int_{W^2}\psi_2\right|&\le\big||W^2|-|W^1|\big|+\left|\int_{W^1}\psi_1-\int_{W^2}\psi_2\right|\\ &\le(|W^1|+C_s)d_{\mathcal{W}^s}(W^1,W^2)\le 6C_sd_{\mathcal{W}^s}(W^1,W^2)\,,\end{aligned}\tag{5.10}$$

where we have used (4.9) and $\alpha>\beta$. Hence, recalling Lemma 5.4 and (5.4), $d_{\mathcal{W}^s}(W^1,W^2)\le\delta$ and using (5.6), (5.7) and (5.10), we have

$$\begin{aligned}\left|\frac{\int_{W^1}\mathcal{L}^nf\psi_1}{\fint_{W^1}\psi_1}-\frac{\int_{W^2}\mathcal{L}^nf\psi_2}{\fint_{W^2}\psi_2}\right|&\le\left|\int_{W^1}\mathcal{L}^nf\psi_1-\int_{W^2}\mathcal{L}^nf\psi_2\right|+\left|\int_{W^2}\mathcal{L}^nf\psi_2\right|\left|\frac{|W^2|}{\int_{W^2}\psi_2}-1\right|\\ &\le\left|\int_{W^1}\mathcal{L}^nf\psi_1-\int_{W^2}\mathcal{L}^nf\psi_2\right|+A\left[\frac{\delta}{|W^2|}\right]^{1-q}\left||W^2|-\int_{W^2}\psi_2\right|2|||\mathcal{L}^nf|||_-\\ &\le\left|\int_{W^1}\mathcal{L}^nf\psi_1-\int_{W^2}\mathcal{L}^nf\psi_2\right|+2^{3-1/q}3C_s^qA\delta^{1-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^\gamma|||\mathcal{L}^nf|||_-\,.\end{aligned}\tag{5.11}$$

To conclude it suffices then to compare $\int_{W^1} \mathcal{L}^n f\, \psi_1$ and $\int_{W^2} \mathcal{L}^n f\, \psi_2$. To this end, define $\mathcal{G}_n^\delta(W^k)$ to be the nth generation homogeneous components of $T^{-n}W^k$ with long pieces subdivided to have length between δ and 2δ. We split $\mathcal{G}_n^\delta(W^1)$ and $\mathcal{G}_n^\delta(W^2)$ into matched and unmatched pieces as follows. On each curve $W_i^1 \in \mathcal{G}_n^\delta(W^1)$, we place a foliation of vertical line segments $\{\ell_x\}_{x\in W_i^1}$ of length $C_1\Lambda^{-n}d_{\mathcal{W}^s(W^1,W^2)}$. Due to the uniform hyperbolicity of T, the images $T^i\ell_x$ are unstable curves for $i \geq 1$ and remain uniformly transverse to the stable cone. Thus $T^n\ell_x$ undergoes the uniform expansion given by (3.1) and, if not cut by a singularity, will intersect both W^1 and W^2. When these segments $T^n\ell_x$ survive uncut, we declare the subcurves U_j^1, U_j^2 connected by the original vertical segments ℓ_x to be 'matched.' Note that, by [CM, Proposition 4.47] there must exists two piecewise smooth curves in $\mathcal{S}_n^{\mathbb{H}}$ that connect the boundaries of U_j^1 and U_j^2 forming a *rectangle* that does not contain any element of $\mathcal{S}_n^{\mathbb{H}}$ in its interior.

All other subcurves we label V_j^1, V_j^2 and declare them to be 'unmatched.' It follows that there can be at most one matched curve U_j^k and two unmatched curves V_j^k for each element $W_i^k \in \mathcal{G}_n^\delta(W^k)$, $k = 1, 2$. Thus we have defined a composition $\mathcal{G}_n^\delta(W^k) = \cup_j U_j^k \cup \cup_j V_j^k$, such that U_j^1 and U_j^2 are defined as the graphs of functions $G_{U_j^k}$ over the same r-interval I_j for each j.

Using this decomposition, and writing $\widehat{T}^n_{U_j^k}\psi_k = \psi_k \circ T^n J_{U_j^k}T^n$ and similarly for $\widehat{T}_{V_j^k}\psi_k$, we write

$$\int_{W^k} \mathcal{L}^n f\, \psi_k = \sum_j \int_{U_j^k} f\, \widehat{T}^n_{U_j^k}\psi_k + \sum_j \int_{V_j^k} f\, \widehat{T}^n_{V_j^k}\psi_k. \tag{5.12}$$

We estimate the contribution from unmatched pieces first. To do so, we group the V_j^k as follows. We say V_j^k is 'created' at time $0 \leq i \leq n-1$ if i is the smallest t such that either an endpoint of $T^{n-t}V_j^k$ is created by an intersection with $T(\mathcal{S}_0^{\mathbb{H}})$, or $T^{n-t}V_j^k$ is contained in a larger unmatched piece with this property (this second case can happen when both endpoints of V_j^k are created by subdivision of long pieces rather than cuts due to singularities). Due to the uniform transversality of the stable cone with curves in $T(\mathcal{S}_0^{\mathbb{H}})$ as well as the uniform transversality of the stable and unstable cones, we have $|T^{n-i}V_j^k| \leq \bar{C}_3\Lambda^{-i}d_{\mathcal{W}^s}(W^1, W^2)$, for some constant $\bar{C}_3 > 0$. Define $P(i) = \{j : V_j^1 \text{ created at time } i\}$.

Although we would like to change variables to estimate the contribution on the curves $T^{n-i}V_j^1$ for $j \in P(i)$, this is one time step before such cuts would be introduced according to our definition of $\mathcal{G}_n^\delta(W)$, so Lemma 3.1 would not apply since there may be many such $T^{n-i}V_j^1$ for each $W_\ell^1 \in \mathcal{G}_i^\delta(W^1)$. However, there can be at most two curves $T^{n-i-1}V_j^1$, $j \in P(i)$, per element of $W_\ell^1 \in \mathcal{G}_{i+1}^\delta(W^1)$, so we will change variables to estimate the contribution from curves of the form $T^{n-i-1}V_j^1$ instead. We have two cases.

Case 1. The curve in $T(\mathcal{S}_0^{\mathbb{H}})$ that creates V_j^1 at time i is the preimage of the boundary of a homogeneity strip. Then $T^{n-i-1}V_j^1$ still enjoys uniform transversality with the boundary of the homogeneity strip and the unstable cone, and so $|T^{n-i-1}V_j^1| \leq \bar{C}_3\Lambda^{-i-1}d_{\mathcal{W}^s}(W^1, W^2)$ as before.

Case 2. The curve in $T(\mathcal{S}_0^{\mathbb{H}})$ that creates V_j^1 at time i is not the preimage of the boundary of a homogeneity strip. Then V_j^1 undergoes bounded expansion from time $n-i$ to time $n-i-1$. Thus $|T^{n-i-1}V_j^1| \leq C\bar{C}_3\Lambda^{-i}d_{\mathcal{W}^s}(W^1, W^2)$, where $C > 0$ depends only on our choice of k_0, the minimum index of homogeneity strips.

In either case, we conclude that $|T^{n-i-1}V_j^1| \leq C_3\Lambda^{-i}d_{\mathcal{W}^s}(W^1, W^2)$, for a uniform constant $C_3 > 0$. Since $d_{\mathcal{W}^s}(W^1, W^2) \leq \delta^2$ ($\delta^{1+\frac{\gamma}{q-\gamma}}$ would suffice), it follows that all curves $T^{n-i-1}V_j^k$ have length shorter than 2δ, thus we may apply (4.7).

$$\left|\sum_j \int_{V_j^1} f\,\widehat{T}^n_{V_j^1}\psi_1\right| \le \sum_{i=0}^{n-1}\left|\sum_{j\in P(i)}\int_{T^{n-i-1}V_j^1}\mathcal{L}^{n-i-1}f\cdot\psi_1\circ T^{i+1}\,J_{T^{n-i-1}V_j^1}T^{i+1}\right|$$
$$\le \sum_{i=0}^{n-1}\sum_{j\in P(i)} A\delta^{1-q}|T^{n-i-1}V_j^1|^q|||\mathcal{L}^{n-i-1}f|||_-|\psi_1|_{C^0(W^1)}|J_{T^{n-i-1}V_j^1}T^{i+1}|_{C^0(T^{n-i-1}V_j^1)}$$
$$\le \sum_{i=0}^{n-1} A\delta^{1-q}C_3^q\Lambda^{-iq}d_{\mathcal{W}^s}(W^1,W^2)^q|||\mathcal{L}^{n-i-1}f|||_-\,(2\bar{C}_0+C_0\theta_1^{i+1})|\psi_1|_{C^0(W^1)},$$

where we have used Lemma 3.1-(b) for the sum over $j\in P(i)$ since there are at most two curve $T^{n-i-1}V_j^1$ for each element $W_\ell^1\in\mathcal{G}^\delta_{i+1}(W)$.[2]

Since $n\ge 2n_0$, we have either that $i+1\ge n_0$ or $n-(i+1)\ge n_0$. In the former case, $|||\mathcal{L}^{n-i-1}f|||_-\le 2|||\mathcal{L}^n f|||_-$ by Lemma 5.4. In the latter case,

$$|||\mathcal{L}^{n-i-1}f|||_-\le|||\mathcal{L}^{n-i-1}f|||_+\le\tfrac{3}{2}|||f|||_+\le\tfrac{3}{2}\,L|||f|||_-\le 3L|||\mathcal{L}^n f|||_-, \tag{5.13}$$

where we have used Lemma 5.4 twice, once on $|||\mathcal{L}^{n-i-1}f|||_+$ and once on $|||f|||_-$. Since the latter estimate (5.13) is the larger of the two, we may use it for all i.

Also, using the assumption that $d_{\mathcal{W}^s}(W^1,W^2)\le\delta$ and (5.7) yields,

$$\delta^{1-q}d_{\mathcal{W}^s}(W^1,W^2)^q\le\delta^{1-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^\gamma.$$

Collecting these estimates and summing over the exponential factors yields (since the estimate for V_j^2 is the same),

$$\sum_{j,k}\left|\int_{V_j^k} f\,\widehat{T}^n_{V_j^k}\psi_k\right|\le C_4AL\delta^{1-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^\gamma|||\mathcal{L}^n f|||_-, \tag{5.14}$$

for some uniform constant C_4 depending only on T and not on the parameters of the cone.

Next, we estimate the contribution on matched pieces U_j^k. To do this, we will need to change test functions on the relevant curves. Define the following functions on U_j^1,

$$\begin{aligned}\tilde{\psi}_2&=\psi_2\circ T^n\circ G_{U_j^2}\circ G^{-1}_{U_j^1}\ ;\quad \widetilde{J}_{U_j^2}T^n=J_{U_j^2}T^n\circ G_{U_j^2}\circ G^{-1}_{U_j^1},\\ \widetilde{T}^n_{U_j^2}\psi_2&=\tilde{\psi}_2\cdot\widetilde{J}_{U_j^2}T^n\frac{\|G'_{U_j^2}\|\circ G^{-1}_{U_j^1}}{\|G'_{U_j^1}\|\circ G^{-1}_{U_j^1}}.\end{aligned} \tag{5.15}$$

Note that $d_*(\widehat{T}^n_{U_j^2}\psi_2,\widetilde{T}^n_{U_j^2}\psi_2)=0$ by construction. Also we define

$$\begin{aligned}\psi_j^-&=\min\{\widehat{T}^n_{U_j^1}\psi_1,\widetilde{T}^n_{U_j^2}\psi_2\}\\ \psi^\Delta_{1,j}&=\widehat{T}^n_{U_j^1}\psi_1-\psi_j^-\ ;\quad\psi^\Delta_{2,j}=\widetilde{T}^n_{U_j^2}\psi_2-\psi_j^-.\end{aligned} \tag{5.16}$$

We will need the following lemma to proceed.

[2]Notice that since we subdivide curves in $\mathcal{G}^\delta_n(W)$ according to length δ and not δ_0, the estimate of Lemma 3.1-(b) becomes $\bar{C}_0\delta^{-1}|W|+C_0\theta_1^n\le 2\bar{C}_0+C_0\theta_1^n$.

Lemma 5.5. *If $c > 4(1+M_0)^q$, M_0 is defined in (5.26), then there exists $C_5 \geq 1$, independent of n, W^1 and W^2 satisfying (5.6), such that for each j,*

a) $d_{\mathcal{W}^s}(U_j^1, U_j^2) \leq C_5 n \Lambda^{-n} d_{\mathcal{W}^s}(W^1, W^2)$;

b) $e^{-C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha} \leq \dfrac{\widehat{T}^n_{U_j^1}\psi_1(x)}{\widetilde{T}^n_{U_j^2}\psi_2(x)} \leq e^{C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha} \quad \forall x \in U_j^1$;

c) setting $B = 8\left[C_5 a^{-1}\right]^{\frac{\alpha-\beta}{\alpha}} d_{\mathcal{W}^s}(W^1,W^2)^{\alpha-\beta}$ *we have* $\psi^\Delta_{i,j} + B\psi^-_j \in \mathcal{D}_{a,\beta}(U_j^1)$, $i = 1, 2$.

Moreover, $\widetilde{T}^n_{U_j^2}\psi_2$ *and* ψ_j^- *belong to* $\mathcal{D}_{a,\alpha}(U_j^1)$.

We postpone the proof of the lemma and use it to conclude the estimates of this section.
For future use note that Lemma 5.5(b) implies

$$0 \leq \psi^\Delta_{k,j}(x) \leq 2C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha \psi^-_j(x). \tag{5.17}$$

Since $d_*(\widetilde{T}^n_{U_j^2}\psi_2, \widehat{T}^n_{U_j^2}\psi_2) = 0$ by construction, and recalling Remark 4.4, Lemma 5.5(c), condition (4.7), and (5.16), (5.17),

$$\begin{aligned}
&\left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1 - \int_{U_j^2} f\,\widehat{T}^n_{U_j^2}\psi_2\right| \leq \left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1 - \int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2\right| \\
&+\left|\frac{\fint_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2}{\fint_{U_j^1}\widetilde{T}^n_{U_j^2}\psi_2} - \frac{\fint_{U_j^2} f\,\widehat{T}^n_{U_j^2}\psi_2}{\fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2}\right| \fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2 + \left|\frac{\fint_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2}{\fint_{U_j^1}\widetilde{T}^n_{U_j^2}\psi_2}\right| \left|\frac{|U_j^2| - |U_j^1|}{|U_j^1|}\right| \fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2 \\
&\leq A\delta^{1-q}|U_j^1|^q \frac{\left[\fint_{U_j^1}(\psi^\Delta_{1,j} + B\psi^-_j) + \fint_{U_j^1}(\psi^\Delta_{2,j} + B\psi^-_j)\right]}{\fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2} \fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2 |||f|||_- \\
&\quad + d_{\mathcal{W}^s}(U_j^1,U_j^2)^\gamma \delta^{1-\gamma} cA \fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2 |||f|||_- \\
&\quad + A\delta^{1-q}|U_j^1|^q \left|\frac{|U_j^2| - |U_j^1|}{|U_j^1|}\right| \fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2 |||f|||_-\,,
\end{aligned} \tag{5.18}$$

where for the first term, we have used that $|\widehat{T}^n_{U_j^1}\psi_1 - \widetilde{T}^n_{U_j^2}\psi_2| = \psi^\Delta_{1,j} + \psi^\Delta_{2,j}$, and for the second and third terms that $\widetilde{T}^n_{U_j^2}\psi_2 \in \mathcal{D}_{a,\alpha}(U_j^1)$ by Lemma 5.5. Then, recalling Lemma 3.1(b), (5.9) and (4.9), we can estimate

$$\sum_j \fint_{U_j^2} \widehat{T}_{U_j^2}\psi_2 \leq \sum_j \fint_{U_j^2} |J_{U_2^j}T^n|_\infty \psi_2 \circ T^n \leq (\bar{C}_0\delta^{-1}|W^2| + C_0\theta_1^n)2e^{2a(2\delta)^\alpha} \leq 24\bar{C}_0\,. \tag{5.19}$$

Next, recalling (5.8), we have[3]

$$|U_j^2| \leq |U_j^1|(1 + d_{\mathcal{W}^s}(U_j^1,U_j^2)) \leq 2|U_j^1|$$

[3] Since the U_j^k are vertically matched, the term on the right hand side of (5.8) proportional to C_s is absent here.

provided we impose

$$C_5 n_0 \Lambda^{-n_0}\delta \le 1 \tag{5.20}$$

where C_5 is from Lemma 5.5-(a) and Λ is defined in (3.1). Moreover, remembering the definition of B in Lemma 5.5-(c) and equation (5.17),

$$\begin{aligned}\fint_{U_j^1}(\psi_{k,j}^{\Delta}+B\psi_j^-) &\le \fint_{U_j^1} 10C_5\widetilde{T}^n_{U_j^2}\psi_2 d_{\mathcal{W}^s}(W^1,W^2)^\gamma \\ &\le 10C_5\frac{|U_j^2|}{|U_j^1|}\fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2 d_{\mathcal{W}^s}(W^1,W^2)^\gamma \le 20C_5\fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2 d_{\mathcal{W}^s}(W^1,W^2)^\gamma,\end{aligned} \tag{5.21}$$

where we have used the assumption $\alpha-\beta\ge\gamma$.

Again using (5.8) and Lemma 5.5-(a) we have

$$\left|\frac{|U_j^2|-|U_j^1|}{|U_j^1|^{1-q}}\right| \le d_{\mathcal{W}^s}(U_j^2,U_j^1)|U_j^1|^q \le (2\delta)^q C_5 n\Lambda^{-n} d_{\mathcal{W}^s}(W^2,W^1). \tag{5.22}$$

Inserting (5.19), (5.21) and (5.22) in (5.18) and recalling Lemmas 5.4 and 5.5-(a) yields,

$$\begin{aligned}&\sum_j\left|\int_{U_j^1} f\,\widehat{T}_{U_j^1}\psi_1-\int_{U_j^2} f\,\widehat{T}_{U_j^2}\psi_2\right| \\ &\le 48\bar{C}_0 A\delta^{1-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^\gamma |||\mathcal{L}^n f|||_-\left(2^q 40C_5\delta^{q-\gamma}+cC_5 n^\gamma\Lambda^{-n\gamma}+2^qC_5 n\Lambda^{-n}\delta\right)\end{aligned} \tag{5.23}$$

Then using this estimate in (5.11), and recalling (5.12) and (5.14) yields

$$\begin{aligned}\left|\frac{\int_{W^1}\mathcal{L}^n f\,\psi_1}{\fint_{W^1}\psi_1}-\frac{\int_{W^2}\mathcal{L}^n f\,\psi_2}{\fint_{W^2}\psi_2}\right| &\le \Big\{2^{3-1/q}3C_s^q+C_4L \\ &+48\bar{C}_0\left(2^q40C_5\delta^{q-\gamma}+cC_5n^\gamma\Lambda^{-n\gamma}+2^qC_5n\Lambda^{-n}\delta\right)\Big\}A\delta^{1-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^\gamma|||\mathcal{L}^nf|||_-\end{aligned} \tag{5.24}$$

which yields the wanted estimate, provided

$$2^{3-1/q}C_s^q+C_4L+48\bar{C}_0\left(2^q40C_5\delta^{q-\gamma}+cC_5n^\gamma\Lambda^{-n\gamma}+2^qC_5n\Lambda^{-n}\delta\right)<c. \tag{5.25}$$

5.2.4 Proof of Lemma 5.5

Proof. (a) This is [DZ1, Lemma 4.2].
(b) Recall that U_j^k is defined as the graph of a function $G_{U_j^k}(r)=(r,\varphi_{U_j^k}(r))$, for $r\in I_j^k$, $k=1,2$. Due to the vertical matching, we have $I_j^1=I_j^2$.

Now for $x\in U_j^1$, let $r\in I_j^1$ be such that $G_{U_j^1}(r)=x$. Set $\bar{x}=G_{U_j^2}(r)$ and note that x and $\bar{x}$ lie on the same vertical line in M since U_j^1 and U_j^2 are matched. Thus by (3.4),

$$\frac{J_{U_j^1}T^n(x)}{\widetilde{J}_{U_j^2}T^n(x)}=\frac{J_{U_j^1}T^n(x)}{J_{U_j^2}T^n(\bar{x})}\le e^{C_d(d(T^nx,T^n\bar{x})^{1/3}+\phi(x,\bar{x}))}\le e^{C_dM_0d_{\mathcal{W}^s}(W^1,W^2)^{1/3}}, \tag{5.26}$$

where M_0 is a constant depending only on the maximum and minimum slopes in C^s and C^u.

Next, for $x\in U_j^1$ consider

$$\frac{\psi_1\circ T^n(x)}{\widetilde{\psi}_2(x)}\frac{\|G'_{U_j^1}\|\circ G^{-1}_{U_j^1}(x)}{\|G'_{U_j^2}\|\circ G^{-1}_{U_j^1}(x)}.$$

Let $T^n(x) = (r, G_{W^1}(r))$ and $T^n(\bar{x}) = (\bar{r}, G_{W^2}(\bar{r}))$, then

$$|r - \bar{r}| \le M_0 d_{\mathcal{W}^s}(W^1, W^2)\,.$$

If $r \in I_{W^2}$, then since $d_*(\psi_1, \psi_2) = 0$,

$$\frac{\psi_1 \circ G_{W^1}(r)}{\psi_2 \circ G_{W^2}(\bar{r})} = \frac{\psi_1 \circ G_{W^1}(r)}{\psi_2 \circ G_{W^2}(r)} \frac{\psi_2 \circ G_{W^2}(r)}{\psi_2 \circ G_{W^2}(\bar{r})} \le \frac{\|G'_{W^2}(r)\|}{\|G'_{W^1}(r)\|} e^{ad(G_{W_1}(r), G_{W^2}(\bar{r}))^\alpha}.$$

Next, since $\|G'_{W^1} - G'_{W^2}\| = |\varphi'_{W^1} - \varphi'_{W^2}|$ and $\|G'_{W^k}\| \ge 1$, we have

$$\frac{\|G'_{W^2}(r)\|}{\|G'_{W^1}(r)\|} \le e^{\|G'_{W^1} - G'_{W^2}\|} \le e^{d_{\mathcal{W}^s}(W^1, W^2)}\,.$$

Similarly, $\frac{\|G'_{U_j^1}\| \circ G^{-1}_{U_j^1}(x)}{\|G'_{U_j^2}\| \circ G^{-1}_{U_j^1}(x)} \le e^{d_{\mathcal{W}^s}(U_j^1, U_j^2)}$. Hence, using part (a) of the lemma and assuming

$$C_5 n_0 \Lambda^{-n_0} \delta^{1-\alpha} \le 1, \tag{5.27}$$

yields

$$\frac{\psi_1 \circ T^n(x)}{\tilde{\psi}_2(x)} \frac{\|G'_{U_j^1}\| \circ G^{-1}_{U_j^1}(x)}{\|G'_{U_j^2}\| \circ G^{-1}_{U_j^1}(x)} \le e^{(aM_0^\alpha + 2) d_{\mathcal{W}^s}(W^1, W^2)^\alpha}\,.$$

The same estimate holds if $\bar{r} \in I_{W^1}$. Otherwise it must be that

$$|I_{W^1} \cap I_{W^2}| \le M_0 d_{\mathcal{W}^s}(W^1, W^2)$$

but then, since $|I_{W^1} \Delta I_{W^2}| \le d_{\mathcal{W}^s}(W^1, W^2)$ we would have $|W^2| \le (1 + M_0) d_{\mathcal{W}^s}(W^1, W^2)$, which violates (5.6) together with the assumption, provided

$$c > 4(1 + M_0)^q. \tag{5.28}$$

The estimates with the opposite sign follow similarly. Putting together these estimates yields part (b) of the lemma with $C_5 = M_0 C_d \delta^{1/3-\alpha} + aM_0^\alpha + 2$.
(c) As noted in (5.17), by (b) it immediately follows that

$$\left|\psi^\Delta_{i,j}(x)\right| \le \left|\widehat{T}^n_{U_j^1} \psi_1(x) - \widetilde{T}^n_{U_j^2} \psi_2(x)\right| \le 2C_5 d_{\mathcal{W}^s}(W^1, W^2)^\alpha \psi_j^-(x).$$

Next, for $x, y \in U_j^1$, let $\bar{x} = G_{U_j^2} \circ G^{-1}_{U_j^1}(x)$, $\bar{y} = G_{U_j^2} \circ G^{-1}_{U_j^1}(y)$, and note these are well-defined due to the vertical matching between U_j^1 and U_j^2. Let $r = G^{-1}_{U_j^1}(x)$ and $s = G^{-1}_{U_j^1}(y)$. Recalling (4.4), we have

$$\frac{\|G'_{U_j^1}(r)\|}{\|G'_{U_j^1}(s)\|} \le e^{\|G'_{U_j^1}(r) - G'_{U_j^1}(s)\|} \le e^{B_*|r-s|} \le e^{B_* d(x,y)}\,,$$

and similarly for $\|G'_{U_j^2}\|$. Using this estimate together with the proof of Lemma 5.2-(a),

$$\begin{aligned}\frac{\widetilde{T}^n_{U_j^2}\psi_2(x)}{\widetilde{T}^n_{U_j^2}\psi_2(y)} &= \frac{\widehat{T}^n_{U_j^2}\psi_2(\bar{x})}{\widehat{T}^n_{U_j^2}\psi_2(\bar{y})} \frac{\|G'_{U_j^2}(r)\|}{\|G'_{U_j^1}(r)\|} \frac{\|G'_{U_j^1}(s)\|}{\|G'_{U_j^2}(s)\|} \\ &\le e^{(aC_1^{-1}\Lambda^{-\alpha n} + C_d(2\delta)^{1/3-\alpha}) d(\bar{x}, \bar{y})^\alpha + 2B_* d(x,y)} \le e^{ad(x,y)^\alpha},\end{aligned} \tag{5.29}$$

since $d(\bar{x},\bar{y}) \le M_0 d(x,y)$ and provided

$$(aC_1^{-1}\Lambda^{-\alpha n_0} + C_d(2\delta)^{1/3-\alpha})M_0^\alpha + B_*(2\delta)^{1-\alpha} < a. \tag{5.30}$$

To abbreviate what follows, let us denote $g_1 = \widehat{T}^n_{U_j^1}\psi_1$ and $g_2 = \widetilde{T}^n_{U_j^2}\psi_2$. Then, given $x, y \in U_j^1$, we have $\psi_j^-(x) = g_{k(x)}$, $\psi_j^-(y) = g_{k(y)}$. If $k(x) = k(y)$, then, by Lemma 5.2(a) and (5.29),

$$\frac{\psi_j^-(x)}{\psi_j^-(y)} = \frac{g_{k(y)}(x)}{g_{k(y)}(y)} \le e^{ad(x,y)^\alpha}.$$

If $k(x) \ne k(y)$, then without loss of generality, we can take $k(x) = 1$ and $k(y) = 2$. By definition, $g_1(x) \le g_2(x)$ and $g_2(y) \le g_1(y)$. Hence,

$$e^{-ad(x,y)^\alpha} \le \frac{g_1(x)}{g_1(y)} \le \frac{\psi_j^-(x)}{\psi_j^-(y)} = \frac{g_1(x)}{g_2(y)} \le \frac{g_2(x)}{g_2(y)} \le e^{ad(x,y)^\alpha}.$$

It follows that $\psi_j^- \in \mathcal{D}_{a,\alpha}(U_j^1)$, and by (5.29), $\widetilde{T}^n_{U_j^2}\psi_2 \in \mathcal{D}_{a,\alpha}(U_j^1)$.

Then, for each $1 > B \ge 2C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha$ and $x, y \in U_j^1$,

$$\frac{\psi^\Delta_{i,j}(x) + B\psi_j^-(x)}{\psi^\Delta_{i,j}(y) + B\psi_j^-(y)} \le \frac{(B + 2C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha)\psi_j^-(x)}{(B - 2C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha)\psi_j^-(y)} \le e^{ad(x,y)^\alpha + 4B^{-1}C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha} \le e^{ad(x,y)^\beta}$$

provided $8B^{-1}C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha \le ad(x,y)^\beta$ and

$$(2\delta)^{\alpha-\beta} \le \frac{1}{2}. \tag{5.31}$$

It remains to consider the case $8B^{-1}C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha \ge ad(x,y)^\beta$. Again we must split into two cases. If $k(x) = k(y) = k$, then, setting $\{\ell\} = \{1,2\} \setminus \{k\}$,

$$\begin{aligned}\frac{\psi^\Delta_{\ell,j}(x) + B\psi_j^-(x)}{\psi^\Delta_{\ell,j}(y) + B\psi_j^-(y)} &\le \frac{g_\ell(x) + (B-1)g_k(x)}{g_\ell(y) + (B-1)g_k(y)} \le \frac{e^{ad(x,y)^\alpha}g_\ell(y) + e^{-ad(x,y)^\alpha}(B-1)g_k(y)}{g_\ell(y) + (B-1)g_k(y)} \\ &\le e^{ad(x,y)^\alpha}\left[1 + \frac{2ad(x,y)^\alpha}{B}\right] \le e^{a[d(x,y)^{\alpha-\beta}(1+2B^{-1})]d(x,y)^\beta} \le e^{\frac{a}{2}d(x,y)^\beta}\end{aligned} \tag{5.32}$$

provided that

$$d(x,y)^{\alpha-\beta}(1+2B^{-1}) \le 4B^{-\frac{\alpha}{\beta}}\left[8C_5 d_{\mathcal{W}^s}(W^1,W^2)^\alpha a^{-1}\right]^{\frac{\alpha-\beta}{\beta}} \le \frac{1}{2}.$$

That is,

$$B \ge 8\left[C_5 a^{-1}\right]^{\frac{\alpha-\beta}{\alpha}} d_{\mathcal{W}^s}(W^1,W^2)^{\alpha-\beta}.$$

The second case is $k = k(x) \ne k(y) = \ell$. In this case, there must exist $\bar{x} \in [x,y]$ such that $\psi_j^-(\bar{x}) = g_1(\bar{x}) = g_2(\bar{x})$. Then,

$$\frac{\psi^\Delta_{\ell,j}(x) + B\psi_j^-(x)}{\psi^\Delta_{\ell,j}(y) + B\psi_j^-(y)} = \frac{g_\ell(x) + (B-1)g_k(x)}{Bg_\ell(\bar{x})}\,\frac{g_\ell(\bar{x}) + (B-1)g_k(\bar{x})}{g_\ell(\bar{x}) + (B-1)g_k(\bar{x})} \le e^{ad(x,y)^\beta}$$

by the estimate (5.32). A similar estimate holds for $\psi^\Delta_{k,j}$. It follows that we can choose

$$B = 8\left[C_5 a^{-1}\right]^{\frac{\alpha-\beta}{\alpha}} d_{\mathcal{W}^s}(W^1,W^2)^{\alpha-\beta} \tag{5.33}$$

and have $\psi^\Delta_{i,j} + B\psi_j^- \in \mathcal{D}_{a,\beta}(U_j^1)$. □

5.3 Conditions on parameters

In this section, we collect the conditions imposed on the cone parameters during the proof of Proposition 5.1. Recall the conditions on the exponents stated before the definition of $\mathcal{C}_{c,A,L}(\delta)$: $\alpha \in (0, 1/3]$, $q \in (0, 1/2)$, $\beta < \alpha$ and $\gamma \le \min\{\alpha - \beta, q\}$.

From (4.9) and Lemma 5.4 we require,

$$e^{a(2\delta)^\beta} < e^{2a\delta_0^\beta} \le 2 \quad \text{and} \quad 4A\bar{C}_0 \delta \delta_0^{-1} \le 1/4 \,.$$

From the proof of Lemma 5.4 and Lemma 5.2, we require the following conditions on n_0,

$$AC_0 \theta_1^{n_0} \le 1/16 \quad \text{and} \quad C_1^{-1} \Lambda^{-\beta n_0} < 1 \,.$$

From Lemma 5.2, Corollary 5.3 and the proof of Lemma 5.5, we require

$$a > aC_1^{-1}\Lambda^{-\beta n_0} + C_d \delta_0^{1/3-\beta} \quad \text{and} \quad a > (aC_1^{-1}\Lambda^{-\alpha n_0} + C_d(2\delta)^{1/3-\alpha}) M_0^\alpha + B_*(2\delta)^{1-\alpha}$$

(recall that we have chosen $n_0 \ge n_1$ after Corollary 5.3).

From the bound on (4.7), we require in (5.5),

$$A > 4L \,.$$

For the contraction of c, we require (see (5.7), the proof of Lemma 5.5 and (5.25))

$$\begin{aligned} &c > \max\{8C_s^q, 4(1+M_0)^q\} \;; \quad C_5 n_0 \Lambda^{-n_0} \delta^{1-\alpha} \le 1 \;; \quad (2\delta)^{\alpha-\beta} \le \tfrac{1}{2} \;; \\ &2^{3-1/q} 3C_s^q + C_4 L + 48\bar{C}_0 \left(2^q 40 C_5 \delta^{q-\gamma} + cC_5 n_0^\gamma \Lambda^{-n_0\gamma} + 2^q C_5 n_0 \Lambda^{-n_0} \delta \right) < c. \end{aligned}$$

Finally, in anticipation of (6.19), we require,

$$cA > 2C_s \,. \tag{5.34}$$

These are all the conditions we shall place on the parameters for the cone, except for δ, which we will take as small as required for the mixing arguments of Section 6.

6 Contraction of L and Finite Diameter

In this section, we use the mixing property of T to prove that the parameter L also contracts. This is done in two steps. In Section 6.1, we use a length scale $\delta_0 \ge \sqrt{\delta}$ and compare averages on the two length scales, δ and δ_0, culminating in Proposition 6.3. In Section 6.2, we obtain a bound on averages in the length scale δ_0. This leads to the strict contraction of L established in Theorem 6.10, which proves the first statement of Theorem 2.1. We prove the second statement of Theorem 2.1 in Section 6.3, showing that the cone $\mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$ has finite diameter in the cone $\mathcal{C}_{c,A,L}(\delta)$ (Proposition 6.11).

6.1 Comparing averages on different length scales

Recall the length scale δ_0 from (3.6) and that $\delta < \delta_0/2$. We choose δ so that $\delta \le \delta_0^2$. Define

$$|||f|||_+^0 = \sup_{\substack{W \in \mathcal{W}^s(\delta_0/2) \\ \psi \in \mathcal{D}_{a,\beta}(W)}} \frac{\int_W f\, \psi \, dm_W}{\int_W \psi \, dm_W}, \qquad |||f|||_-^0 = \inf_{\substack{W \in \mathcal{W}^s(\delta_0/2) \\ \psi \in \mathcal{D}_{a,\beta}(W)}} \frac{\int_W f\, \psi \, dm_W}{\int_W \psi \, dm_W}.$$

Recall that $\mathcal{W}^s(\delta_0/2)$ denotes those curves in $\mathcal{W}^s$ with length between $\delta_0/2$ and δ_0. By subdividing curves of with length in $[\delta_0/2, \delta_0]$ into curves with length in $[\delta, 2\delta]$, we immediately deduce the relations,

$$|||f|||_- \le |||f|||_-^0 \le |||f|||_+^0 \le |||f|||_+ . \tag{6.1}$$

Lemma 6.1. *Recall $e^{a\delta_0^\beta} \le 2$ from (4.9) and $A\delta \le \delta_0/4$ from Lemma 5.4. For all $n \in \mathbb{N}$,*[4]

$$|||\mathcal{L}^n f|||_+^0 \le |||f|||_+^0 + 3C_0 \sum_{i=1}^n \theta_1^i |||f|||_+ \le |||f|||_+^0 + \frac{1}{4}|||f|||_+ , \tag{6.2}$$

$$|||\mathcal{L}^n f|||_-^0 \ge \frac{3}{4}|||f|||_-^0 . \tag{6.3}$$

Proof. We prove (6.2) by induction on n. It holds trivially for $n = 0$. We assume the inequality holds for $0 \le k \le n-1$ and prove the statement for n.

Let $W \in \mathcal{W}^s(\delta_0/2)$. Define $\widehat{L}_1(W)$ to be those elements of $\mathcal{G}_1(W)$ having length at least $\delta_0/2$. For $k > 1$, let $\widehat{L}_k(W)$ denote those curves of length at least $\delta_0/2$ in $\mathcal{G}_k(W)$ that are not already contained in an element of $\widehat{L}_i(W)$ for any $i = 1, \dots, k-1$. For $V_j \in \widehat{L}_k(W)$, let $P_k(j)$ be the collection of indices i such that $W_i \in \mathcal{G}_n(W)$ satisfies $T^{n-k}W_i \subset V_j$. Denote by $\mathcal{I}_n^0(W)$ those indices i for which $T^{n-k}W_i$ is never contained in an element of $\mathcal{G}_k(W)$ of length at least $\delta_0/2$, $1 \le k \le n$, and $\delta \le |W_i| < \delta_0/2$. Let $\mathcal{I}_n(W)$ denote the remainder of the indices i for curves in $\mathcal{G}_n(W)$, i.e. those curves W_i of length shorter than δ and for which $T^{n-k}W_i$ is not contained in an element of $\mathcal{G}_k(W)$ of length at least $\delta_0/2$. By construction, each $W_i \in \mathcal{G}_n(W)$ belongs to precisely one $P_k(j)$ or $\mathcal{I}_n^0(W)$ or $\mathcal{I}_n(W)$.

Now, for $\psi \in \mathcal{D}_{a,\beta}(W)$, note that

$$\sum_{i \in P_k(j)} \int_{W_i} f\, \psi \circ T^n \, J_{W_i}T^n = \int_{V_j} \mathcal{L}^{n-k} f\, \psi \circ T^k \, J_{V_j}T^k.$$

Using this equality, we estimate,

$$\begin{aligned}
\int_W \mathcal{L}^n f\, \psi &= \sum_{k=1}^n \sum_{V_j \in \widehat{L}_k(W)} \int_{V_j} \mathcal{L}^{n-k} f\, \psi \circ T^k \, J_{V_j}T^k \;+ \sum_{i \in \mathcal{I}_n^0(W)} \int_{W_i} f\, \psi \circ T^n \, J_{W_i}T^n \\
&\quad + \sum_{i \in \mathcal{I}_n(W)} \int_{W_i} f\, \psi \circ T^n \, J_{W_i}T^n \\
&\le \sum_{k=1}^n \sum_{V_j \in \widehat{L}_k(W)} |||\mathcal{L}^{n-k} f|||_+^0 \int_{V_j} \psi \circ T^k \, J_{V_j}T^k \;+ \sum_{i \in \mathcal{I}_n^0(W)} |||f|||_+ \int_{W_i} \psi \circ T^n \, J_{W_i}T^n \\
&\quad + \sum_{i \in \mathcal{I}_n(W)} A\delta^{1-q}|W_i|^q |||f|||_- |\psi|_{C^0(W)} |J_{W_i}T^n|_{C^0(W_i)} \\
&\le \sum_{k=1}^n \sum_{V_j \in \widehat{L}_k(W)} \Big(|||f|||_+^0 + 3\sum_{i=1}^{n-k} C_0\theta_1^i |||f|||_+ \Big) \int_{T^k V_j} \psi \\
&\quad + \sum_{i \in \mathcal{I}_n^0(W)} |||f|||_+ \frac{\delta_0}{2} |\psi|_{C^0(W)} |J_{W_i}T^n|_{C^0(W_i)} + A\frac{\delta}{\delta_0}\delta_0 |\psi|_{C^0(W)} |||f|||_+ C_0\theta_1^n \\
&\le \int_W \psi \Big(|||f|||_+^0 + 3\sum_{i=1}^{n-1} C_0\theta_1^i |||f|||_+ \Big) + \Big(1 + 2A\frac{\delta}{\delta_0}\Big) e^{a\delta_0^\beta} \int_W \psi \; |||f|||_+ C_0\theta_1^n,
\end{aligned}$$

[4]The second inequality in (6.2) follows from equation (3.6).

where for the second inequality we have used the inductive hypothesis, and for the second and third we have used Lemmas 3.1-(a) and 4.2. This proves the required inequality if δ_0 is small enough that $e^{a\delta_0^\beta} \le 2$ and δ is small enough that $A\delta \le \delta_0/4$, both of which we have assumed.

We prove (6.3) similarly, although now the inductive hypothesis is $|||\mathcal{L}^k f|||_-^0 \ge (1 - 3\sum_{i=1}^k C_0\theta_1^i)$ for each $k = 0, \dots, n-1$. We begin with the same decompostion of $\mathcal{G}_n(W)$, although we simply drop the terms in $\mathcal{I}_n^0(W)$ since they are all positive.

$$
\begin{aligned}
\int_W \mathcal{L}^n f\, \psi &= \sum_{k=1}^n \sum_{V_j \in \widehat{L}_k(W)} \int_{V^j} \mathcal{L}^{n-k} f\, \psi \circ T^k\, J_{V_j}T^k \;+ \sum_{i \in \mathcal{I}_n^0(W)} \int_{W_i} f\, \psi \circ T^n\, J_{W_i}T^n \\
&\quad + \sum_{i \in \mathcal{I}_n(W)} \int_{W_i} f\, \psi \circ T^n\, J_{W_i}T^n \\
&\ge \sum_{k=1}^n \sum_{V_j \in \widehat{L}_k(W)} |||\mathcal{L}^{n-k} f|||_-^0 \int_{V^j} \psi \circ T^k\, J_{V_j}T^k - \sum_{i \in \mathcal{I}_n(W)} A\delta^{1-q}|W_i|^q |||f|||_- |\psi|_{C^0(W)} |J_{W_i}T^n|_{C^0(W_i)} \\
&\ge \sum_{k=1}^n \sum_{V_j \in \widehat{L}_k(W)} \int_{T^kV_j} \psi\; |||f|||_-^0 \Big(1 - 3\sum_{i=1}^{n-k} C_0\theta_0^i\Big) - A\frac{\delta}{\delta_0}\delta_0 |\psi|_{C^0(W)} |||f|||_- C_0\theta_1^n \\
&\ge \int_W \psi\; |||f|||_-^0 \Big(1 - 3\sum_{i=1}^{n-1} C_0\theta_1^i\Big) - 2A\frac{\delta}{\delta_0} e^{a\delta_0^\beta} \int_W \psi\; |||f|||_-^0 C_0\theta_1^n \\
&\quad - |||f|||_-^0 \Big(1 - 3\sum_{i=1}^{n-1} C_0\theta_1^i\Big) \sum_{i \in \mathcal{I}_n(W) \cup \mathcal{I}_n^0(W)} |W_i| |\psi|_{C^0(W)} |J_{W_i}T^n|_{C^0(W_i)} \\
&\ge \int_W \psi\; |||f|||_-^0 \Big(1 - 3\sum_{i=1}^{n-1} C_0\theta_1^i - 2A\frac{\delta}{\delta_0} e^{a\delta_0^\beta} C_0\theta_1^n - e^{a\delta_0^\beta} C_0\theta_1^n\Big),
\end{aligned}
$$

where again we have used Lemmas 3.1(a) and 4.2 as well as the bound $|||f|||_- \le |||f|||_-^0$. This proves the inductive claim, and from this, (6.3) follows from (3.6). □

Next, we have a partial converse of Lemma 6.1.

Lemma 6.2. *For all $n \ge \frac{\log(8C_0(L\delta_0\delta^{-1} + 2A))}{|\log\theta_1|}$, we have*

$$
\begin{aligned}
|||\mathcal{L}^n f|||_+ &\le \max_{k=0,\dots n-1} |||\mathcal{L}^k f|||_+^0 + \frac{1}{8}|||f|||_- \\
|||\mathcal{L}^n f|||_- &\ge \frac{3}{4} \min_{k=0,\dots n-1} |||\mathcal{L}^k f|||_-^0 - \frac{1}{8}|||f|||_-
\end{aligned}
$$

Proof. The proof follows along the lines of the proof of Lemma 6.1, using the same decomposition into $\hat{L}_k(W)$, $\mathcal{I}_n^0(W)$ and $\mathcal{I}_n(W)$, except that now we begin with $W \in \mathcal{W}^s(\delta)$ and $\psi \in \mathcal{D}_{a,\beta}(W)$. We have,

$$\begin{aligned}
\int_W \mathcal{L}^n f\,\psi &\le \sum_{k=1}^n \sum_{V_j\in \widehat{L}_k(W)} |||\mathcal{L}^{n-k} f|||_+^0 \int_{V^j} \psi\circ T^k\, J_{V_j}T^k \;+\; \sum_{i\in \mathcal{I}_n^0(W)} |||f|||_+ \int_{W_i} \psi\circ T^n\, J_{W_i}T^n \\
&\quad + \sum_{i\in\mathcal{I}_n(W)} A\delta^{1-q}|W_i|^q |||f|||_- |\psi|_{C^0(W)} |J_{W_i}T^n|_{C^0(W_i)} \\
&\le \int_W \psi \max_{k=0,\dots n-1} |||\mathcal{L}^k f|||_+^0 + |||f|||_+ C_0\theta_1^n \frac{\delta_0}{\delta}\int_W \psi + 2A|||f|||_- C_0\theta_1^n \int_W \psi \\
&\le \int_W \psi\,\Big(\max_{k=0,\dots n-1} |||\mathcal{L}^k f|||_+^0 + |||f|||_- C_0\theta_1^n (L\delta_0\delta^{-1} + 2A)\Big)\,,
\end{aligned}$$

which proves the first inequality, given our assumed bound on n.

The second inequality follows similarly, again along the lines of Lemma 6.1.

$$\begin{aligned}
\int_W \mathcal{L}^n f\,\psi &\ge \sum_{k=1}^n \sum_{V_j\in \widehat{L}_k(W)} |||\mathcal{L}^{n-k} f|||_-^0 \int_{V^j} \psi\circ T^k\, J_{V_j}T^k - \sum_{i\in\mathcal{I}_n(W)} A\delta^{1-q}|W_i|^q |||f|||_- |\psi|_{C^0(W)} |J_{W_i}T^n|_{C^0(W_i)} \\
&\ge \min_{k=0,\dots n-1} |||\mathcal{L}^k f|||_-^0 \left(\int_W \psi - \sum_{i\in\mathcal{I}_n(W)\cup\mathcal{I}_n^0(W)} |W_i||\psi|_{C^0(W)}|J_{W_i}T^n|_{C^0(W_i)}\right) - 2A\int_W \psi\; |||f|||_- C_0\theta_1^n \\
&\ge \int_W \psi\,\left(\min_{k=0,\dots n-1} |||\mathcal{L}^k f|||_-^0 (1-\delta_0\delta^{-1}C_0\theta_1^n) - 2AC_0\theta_1^n |||f|||_-\right)\,,
\end{aligned}$$

and our bound on n suffices to complete the proof of the lemma. □

Finally, we collect these estimates in the following proposition. Set

$$N(\delta)^- = \frac{\log(8C_0(L\delta_0\delta^{-1}+2A))}{|\log\theta_1|}, \tag{6.4}$$

from Lemma 6.2.

Proposition 6.3. *For all $n \ge N(\delta)^-$, either,*

$$\frac{|||\mathcal{L}^n f|||_+}{|||\mathcal{L}^n f|||_-} \le \frac{8}{9}\frac{|||f|||_+}{|||f|||_-}\,,$$

or

$$|||\mathcal{L}^n f|||_+ \le 8|||f|||_+^0 \quad \textit{and} \quad |||\mathcal{L}^n f|||_- \ge \frac{9}{20}|||f|||_-^0\,.$$

Proof. Since $n \ge N(\delta)^- \ge n_0$, we may apply both Lemmas 5.4 and 6.2. Now, by Lemma 6.2,

$$|||\mathcal{L}^n f|||_- \ge \frac{3}{4}\max_{k=0,\dots n-1} |||\mathcal{L}^k f|||_-^0 - \frac{1}{8}|||f|||_- \ge \frac{9}{16}|||f|||_-^0 - \frac{1}{4}|||\mathcal{L}^n f|||_-\,,$$

applying Lemma 6.1 to the first term and Lemma 5.4 to the second. This yields immediately, $|||\mathcal{L}^n f|||_- \ge \frac{9}{20}|||f|||_-^0$, which is the final inequality in the statement of the lemma.

Now consider the following alternatives. If $|||\mathcal{L}^n f|||_+ \le \frac{2}{5}|||f|||_+$, then

$$\frac{|||\mathcal{L}^n f|||_+}{|||\mathcal{L}^n f|||_-} \le \frac{\frac{2}{5}|||f|||_+}{\frac{9}{20}|||f|||_-^0} \le \frac{8}{9}\frac{|||f|||_+}{|||f|||_-}$$

proving the first alternative. On the other hand, if $|||\mathcal{L}^n f|||_+ \geq \frac{2}{5}|||f|||_+$, then using Lemmas 6.2, 6.1 and 5.4,

$$|||\mathcal{L}^n f|||_+ \leq \max_{k=0,\ldots n-1} |||\mathcal{L}^k f|||_+^0 + \frac{1}{8}|||f|||_- \leq |||f|||_+^0 + \frac{1}{4}|||f|||_+ + \frac{1}{4}|||\mathcal{L}^n f|||_-$$
$$\leq |||f|||_+^0 + \frac{7}{8}|||\mathcal{L}^n f|||_+ \,,$$

which yields the second alternative. □

6.2 Mixing implies contraction of L

The importance of Proposition 6.3 is that either L contracts within $N(\delta)^-$ iterates or we can compare ratios of integrals on the length scale δ_0 (which is fixed independently of δ). In the latter case we will use the mixing property of T in order to compare the value of $\int_W \mathcal{L}^n f \psi$ for different W of length approximately δ_0. To this end, we will define a Cantor set R_* comprised of local stable and unstable manifolds of a certain length in order to make our comparison when curves cross this set.

We construct an approximate rectangle D in M, contained in a single homogeneity strip, whose boundaries are comprised of two local stable and two local unstable manifolds as follows. Choose $\bar{\delta}_0 > 0$ and $x \in M$ such that $\mathrm{dist}(T^{-n}x, \mathcal{S}_1^{\mathbb{H}}) \geq \bar{\delta}_0 \Lambda^{-|n|}$ for all $n \in \mathbb{Z}$. This implies that the homogenous local stable and unstable manifolds of x, $W^s_{\mathbb{H}}(x)$ and $W^u_{\mathbb{H}}(x)$, have length at least $\bar{\delta}_0$ on either side of x. By the Sinai Theorem applied to homogeneous unstable manifolds (see, for example, [CM, Theorem 5.70]), we may choose $\delta_0 < \bar{\delta}_0$ such that more than 9/10 of the measure of points in $W^u_{\mathbb{H}}(x) \cap B_{\delta_0}(x)$ have homogeneous local stable manifolds longer than $2\delta_0$ on both sides of $W^u_{\mathbb{H}}(x)$, and analogously for the points in $W^s_{\mathbb{H}}(x) \cap B_\delta(x)$. Let D'_{δ_0} denote the minimal solid rectangle containing this set of stable and unstable manifolds. There must exist a rectangle D fully crossing D' in the stable direction and with boundary comprising two stable and two unstable manifolds, such that the unstable diameter of D is between δ_0^4 and $2\delta_0^4$ and the set of local homogeneous stable and unstable manifolds fully crossing D comprise at least 3/4 of the measure of D with respect to μ_{SRB}; otherwise, at most 3/4 of the measure of $W^u_{\mathbb{H}}(x) \cap B_{\delta_0}(x)$ would have long stable manifolds on either side of $W^u_{\mathbb{H}}(x)$, contradicting our choice of δ_0.

Let R_* denote the maximal set of homogeneous stable and unstable manifolds in D that fully cross D. By construction, $\mu_{\mathrm{SRB}}(R_*) > (3/4)\mu_{\mathrm{SRB}}(D) \approx \delta_0^5$. Below, we denote D by $D(R_*)$ since it is the minimal solid rectangle that defines R_*.

We say that a stable curve W *properly crosses* a Cantor rectangle R (in the stable direction) if W intersects the interior of the solid rectangle $D(R)$, but does not terminate in $D(R)$, and does not intersect the two stable manifolds contained in $\partial D(R)$.

Lemma 6.4. *There exists $n_* \in \mathbb{N}$, depending only on δ_0, such that for all $W \in \mathcal{W}^s$ with[5] $|W| \geq \delta_0/(6\bar{C}_0)$, and all $n \geq n_*$, $T^{-n}W$ contains a connected, homogeneous component that properly crosses R_*.*

Proof. By [CM, Lemma 7.87], there exist finitely many Cantor rectangles $\mathcal{R}(\delta_0) = \{R_1, \ldots, R_k\}$, with $\mu_{\mathrm{SRB}}(R_i) > 0$ for each i, such that any stable curve $W \in \mathcal{W}^s$ with $|W| \geq \delta_0/(6\bar{C}_0)$ properly crosses at least one of them. Let $\varepsilon_{\mathcal{R}}$ to be the minimum length of an unstable manifold in R_i, for any $R_i \in \mathcal{R}(\delta_0)$.

Consider the solid rectangle $D'(R_*) \subset D(R_*)$ which crosses $D(R_*)$ fully in the stable direction, but comprises the approximate middle 1/2 of $D(R_*)$ in the unstable direction, with approximately

[5] Recall that $\bar{C}_0$ is from Lemma 3.1.

1/4 of the unstable diameter of $D(R_*)$ on each side of $D'(R_*)$. Let $R'_* := R_* \cap D'(R_*)$ and note that $\mu_{\rm SRB}(R'_*) > 0$ since $\mu_{\rm SRB}(R_*) > (3/4)\mu_{\rm SRB}(D)$ by construction.

Now given $W \in \mathcal{W}^s$ with $|W| \ge \delta_0/(6\bar{C}_0)$, let $R_i \in \mathcal{R}(\delta_0)$ denote the Cantor rectangle which W crosses properly. By the mixing property of T, there exists $n_i^* > 0$ such that for all $n \ge n_i^*$, $T^n(R'_*) \cap R_i \neq \emptyset$. We may increase n_i^* if necessary so that $\Lambda^{n_i^*}\delta_0^4/8 \ge \varepsilon_{\mathcal{R}}$. We claim that $T^n(R_*)$ properly crosses R_i in the unstable direction for all $n \ge n_i^*$. If not, then the unstable manifolds comprising R_* must be cut by a singularity curve in $\mathcal{S}_1^{\mathbb{H}}$ before time n_i^* (since otherwise they would be longer than $2\varepsilon_{\mathcal{R}}$ by choice of n_i^*), and the images of those unstable manifolds must terminate on the unstable manifolds in R_i. But this implies that some unstable manifolds in R_i will be cut under T^{-n}, a contradiction.

Since $T^n(R_*)$ properly crosses R_i in the unstable direction, it follows that $T^n(D(R_*))$ contains a subinterval of W (here we use the fact that the stable manifolds of R_* cannot be cut under T^n, as well as that the singularity curves of T^n can only terminate on other elements of $\mathcal{S}_n^{\mathbb{H}}$ [CM, Proposition 4.47]), call it V. Thus $T^{-n}V$ properly crosses R_*, as required.

Since $\mathcal{R}(\delta_0)$ is finite, setting $n_* = \max_{1\le i\le k}\{n_i^*\} < \infty$ completes the proof of the lemma. □

Lemma 6.5. *Let $W^1, W^2 \in \mathcal{W}^s$ and $n \ge 0$. Suppose $U_1 \in \mathcal{G}_n(W^1)$ and $U_2 \in \mathcal{G}_n(W^2)$ properly cross R_* and define $\bar{U}_i = U_i \cap D(R_*)$, $i = 1, 2$. Then there exists $C_7 > 0$, depending only on the maximum slope and maximum curvature $\bar{B}$ of curves in $\mathcal{W}^s$, such that $d_{\mathcal{W}^s}(\bar{U}_1, \bar{U}_2) \le C_7\delta_0^2$.*

Proof. Define a foliation of vertical line segments covering $D(R_*)$. Due to the uniform transversality of the stable cone with the vertical direction, it is clear that the length of the segments connecting $\bar{U}_1$ and $\bar{U}_2$ have length at most $C_3\delta_0^4$, where $C_3 > 0$ depends only on the maximum slope in $C^s(x)$. Moreover, the unmatched parts of $\bar{U}_1$ and $\bar{U}_2$ near the boundary of $D(R_*)$ also have length at most $C_3\delta_0^4$.

Recalling the definition of $d_{\mathcal{W}^s}(\cdot,\cdot)$, it remains to estimate the C^1 distance between the graphs of $\bar{U}_1$ and $\bar{U}_2$. Denote by $\varphi_1(r)$ and $\varphi_2(r)$ the functions defining $\bar{U}_1$ and $\bar{U}_2$ on a common interval $I = I_{\bar{U}_1} \cap I_{\bar{U}_2}$. Let $\varphi_i' = \frac{\varphi_i}{dr}$. For $x \in \bar{U}_1$ over I, let $\bar{x} \in \bar{U}_2$ denote the point on the same vertical line segment as x.

Suppose there exists $x \in \bar{U}_1$ over I such that $|\varphi_1'(r(x)) - \varphi_2'(r(\bar{x}))| > C\delta_0^2$ for some $C > 0$, where $r(x)$ denotes the r-coordinate of $x = (r, \varphi)$. Since the curvature of each U_i is bounded by $\bar{B}$ by definition, we have $|\varphi_i''| \le \bar{B}(1 + (K_{\max} + \tau_{\min}^{-1})^2)^{3/2} =: \bar{C}_7$.

Now consider an interval $J \subset I$ of radius δ_0^2 centered at $r(x)$. Then $|\varphi_1'(r) - \varphi_1'(r(x))| \le \bar{C}_7|r - r(x)|$ for all $r \in J$, and similarly for φ_2'. Thus,

$$|\varphi_1'(r) - \varphi_2'(r)| \ge C\delta_0^2 - 2\bar{C}_7\delta_0^2 = (C - 2\bar{C}_7)\delta_0^2 \text{ for all } r \in J.$$

This in turn implies that there exists $r \in J$ such that $|\varphi_1(r) - \varphi_2(r)| \ge (C - 2\bar{C}_7)\delta_0^4$, which is a contradiction if $C - 2\bar{C}_7 > C_3$. This proves the lemma with $C_7 = 2\bar{C}_7 + C_3$. □

Recall that by Lemma 4.1, for $W \in \mathcal{W}^s$ the cone $\mathcal{D}_{a,\alpha}(W)$ has finite diameter in $\mathcal{D}_{a,\beta}(W)$ for $\alpha > \beta$, so that

$$\rho_{W,a,\beta}(g_1, g_2) \le D_0 \quad \text{for all } g_1, g_2 \in \mathcal{D}_{a,\alpha}(W) \tag{6.5}$$

for some constant $D_0 > 0$ depending only on a, α and β. Without loss of generality, we take $D_0 \ge 1$.

Lemma 6.6. *Suppose $W^1, W^2 \in \mathcal{W}^s$ with $|W^1|, |W^2| \in [\delta_0/3, \delta_0]$ and $d_{\mathcal{W}^s}(W^1, W^2) \le C_7\delta_0^2$. Assume $\psi_\ell \in \mathcal{D}_{a,\alpha}(W^\ell)$ with $\int_{W^1}\psi_1 = \int_{W^2}\psi_2 = 1$.*

Recall that $\delta \le \delta_0^2$. Let $C > 0$ be such that if $n \ge C\log(\delta_0/\delta)$ then $C_5 n\Lambda^{-n} \le \delta/\delta_0^2$, where C_5 is from Lemma 5.5. For all n such that $n \ge C\log(\delta_0/\delta) \ge 2n_0$, we have

$$\frac{\int_{W^1} \mathcal{L}^n f\,\psi_1}{\int_{W^2} \mathcal{L}^n f\,\psi_2} \le 2$$

provided

$$\left[\frac{2\bar{C}_0 C_3 C_7 (3LA\delta^{1-q}\delta_0^{2q} + 3L\delta_0^2)}{1-\Lambda^{-q}} + 2\bar{C}_0 A\delta^{1-q}(2\delta^q + c\delta^{\gamma+q} + D_0\delta^q + 3\delta_0^q)\right] 6e^{2a\delta_0^\alpha} \le \delta_0.$$

Remark 6.7. *Since $\delta \le \delta_0^2$, the condition of Lemma 6.6 will be satisfied if*

$$\left[\frac{2\bar{C}_0 C_3 C_7 3LA\delta_0 + 3L\delta_0)}{1-\Lambda^{-q}} + 2\bar{C}_0 A\delta_0^{1-q}(2\delta_0^q + c\delta_0^{2\gamma+q} + D_0\delta_0^q + 3)\right] 6e^{2a\delta_0^\alpha} \le 1. \tag{6.6}$$

This will determine our choice of δ_0.

Proof. We will change variables to integrate on $T^{-n}W^\ell$, $\ell = 1, 2$. As in Section 5.2.3, we split $\mathcal{G}_n(W^\ell)$ into matched pieces $\{U_j^\ell\}_j$ and unmatched pieces $\{V_j^\ell\}_j$. Corresponding matched pieces U_j^1 and U_j^2 are defined as graphs $G_{U_j^\ell}$ over the same r-interval I_j and are connected by a foliation of vertical line segments. Following (5.12), we write,

$$\int_{W^\ell} \mathcal{L}^n f\,\psi_\ell = \sum_j \int_{U_j^\ell} f\,\widehat{T}^n_{U_j^\ell}\psi_\ell + \sum_j \int_{V_j^\ell} f\,\widehat{T}^n_{V_j^\ell}\psi_\ell,$$

where $\widehat{T}^n_{U_j^\ell}\psi_\ell := \psi_\ell \circ T^n\, J_{U_j^\ell}T^n$, and similarly for $\widehat{T}^n_{V_j^\ell}\psi_\ell$, $\ell = 1, 2$.

We perform the estimate over unmatched pieces first, following the same argument as in Section 5.2.3 to conclude that $|T^{n-i-1}V_j^1| \le C_3\Lambda^{-i} d_{\mathcal{W}^s}(W^1, W^2) \le C_3 C_7 \Lambda^{-i}\delta_0^2$, for any curve V_j^1 created at time i, $0 \le i \le n-1$.

Recalling the sets $P(i)$ from Section 5.2.3 of unmatched pieces created at time i, we split the estimate into curves $P(i;S)$ if $|T^{n-i-1}V_j^1| < \delta$ and curves $P(i;L)$ if $|T^{n-i-1}V_j^1| \ge \delta$.

The estimate over short unmatched pieces is given by,

$$\begin{aligned}
\sum_{i=0}^{n-1}\sum_{j\in P(i;S)} \left|\int_{V_j^1} f\,\widehat{T}^n_{V_j^1}\psi_1\right| &= \sum_{i=0}^{n-1}\sum_{j\in P(i;S)} \left|\int_{T^{n-i-1}V_j^1} \mathcal{L}^{n-i-1}f\cdot\psi_1\circ T^{i+1}\, J_{T^{n-i-1}V_j^1}T^{i+1}\right| \\
&\le \sum_{i=0}^{n-1}\sum_{j\in P(i;S)} A\delta^{1-q}C_3^q\Lambda^{-iq} d_{\mathcal{W}^s}(W^1,W^2)^q |||\mathcal{L}^{n-i-1}f|||_- |\psi_1|_{C^0} |J_{T^{n-i-1}V_j^1}T^{i+1}|_{C^0} \\
&\le \frac{\bar{C}_0 A}{1-\Lambda^{-q}} C_3^q C_7^q \delta_0^{2q} 3L |||\mathcal{L}^n f|||_- \delta^{1-q} |\psi_1|_{C^0}\,,
\end{aligned} \tag{6.7}$$

where we have used Lemma 3.1-(b), $|W| \in [\delta_0/3, \delta_0]$, and Remark 3.2 to estimate the sum over the Jacobians, as well as (5.13) to estimate $|||\mathcal{L}^{n-i-1}f|||_- \le 3L|||\mathcal{L}^n f|||_-$.

For the estimate over long pieces, we subdivide them into curves of length between δ and 2δ

and estimate them by $\|\mathcal{L}^{n-i-1}f\|_+$, then we recombine them to obtain,

$$\begin{aligned}\sum_{i=0}^{n-1}\sum_{j\in P(i;L)}\left|\int_{V_j^1} f\,\widehat{T}^n_{V_j^1}\psi_1\right| &= \sum_{i=0}^{n-1}\sum_{j\in P(i;L)}\left|\int_{T^{n-i-1}V_j^1}\mathcal{L}^{n-i-1}f\cdot\psi_1\circ T^{i+1}\,J_{T^{n-i-1}V_j^1}T^{i+1}\right|\\ &\le \sum_{i=0}^{n-1}\sum_{j\in P(i;L)}\|\mathcal{L}^{n-i-1}f\|_+\int_{T^{n-i-1}V_j^1}\psi_1\circ T^{i+1}\,J_{T^{n-i-1}V_j^1}T^{i+1}\\ &\le 3L\|\mathcal{L}^n f\|_-\sum_{i=0}^{n-1}\sum_{j\in P(i;L)}|T^{n-i-1}V_j^1||\psi_1|_{C^0}|J_{T^{n-i-1}V_j^1}T^{i+1}|_{C^0}\\ &\le \frac{C_3C_7\bar{C}_0}{1-\Lambda^{-1}}\delta_0^2 3L\|\mathcal{L}^n f\|_-|\psi_1|_{C^0}\,,\end{aligned}\tag{6.8}$$

where, in third line we used (5.13), and in the fourth line, since $|W^1|\ge\delta_0/3$, we used Remark 3.2 to drop the second term in Lemma 3.1(b).

Next, we estimate the integrals over the matched pieces U_j^1. We argue as in Section 5.2.3, but our estimates here are somewhat simpler since we do not need to show that parameters contract.

We first treat the matched short pieces with $|U_j^1|<\delta$ much as we did the unmatched ones. By Lemma 5.5, $d_{\mathcal{W}^s}(U_j^1,U_j^2)\le C_5 n\Lambda^{-n}d_{\mathcal{W}^s}(W^1,W^2)\le\delta$, since we have chosen $n\ge C\log(\delta_0/\delta)$. Thus if $|U_j^1|<\delta$ then $|U_j^2|<2\delta$, and the analogous fact holds for short curves $|U_j^2|<\delta$. With this perspective, we call U_j^ℓ short if either $|U_j^1|<\delta$ or $|U_j^2|<\delta$. On short pieces, we apply (4.7)

$$\sum_{j\text{ short}}\left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1\right|\le\sum_{j\text{ short}}2A\delta\|f\|_-|\psi_1|_{C^0}|J_{U_j^1}T^n|_{C^0}\le 4A\delta\|\mathcal{L}^n f\|_-\bar{C}_0|\psi_1|_{C^0}\,,\tag{6.9}$$

where we have again used Lemmas 3.1(b) and 5.4 for the second inequality. Remark that the same argument holds for W^2 with test function ψ_2.

Finally, to estimate the integrals over matched curves with $|U_j^1|,|U_j^2|\ge\delta$ we follow equation (5.18), recalling (5.15), although we no longer have Lemma 5.5(c) at our disposal,

$$\begin{aligned}&\left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1-\int_{U_j^2} f\,\widehat{T}^n_{U_j^2}\psi_2\right|\le\left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1-\int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2\right|\\ &+\left|\frac{\int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2}{\fint_{U_j^1}\widetilde{T}^n_{U_j^2}\psi_2}-\frac{\int_{U_j^2} f\,\widehat{T}^n_{U_j^2}\psi_2}{\fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2}\right|\fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2+\left|\frac{\int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2}{\fint_{U_j^1}\widetilde{T}^n_{U_j^2}\psi_2}\right|\left|\frac{|U_j^2|-|U_j^1|}{|U_j^1|}\right|\fint_{U_j^2}\widehat{T}^n_{U_j^2}\psi_2\\ &\le\left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1-\int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2\right|+d_{\mathcal{W}^s}(U_j^1,U_j^2)^\gamma\delta^{1-\gamma}cA\|f\|_-|J_{U_j^2}T^n|_{C^0}|\psi_2|_{C^0}\\ &\quad+A\delta d_{\mathcal{W}^s}(U_j^1,U_j^2)\|f\|_-|J_{U_j^2}T^n|_{C^0}|\psi_2|_{C^0}\,,\end{aligned}\tag{6.10}$$

where we have used (5.22) to estimate $\left|\frac{|U_j^2|-|U_j^1|}{|U_j^1|}\right|$.

To estimate the first term on the right side above, we use (4.7) and Lemma 4.7,

$$\begin{aligned}\left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1 - \int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2\right| &\le \left|\frac{\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1}{\fint_{U_j^1}\widehat{T}^n_{U_j^1}\psi_1} - \frac{\int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2}{\fint_{U_j^1}\widetilde{T}^n_{U_j^2}\psi_2}\right| \fint_{U_j^1}\widehat{T}^n_{U_j^1}\psi_1 \\ &\quad + \frac{\int_{U_j^1} f\,\widetilde{T}^n_{U_j^2}\psi_2}{\fint_{U_j^1}\widetilde{T}^n_{U_j^2}\psi_2}\left|\fint_{U_j^1}\widehat{T}^n_{U_j^1}\psi_1 - \fint_{U_j^1}\widetilde{T}^n_{U_j^2}\psi_2\right| \\ &\le 2\delta L\rho(\widehat{T}^n_{U_j^1}\psi_1, \widetilde{T}^n_{U_j^2}\psi_2)|\!|\!|f|\!|\!|_-|J_{U_j^1}T^n|_{C^0}|\psi_1|_{C_0} \\ &\quad + A\delta^{1-q}\delta_0^q|\!|\!|f|\!|\!|_-\big(|J_{U_j^1}T^n|_{C^0}|\psi_1|_{C_0} + 2|J_{U_j^2}T^n|_{C^0}|\psi_2|_{C_0}\big)\,,\end{aligned}$$

where we have used $|U_j^1| \le \delta_0$ in the last line. We may apply (6.5) since $\widehat{T}^n_{U_j^1}\psi_1, \widetilde{T}^n_{U_j^2}\psi_2 \in \mathcal{D}_{a,\alpha}(U_j^1)$ by Lemma 5.5. Now putting the above estimate together with (6.10), recalling $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le \delta$, and using Lemma 3.1-(b) and Remark 3.2 as well as Lemma 5.4, we sum over j to obtain,

$$\begin{aligned}&\sum_{j\ \text{long}}\left|\int_{U_j^1} f\,\widehat{T}^n_{U_j^1}\psi_1 - \int_{U_j^2} f\,\widehat{T}^n_{U_j^2}\psi_2\right| \\ &\qquad \le 2A\delta^{1-q}|\!|\!|\mathcal{L}^n f|\!|\!|_-\bar{C}_0\left(c\delta^{\gamma+q} + \delta^{1+q} + \frac{2LD_0\delta^q}{A} + 3\delta_0^q\right)(|\psi_1|_{C^0} + |\psi_2|_{C^0}).\end{aligned} \tag{6.11}$$

Collecting (6.7), (6.8), (6.9) and (6.11), and recalling $D_0 \ge 1$ and $A > 4L$, yields

$$\begin{aligned}\int_{W^1}\mathcal{L}^n f\,\psi_1 &\le \frac{\bar{C}_0C_3C_7(3LA\delta^{1-q}\delta_0^{2q} + 3L\delta_0^2)}{1-\Lambda^{-q}}|\!|\!|\mathcal{L}^n f|\!|\!|_-|\psi_1|_{C^0} + 4\bar{C}_0A\delta|\!|\!|\mathcal{L}^n f|\!|\!|_-|\psi_1|_{C^0} \\ &\quad + \sum_j\int_{U_j^2} f\,\widehat{T}_{U_j^2}\psi_2 + 2A\delta^{1-q}|\!|\!|\mathcal{L}^n f|\!|\!|_-\bar{C}_0(c\delta^{\gamma+q} + D_0\delta^q + 3\delta_0^q)(|\psi_1|_{C^0} + |\psi_2|_{C^0}) \\ &\le \left\{1 + \Big[\frac{2\bar{C}_0C_3C_7(3LA\delta^{1-q}\delta_0^{2q} + 3L\delta_0^2)}{1-\Lambda^{-q}}\right. \\ &\quad \left. + 2\bar{C}_0A\delta^{1-q}(2\delta^q + c\delta^{\gamma+q} + D_0\delta^q + 3\delta_0^q)\Big]\frac{|\psi_1|_{C^0} + |\psi_2|_{C^0}}{\int_{W^2}\psi_2}\right\}\int_{W^2}\mathcal{L}^n f\,\psi_2\,.\end{aligned}$$

Now since $\int_{W^i}\psi_i = 1$, we have $e^{-a\delta_0^\alpha} \le |W^i|\psi_i \le e^{a\delta_0^\alpha}$. Thus since $|W^i| \ge \delta_0/3$,

$$\frac{|\psi_1|_{C^0} + |\psi_2|_{C^0}}{\int_{W^2}\psi_2} \le \frac{6}{\delta_0}e^{2a\delta_0^\alpha}\,,$$

which proves the Lemma. □

Our strategy will be the following. For W^1, $W^2 \in \mathcal{W}^s(\delta_0/2)$ and n sufficiently large, we wish to compare $\int_{W^1}\mathcal{L}^n f\,\psi_1$ with $\int_{W^2}\mathcal{L}^n f\,\psi_2$, where we normalize $\int_{W^1}\psi_1 = \int_{W^2}\psi_2 = 1$. By Lemmas 6.4 and 6.5, we find $U_i^\ell \in \mathcal{G}_{n_*}(W^\ell)$, $\ell = 1, 2$, such that U_i^ℓ properly crosses R_*, and $d_{\mathcal{W}^s}(\bar{U}_i^1, \bar{U}_i^2) \le C_7\delta_0^2$, where $\bar{U}_i^\ell = U_i^\ell \cap D(R_*)$.

Next, for each i, we wish to compare $\int_{\bar{U}_i^1}\mathcal{L}^{n-n_*}f\,\widehat{T}^{n_*}_{U_i^1}\psi_1$ with $\int_{\bar{U}_i^2}\mathcal{L}^{n-n_*}f\,\widehat{T}^{n_*}_{U_i^2}\psi_2$, where, as usual, $\widehat{T}^{n_*}_{U_i^\ell}\psi_\ell = \psi_\ell\circ T^{n_*}J_{U_i^\ell}T^{n_*}$. However, the weights $\int_{\bar{U}_i^\ell}\widehat{T}^{n_*}_{U_i^\ell}\psi_\ell$ may be very different for $\ell = 1, 2$ since the stable Jacobians along the respective orbits before time n_* may not be comparable. To remedy this, we adopt the following strategy for matching integrals on curves.

For each curve $U_i^\ell \in \mathcal{G}_{n_*}(W)$ which properly crosses R_*, we redefine $\bar U_i^\ell$ to denote the middle third of $U_i^\ell \cap D(R_*)$. Let M^ℓ denote the index set of such i.

Let $p_i^\ell = \int_{\bar U_i^\ell} \widehat T^{n_*}_{U_i^\ell} \psi_\ell$, and let $m_\ell = \sum_{i \in M^\ell} p_i^\ell$. Without loss of generality, assume $m_2 \ge m_1$.

We will match the integrals $\sum_{i\in M^1} \int_{\bar U_i^1} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_i^1}\psi_1$ with $\sum_{j\in M^2} \frac{m_1}{m_2} \int_{\bar U_j^2} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_j^2}\psi_2$. The remainder of the integrals $\sum_{j\in M^2} \frac{m_2-m_1}{m_2} \int_{\bar U_j^2} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_j^2}\psi_2$ as well as any unmatched pieces (including the outer two-thirds of each U_i^ℓ) we continue to iterate until such time as they can be matched as the middle third of a curve that properly crosses R_*.

Set $\widehat T^{n_*}_{U_j^2}\tilde\psi_2 = \frac{m_1}{m_2}\widehat T^{n_*}_{U_j^2}\psi_2$, and consider the following decomposition of the integrals we want to match,

$$\sum_{\substack{i\in M^1\\ j\in M^2}} \int_{\bar U_i^1} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_i^1}\psi_1 \, \frac{p_j^2}{m_2} \quad \text{and} \quad \sum_{\substack{i\in M^1\\ j\in M^2}} \int_{\bar U_j^2} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_j^2}\tilde\psi_2 \, \frac{p_i^1}{m_1}$$

For each pair i,j in the first sum, the test function has integral weight $\frac{p_i^1 p_j^2}{m_2}$, and the same is true for the corresponding pair in the second sum. Thus these integrals are paired precisely according to the assumptions of Lemma 6.6. It follows that if $n - n_* \ge C\log(\delta_0/\delta)$, then

$$\begin{aligned}
\sum_{i\in M^1} \int_{\bar U_i^1} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_i^1}\psi_1 &= \sum_{\substack{i\in M^1\\ j\in M^2}} \int_{\bar U_i^1} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_i^1}\psi_1 \frac{p_j^2}{m_2} \\
&\le 2 \sum_{\substack{i\in M^1\\ j\in M^2}} \int_{\bar U_j^2} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_j^2}\tilde\psi_2 \, \frac{p_i^1}{m_1} = 2 \sum_{j\in M^2} \int_{\bar U_j^2} \mathcal{L}^{n-n_*} f\, \widehat T^{n_*}_{U_j^2}\tilde\psi_2 \, .
\end{aligned} \tag{6.12}$$

We want to repeat the above construction until most of the mass has been compared. To this end we set up an inductive scheme. Consider the family of curves $W_i^\ell \in \mathcal{G}_{n_*}(W^\ell)$ that have not been matched. Each carries a test function $\psi_{\ell,i} := \widehat T^{n_*}_{W_i^\ell}\tilde\psi_\ell$. Renormalizing by a factor $\mathfrak{r}_{\ell,1} < 1$, we have $\sum_i \int_{W_i^\ell} \psi_{\ell,i} = 1$.

Definition 6.8. *Given a countable collection of curves and test functions, $\mathcal{F} = \{W_i, \psi_i\}_i$, with $W_i \in \mathcal{W}^s$, $|W_i| \le \delta_0$, $\psi_i \in \mathcal{D}_{a,\alpha}(W_i)$ and $\sum_i \int_{W_i} \psi_i = 1$, we call $\mathcal{F}$ an* admissible family *if*

$$\sum_i \fint_{W_i} \psi_i \le C_* \, , \quad \text{where } C_* := 3\bar C_0 \delta_0^{-1}. \tag{6.13}$$

Notice that any stable curve $W \in \mathcal{W}^s(\delta_0/2)$ together with test function $\psi \in \mathcal{D}_{a,\alpha}(W)$ normalized so that $\int_W \psi = 1$ forms an admissible family since $|W| \ge \delta_0/2$. The content of the following lemma is that an admissible family can be iterated and remain admissible; moreover, a family with larger average integral in (6.13) can be made admissible under iteration.

Lemma 6.9. *Let $\{W_i, \psi_i\}_i$ be a countable collection of curves $W_i \in \mathcal{W}^s$, $|W_i| \le \delta_0$, with functions $\psi_i \in \mathcal{D}_{a,\alpha}(W_i)$, normalized so that $\sum_i p_i = 1$, where $p_i = \int_{W_i}\psi_i$. Suppose that $\sum_i |W_i|^{-1} p_i = C_\sharp$.*

Choose $n_\sharp \in \mathbb{N}$ so that $C_0 \theta_1^{n_\sharp} \frac{C_\sharp}{C_*} \le 1/6$. Then for all $n \ge n_\sharp$, the dynamically iterated family $\{V_j^i \in \mathcal{G}_n(W_i), \widehat T^n_{V_j^i}\psi_i\}_{i,j}$ is admissible.

Proof. Setting $p_j^i = \int_{V_j^i} \widehat T^n_{V_j^i}\psi_i = \int_{V_j^i} \psi_i \circ T^n J_{V_j^i} T^n$, it is immediate that $\sum_{i,j} p_j^i = 1$.

Now fix W_i and consider $V_j^i \in \mathcal{G}_n(W_i)$. Then using Lemmas 3.1 and 4.2 we estimate,

$$\sum_j |V_j^i|^{-1} p_j^i = \sum_j \fint_{V_j^i} \psi_i \circ T^n \, J_{V_j^i} T^n \le \sum_j |\psi_i|_{C^0(W_i)} |J_{V_j^i} T^n|_{C^0(V_j^i)}$$
$$\le |\psi_i|_{C^0} (\bar{C}_0 \delta_0^{-1} |W_i| + C_0 \theta_1^n) \le \bar{C}_0 \delta_0^{-1} e^{a\delta_0^\alpha} p_i + C_0 \theta_1^n e^{a\delta_0^\alpha} |W_i|^{-1} p_i \,.$$

Using that $e^{a\delta_0^\alpha} \le 2$, we sum over i and use the assumption on the family $\{W_i, \psi_i\}_i$ to obtain,

$$\sum_{i,j} \sum_j |V_j^i|^{-1} p_j^i \le \sum_i \left(2\bar{C}_0 \delta_0^{-1} p_i + 2C_0 \theta_1^n |W_i|^{-1} p_i \right) \le 2\bar{C}_0 \delta_0^{-1} + 2C_0 \theta_1^n C_\sharp \,. \tag{6.14}$$

Thus if $n \ge n_\sharp$, the above expression is bounded by C_*, as required. □

Theorem 6.10. *Let $L \ge 60$. Suppose a, c, A and L satisfy the conditions of Section 5.3, and that in addition, $\delta \le \delta_0^2$ satisfy (6.6) and (6.17). Then there exists $\chi < 1$ and $k_* \in \mathbb{N}$ such that if $n \in \mathbb{N}$ satisfies $n \ge N(\delta)^- + k_* n_*$,[6] with k_* depending only on δ_0, L, n_* (see equation (6.16)), then $\mathcal{L}^n \mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$.*

Proof. As before, we take $f \in \mathcal{C}_{c,A,L}(\delta)$, W^1, $W^2 \in \mathcal{W}^s(\delta_0/2)$ and test functions $\psi_\ell \in \mathcal{D}_{a,\beta}(W^\ell)$ such that $\int_{W^1} \psi_1 = \int_{W^2} \psi_2 = 1$. In order to iterate the matching argument described above, we need upper and lower bounds on the amount of mass matched via the process described by (6.12).

Upper Bound on Matching. By definition of $\bar{U}_i^\ell$, for each curve U_i^ℓ that properly crosses R_* at time n_*, at least $2/3$ of the length of that curve remains not matched. Thus if $p_i = \int_{U_i^\ell} \widehat{T}^n_{U_i^\ell} \tilde{\psi}_i$, then at least $(1 - e^{a\delta_0^\alpha}/3) p_i$ remains unmatched. Using $e^{a\delta_0^\alpha} \le 2$, we conclude that at least $(1/3) p_i$ of the mass remains unmatched. Thus if $\mathfrak{r}$ denotes the total mass remaining after matching at time n_*, we have $\mathfrak{r} \ge 1/3$. Renormalizing the family by $\mathfrak{r}$, we have $\sum_i |W_i|^{-1} \frac{p_i}{\mathfrak{r}} \le 3C_*$.

By the proof of Lemma 6.9 with $C_\sharp = 3C_*$, we see that choosing $n_\sharp$ such that $6C_0 \theta_1^{n_\sharp} \le 1/3$, then the bound in (6.14) is less than C_*, and the family recovers its regularity in the sense of Lemma 6.9 after this number of iterates.

Lower Bound on Matching. By definition of admissible family, for each $\varepsilon > 0$, $\sum_{|W_i| < \varepsilon} p_i \le C_* \varepsilon$. So choosing $\varepsilon = \delta_0/(6\bar{C}_0)$, we have that

$$\sum_{|W_i| \ge \delta_0/(6\bar{C}_0)} p_i \ge \frac{1}{2} \,.$$

On each W_i with $|W_i| \ge \delta_0/(6\bar{C}_0)$, we have at least one $U_j^i \in \mathcal{G}_{n_*}(W_i)$ that properly crosses R_* by Lemma 6.4. Then denoting by $\bar{U}_j^i$ the matched part (middle third) of U_j^i, we have

$$\int_{\bar{U}_j^i} \widehat{T}^{n_*}_{U_j^i} \tilde{\psi}_i = \int_{\bar{U}_j^i} \tilde{\psi}_i \circ T^{n_*} \, J_{U_j^i} T^{n_*} \ge \tfrac{\delta_0}{3} \inf \tilde{\psi}_i \inf J_{U_j^i} T^{n_*}$$
$$\ge \tfrac{1}{3} e^{-a\delta_0^\alpha} p_i e^{-C_d \delta_0^{1/3}} \frac{|T^{n_*} U_j^i|}{|U_j^i|} \ge \frac{1}{12} p_i \frac{C^{n_*} \delta_0^{(5/3)^{n_*}}}{\delta_0} =: \varepsilon_{n_*} p_i \,,$$

where we have used the fact that if $W \in \mathcal{W}^s$ and $T^{-1}W$ is a homogeneous stable curve, then $|T^{-1}W| \le C^{-1} |W|^{3/5}$ for some constant $C > 0$ (see, for example [DZ3, eq. (6.9)]).

[6] Recall that n_* is defined in Lemma 6.4 while $N(\delta)^-$ is defined in equation (6.4).

Thus a lower bound on the amount of mass coupled at time n_* is $\frac{\varepsilon_{n_*}}{2} > 0$.

We are finally ready to put these elements together. For $k_* \in \mathbb{N}$ and $k = 1, \dots k_*$, let $M^\ell(k)$ denote the index set of curves in $\mathcal{G}_{kn_*}(W^\ell)$ which are matched at time kn_*. By choosing δ_0 small, we can ensure that $n_\sharp \le n_*$, where $n_\sharp$ from Lemma 6.9 corresponds to $C_\sharp = 3C_*$. Thus the family of remaining curves is always admissible at time kn_*. Let $M^\ell(\sim)$ denote the index set of curves that are not matched by time k_*n_*. We estimate using (6.12) at each time $n = kn_*$,

$$\begin{aligned}\int_{W^1} \mathcal{L}^n f\, \psi_1 &= \sum_{k=1}^{k_*} \sum_{i \in M^1(k)} \int_{\bar U_i^1} \mathcal{L}^{n-kn_*} f\, \widehat{T}^{kn_*}_{U_i^1} \tilde\psi_1 + \sum_{i \in M^1(\sim)} \int_{V_i^1} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^1} \tilde\psi_1 \\ &\le \sum_{k=1}^{k_*} \sum_{i \in M^2(k)} 2 \int_{\bar U_i^2} \mathcal{L}^{n-kn_*} f\, \widehat{T}^{kn_*}_{U_i^2} \tilde\psi_2 + \sum_{i \in M^1(\sim)} \int_{V_i^1} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^1} \tilde\psi_1\end{aligned} \tag{6.15}$$

We estimate the sum over unmatched pieices $M^\ell(\sim)$ by splitting the estimate in curves longer than δ, $M^\ell(\sim; Lo)$, and curves shorter than δ, $M^\ell(\sim; Sh)$.

$$\begin{aligned}\sum_{i \in M^\ell(\sim)} \int_{V_i^\ell} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^\ell} \tilde\psi_\ell &= \sum_{i \in M^\ell(\sim; Lo)} \int_{V_i^\ell} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^\ell} \tilde\psi_\ell + \sum_{i \in M^\ell(\sim; Sh)} \int_{V_i^\ell} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^\ell} \tilde\psi_\ell \\ &\le \sum_{i \in M^\ell(\sim; Lo)} |||\mathcal{L}^{n-k_*n_*} f|||_+ \int_{V_i^\ell} \widehat{T}^{k_*n_*}_{V_i^\ell} \tilde\psi_\ell + \sum_{i \in M^\ell(\sim; Sh)} A |||\mathcal{L}^{n-k_*n_*} f|||_- \delta |\psi_\ell|_{C^0} |J_{V_i^\ell} T^{k_*n_*}|_{C^0} \\ &\le (1 - \tfrac{\varepsilon_{n_*}}{2})^{k_*} 3L |||\mathcal{L}^n f|||_- + A2 |||\mathcal{L}^n f|||_- \delta |\psi_\ell|_{C^0} \bar C_0 \,.\end{aligned}$$

where we have used (5.13) and the fact that $k_*n_* \ge n_0$. For the sum over long pieces, we used that the total mass of unmatched pieces decays exponentially in k, while for the sum over short pieces, we used Lemma 3.1 and Remark 3.2 to sum over the Jacobians since $|W^1| \ge \delta_0/2$. Finally, since $|\psi_1|_{C^0} \le e^{a\delta_0^\alpha} \fint_{W^1} \psi_1 \le \frac{4}{\delta_0}$, we conclude,

$$\begin{aligned}\sum_{i \in M^1(\sim)} \left| \int_{V_i^1} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^1} \tilde\psi_1 \right| &\le \Big(3L(1 - \tfrac{\varepsilon_{n_*}}{2})^{k_*} + 8A\bar C_0 \tfrac{\delta}{\delta_0} \Big) |||\mathcal{L}^n f|||_- \\ &\le \Big(3L(1 - \tfrac{\varepsilon_{n_*}}{2})^{k_*} + 8A\bar C_0 \tfrac{\delta}{\delta_0} \Big) \int_{W^2} \mathcal{L}^n f\, \psi_2 \,,\end{aligned}$$

using the fact that $\fint_{W^2} \psi_2 = 1$. A similar estimate holds for the sum over curves in $M^2(\sim)$. Finally, we put together this estimate with (6.15) to obtain,

$$\begin{aligned}\int_{W^1} \mathcal{L}^n f\, \psi_1 &\le \sum_{k=1}^{k_*} \sum_{i \in M^2(k)} 2 \int_{\bar U_i^2} \mathcal{L}^{n-kn_*} f\, \widehat{T}^{kn_*}_{U_i^2} \tilde\psi_2 + \sum_{i \in M^1(\sim)} \int_{V_i^1} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^1} \tilde\psi_1 \\ &\le 2 \int_{W^2} \mathcal{L}^n f\, \psi_2 + 2 \sum_{j \in M^2(\sim)} \left| \int_{V_j^2} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_j^2} \tilde\psi_2 \right| \\ &\quad + \sum_{i \in M^1(\sim)} \left| \int_{V_i^1} \mathcal{L}^{n-k_*n_*} f\, \widehat{T}^{k_*n_*}_{V_i^1} \tilde\psi_1 \right| \\ &\le \int_{W^2} \mathcal{L}^n f\, \psi_2 \left(2 + 3\big(3L(1 - \tfrac{\varepsilon_{n_*}}{2})^{k_*} + 8A\bar C_0 \tfrac{\delta}{\delta_0}\big) \right) .\end{aligned}$$

We choose k_* such that

$$3L(1 - \frac{\varepsilon_{n_*}}{2})^{k_*} < \frac{1}{6}. \tag{6.16}$$

Note that this choice of k_* depends only on δ_0 via ε_{n_*}, and not on δ. Next, choose $\delta > 0$ sufficiently small that

$$8A\bar{C}_0\delta/\delta_0 < \frac{1}{6}. \tag{6.17}$$

These choices imply that

$$\int_{W^1} \mathcal{L}^n f\, \psi_1 \le 3 \int_{W^2} \mathcal{L}^n f\, \psi_2\,.$$

Finally we prove that the first alternative of Proposition 6.3 must happen. Suppose the contrary. Since this bound holds for all $W^1, W^2 \in \mathcal{W}^s(\delta_0/2)$ and test functions ψ_1, ψ_2 with $\int_{W^1} \psi_1 = \int_{W^2} \psi_2 = 1$, we conclude that, for $k \ge k_*$ and $m \ge N(\delta)^-$,

$$\frac{|||\mathcal{L}^{kn_*+m} f|||_+}{|||\mathcal{L}^{kn_*+m} f|||_-} \le \frac{160}{9} \frac{|||\mathcal{L}^{kn_*} f|||^0_+}{|||\mathcal{L}^{kn_*} f|||^0_-} \le \frac{160}{3} \le \frac{8}{9} L\,,$$

if we choose $L \ge 60$. □

6.3 Finite diameter

In this section we prove the following proposition, which completes the proof of Theorem 2.1.

Proposition 6.11. *For any* $\chi \in \left(\max\{\frac{1}{2}, \frac{1}{L}, \frac{1}{\sqrt{A-1}}\}, 1\right)$*, the cone* $\mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$ *has diameter less than* $\Delta := \log\left(\frac{(1+\chi)^2}{(1-\chi)^2}\chi L\right) < \infty$ *in* $\mathcal{C}_{c,A,L}(\delta)$*, assuming* $\delta > 0$ *is sufficiently small to satisfy* (6.19)*.*

Proof. For brevity, we will denote $\mathcal{C} = \mathcal{C}_{c,A,L}(\delta)$ and $\mathcal{C}_\chi = \mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$. For $f \in \mathcal{C}_\chi$, we will show that $\rho(f, 1) < \infty$, where ρ denotes distance in the cone $\mathcal{C}$. Fix $f \in \mathcal{C}_\chi$ throughout.

According to (4.1) if we find $\lambda > 0$ such that $f - \lambda \succeq 0$, then $\bar{\alpha}(1, f) \ge \lambda$.

Notice that $|||f - \lambda|||_\pm = |||f|||_\pm - \lambda$. Hence $f - \lambda$ satisfies (4.6) if

$$|||f|||_+ - \lambda \le L(|||f|||_- - \lambda) \quad \Longleftarrow \quad \lambda \le \frac{L(1-\chi)}{L-1} |||f|||_- =: \bar{\alpha}_1\,,$$

where we have used that $f \in \mathcal{C}_\chi$.

Similarly, $f - \lambda$ satisfies (4.7) if, for all $W \in \mathcal{W}^s_-(\delta)$ and $\psi \in \mathcal{D}_{a,\beta}(W)$,

$$|W|^{-q} \frac{\left|\int_W f\psi - \lambda \int_W \psi\right|}{⨍_W \psi} \le A\delta^{1-q}(|||f|||_- - \lambda) \quad \Longleftarrow \quad \lambda \le \frac{(1-\chi)A|||f|||_-}{A+1} =: \bar{\alpha}_2\,.$$

Next, notice that for any $\lambda \ge 0$, $W^1, W^2 \in \mathcal{W}^s_-(\delta)$ and $\psi_\ell \in \mathcal{D}_{a,\alpha}(W^\ell)$,

$$\begin{aligned} \left| \frac{\int_{W^1}(f-\lambda)\psi_1}{⨍_{W^1}\psi_1} - \frac{\int_{W^2}(f-\lambda)\psi_2}{⨍_{W^2}\psi_2} \right| &= \left| \frac{\int_{W^1} f\,\psi_1}{⨍_{W^1}\psi_1} - \frac{\int_{W^2} f\,\psi_2}{⨍_{W^2}\psi_2} - \lambda(|W^1| - |W^2|) \right| \\ &\le \chi^2 d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} cA|||f|||_- + \lambda(\delta + C_s) d_{\mathcal{W}^s}(W^1, W^2)\,, \end{aligned} \tag{6.18}$$

where we have used (5.8), so that $f - \lambda$ satisfies (4.8) if

$$\chi^2 d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} cA|||f|||_- + \lambda(\delta + C_s)\delta^{1-\gamma} d_{\mathcal{W}^s}(W^1, W^2)^\gamma \le d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} cA(|||f|||_- - \lambda)\,.$$

This occurs whenever

$$\lambda \le \frac{cA|||f|||_-(1-\chi^2)}{\delta + C_s + cA} \quad \Longleftarrow \quad \lambda \le (1-\chi)|||f|||_- =: \bar\alpha_3\,,$$

provided that δ is chosen sufficiently small that

$$\delta + C_s \le \chi c A\,, \tag{6.19}$$

which is possible since $cA > 2C_s$ by (5.34) and $\chi > 1/2$.

Note that $\bar\alpha_2 \le \bar\alpha_3 \le \bar\alpha_1$, so that $\bar\alpha_2 = \min_i\{\bar\alpha_i\}$. Thus if $\lambda \le \bar\alpha_2$, then $f - \lambda \in \mathcal{C}$, i.e. $\bar\alpha(1,f) \ge \bar\alpha_2$.

Next, we proceed to estimate $\bar\beta(1,f)$ for $f \in \mathcal{C}_\chi$. If we find $\mu > 0$ such that $\mu - f \in \mathcal{C}$, this will imply that $\bar\beta(1,f) \le \mu$. Remarking that $|||\mu - f|||_\pm = \mu - |||f|||_\mp$, we have that $\mu - f$ satisfies (4.6) if

$$\mu \ge \frac{L|||f|||_+ - |||f|||_-}{L-1} \quad \Longleftarrow \quad \mu \ge \frac{L}{L-1}|||f|||_+ =: \bar\beta_1\,,$$

while $\mu - f$ satisfies (4.7) if for all $W \in \mathcal{W}^s_-(\delta)$, $\psi \in \mathcal{D}_{a,\beta}(W)$,

$$|W|^{-q}\frac{|\mu\int_W \psi - \int_W f\psi|}{⨍_W \psi} \le A\delta^{1-q}(\mu - |||f|||_+) \quad \Longleftarrow \quad \mu \ge \frac{(1+\chi)A}{A - 2^{1-q}}|||f|||_+ =: \bar\beta_2\,.$$

Finally, recalling (6.18) and again (5.8), we have that $\mu - f$ satisfies (4.8) whenever

$$\chi^2 d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}cA|||f|||_- + \mu(\delta + C_s)\delta^{1-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^\gamma \le d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}cA(\mu - |||f|||_+)\,.$$

This is implied by,

$$\mu \ge \frac{cA(1+\chi^2)}{cA - (\delta + C_s)}|||f|||_+ \quad \Longleftarrow \quad \mu \ge \frac{1+\chi^2}{1-\chi}|||f|||_+ =: \bar\beta_3\,,$$

where again we have assumed (6.19).

Defining $\bar\beta = \max_i\{\bar\beta_i\}$, it follows that if $\mu \ge \bar\beta$, then $\mu - f \in \mathcal{C}$. Thus $\bar\beta \ge \bar\beta(1,f)$. Since $\chi > 1/L$ and $\chi^2 > 1/(A-1)$, it holds that $\bar\beta_3 \ge \bar\beta_2 \ge \bar\beta_1$. Thus $\bar\beta = \bar\beta_3$. Our assumption also implies $\chi > 1/A$, so that $\bar\alpha_2 \ge \frac{1-\chi}{1+\chi}|||f|||_-$.

Finally, recalling (4.1), we have

$$\rho(1,f) = \log\left(\frac{\bar\beta(1,f)}{\bar\alpha(1,f)}\right) \le \log\left(\frac{\bar\beta_3}{\bar\alpha_2}\right) \le \log\left(\frac{\frac{1+\chi^2}{1-\chi}|||f|||_+}{\frac{1-\chi}{1+\chi}|||f|||_-}\right) \le \log\left(\frac{(1+\chi)^2}{(1-\chi)^2}\chi L\right)\,,$$

for all $f \in \mathcal{C}_\chi$, completing the proof of the proposition. □

Remark 6.12. *Note that, setting $\chi_* = \max\{\frac{1}{2}, \frac{1}{L}, \frac{1}{\sqrt{A-1}}\}$, for $\chi \le \chi_*$ Proposition 6.11 implies only that the diameter of $\mathcal{C}_{\chi c,\chi A,\chi L}(\delta) \subset \mathcal{C}_{\chi_* c,\chi_* A,\chi_* L}(\delta)$, in $\mathcal{C}_{c,A,L}(\delta)$, is bounded by $\log\left(\frac{(1+\chi_*)^2}{(1-\chi_*)^2}\chi_* L\right)$. If needed, a more accurate formula can be easily obtained, but it would be more cumbersome.*

7 Convergence to Equilibrium and Decay of Correlations

In this section we show how Theorem 2.1 (i.e. Theorem 6.10 and Proposition 6.11) imply the (by now classical) result on decay of correlations and convergence to equilibrium. To be more precise, the results are comparable with the ones obtained in [DZ1] since they apply to a similar (very) large class of observables (and possibly even distributions). Before stating the exact results (see Theorems 7.3, 7.4 and Corollary 7.5), we establish a key lemma that integration with respect to μ_{SRB} against suitable test functions respects the ordering in our cone. Recall the vector space of functions $\mathcal{A}$ defined in Section 4.3.

Lemma 7.1. *Let $\delta > 0$ be small enough that $2C_\ell C_h(1+A)(\delta^{4/3} + \delta^{1/3+\beta} a\ell_{\max}) < 1$, where $C_\ell, C_h > 0$ are from (7.4) and $\ell_{\max}$ is the maximum diameter of the connected components of M.*

Suppose $\psi \in C^1(M)$ satisfies $2(2\delta)^{1-\beta}|\psi'|_{C^0(M)} \le a \min_M \psi$. If $f, g \in \mathcal{A}$ with $f \preceq g$, then $\int f\, \psi\, d\mu_{SRB} \le \int g\, \psi\, d\mu_{SRB}$.

Proof. Let $\psi_{\min} = \min_M \psi$. The assumption on ψ implies that $\psi \in \mathcal{D}_{\frac{a}{2},\beta}(W)$ for each $W \in \mathcal{W}^s_-(\delta)$ since,

$$\left|\log \frac{\psi(x)}{\psi(y)}\right| \le \frac{1}{\psi_{\min}}|\psi(x) - \psi(y)| \le \frac{|\psi'|_{C^0(M)}}{\psi_{\min}} d(x,y) \le \frac{|\psi'|_{C^0(M)}}{\psi_{\min}}(2\delta)^{1-\beta} d(x,y)^\beta .$$

Suppose $f, g \in \mathcal{A}$ satisfy $f \preceq g$. If $g - f = 0$, then the lemma holds trivially, so suppose instead that $g - f \in \mathcal{C}_{c,A,L}(\delta)$. Then according to (4.5) and (4.7), for all $\psi \in \mathcal{D}_{a,\beta}(W)$,

$$|||g-f|||_- \int_W \psi \;\le\; \int_W (g-f)\psi\, dm_W \;\le\; |||g-f|||_+ \int_W \psi \qquad \forall\, W \in \mathcal{W}^s(\delta) \tag{7.1}$$

$$\left|\int_W (g-f)\psi\, dm_W\right| \;\le\; |||g-f|||_- A\delta^{1-q}|W|^q \fint_W \psi \qquad \forall\, W \in \mathcal{W}^s_-(\delta). \tag{7.2}$$

Next, we disintegrate μ_{SRB} according to a smooth foliation of stable curves as follows. Since the stable cones for T are globally constant, we fix a direction in the stable cone and consider stable curves in the form of line segments with this slope. Let $k_\delta \ge k_0$ denote the minimal index k of a homogeneity strip $\mathbb{H}_k$ such that the stable line segments in $\mathbb{H}_k$ have length less than δ. Due to the fact that the minimum slope in the stable cone is $\mathcal{K}_{\min} > 0$, we have

$$k_\delta = C_h \delta^{-1/3}, \tag{7.3}$$

for some constant $C_h > 0$ independent of δ.

Now for $k < k_\delta$, we decompose $\mathbb{H}_k$ into horizontal bands B_i such that every maximal line segment of the chosen slope in B_i has equal length between δ and 2δ. We do the same on $M \setminus (\cup_{k \ge k_0} \mathbb{H}_k)$. On each B_i, define a foliation of such parallel line segments $\{W_\xi\}_{\xi \in \Xi_i} \subset \mathcal{W}^s(\delta)$. Using the smoothness of this foliation, we disintegrate μ_{SRB} into conditional measures $\cos\varphi(x) dm_{W_\xi}$ on W_ξ and a factor measure $\hat{\mu}$ on the index set Ξ_i. Note that our conditional measures are not normalized - we include this factor in $\hat{\mu}$. Finally, on each homogeneity strip $\mathbb{H}_k$, $k \ge k_\delta$, we carry out a similar decomposition, but using homogeneous parallel line segments of maximal length in $\mathbb{H}_k$ (which are necessarily shorter than length δ). We use the notation $\{W_\xi\}_{\xi \in \Xi_k} \subset \mathcal{W}^s_-(\delta)$ for the foliations in these homogeneity strips. Note that in both cases, we have $\hat{\mu}(\Xi_i), \hat{\mu}(\Xi_k) \le C_\ell$, for some constant C_ℓ depending only on the chosen slope and spacing of homogeneity strips.

Also, it follows as in (3.3), that for $x, y \in W \in \mathcal{W}^s_-(\delta)$,

$$\log \frac{\cos\varphi(x)}{\cos\varphi(y)} \le C_d (2\delta)^{1/3-\beta} d(x,y)^\beta ,$$

so that $\cos\varphi \in \mathcal{D}_{\frac{a}{2},\beta}(W)$ by the assumption of Lemma 5.2. Thus $\psi\cos\varphi \in \mathcal{D}_{a,\beta}(W)$ for all $W \in \mathcal{W}^s_-(\delta)$.

Using this fact and our disintegration of μ_{SRB}, we estimate the integral applying (7.1) on Ξ_i and (7.2) on Ξ_k,

$$\begin{aligned}
\int_M (g-f)\psi \, d\mu_{\text{SRB}} &= \sum_i \int_{\Xi_i}\int_{W_\xi} (g-f)\psi\cos\varphi \, dm_{W_\xi} d\hat{\mu}(\xi) + \sum_{k\ge k_\delta}\int_{\Xi_k}\int_{W_\xi}(g-f)\psi\cos\varphi\, dm_{W_\xi} d\hat{\mu}(\xi) \\
&\ge |||g-f|||_- \left(\sum_i \int_{\Xi_i}\int_{W_\xi} \psi\cos\varphi\, dm_{W_\xi} d\hat{\mu}(\xi) - A\delta \sum_{k\ge k_\delta}\int_{\Xi_k} \fint_{W_\xi}\psi\cos\varphi\, dm_{W_\xi} d\hat{\mu}(\xi)\right) \\
&\ge |||g-f|||_- \left(\psi_{\min}\mu_{\text{SRB}}(M\setminus(\cup_{k\ge k_\delta}\mathbb{H}_k)) - A\delta C_\ell |\psi|_{C^0}\sum_{k\ge k_\delta} k^{-2}\right) \\
&\ge |||g-f|||_- \left(\psi_{\min}(1-2C_\ell C_h \delta^{4/3}) - |\psi|_{C^0} A C_\ell C_h 2\delta^{4/3}\right),
\end{aligned} \tag{7.4}$$

where we have estimated $\sum_{k\ge k_\delta} k^{-2} \le 2k_\delta^{-1}$ and $\mu_{\text{SRB}}(\cup_{k\ge k_\delta}\mathbb{H}_k) \le 2C_\ell C_h\delta^{4/3}$.

Now $|\psi|_{C^0} \le \psi_{\min} + \ell_{\max}|\psi'|_{C^0}$, where $\ell_{\max}$ is the maximum diameter of the connected components of M. Then by the assumption on ψ, we have

$$\begin{aligned}
2C_\ell C_h(1+A)\delta^{4/3}|\psi|_{C^0} &\le 2C_\ell C_h(1+A)\delta^{4/3}\psi_{\min}(1+\ell_{\max}\tfrac{a}{2}(2\delta)^{\beta-1}) \\
&\le \psi_{\min} 2C_\ell C_h (1+A)(\delta^{4/3} + a\ell_{\max}\delta^{1/3+\beta}) \le \psi_{\min},
\end{aligned}$$

where for the last inequality we have used the assumption on δ in the statement of the lemma. We conclude that the lower bound in (7.4) cannot be less than 0. □

Remark 7.2. *Lemma 7.1 implies there exists $\bar{C} \ge 1$ such that $\int_M f\, d\mu_{SRB} \ge \bar{C}^{-1}|||f|||_- > 0$ for all $f \in \mathcal{C}_{c,A,L}(\delta)$.*

Using instead the upper bound in (7.1) and following the estimate of (7.4) yields,

$$0 < \int_M f\psi\, d\mu_{SRB} \le |||f|||_+ C|\psi|_{C^0},$$

for all $f \in \mathcal{C}_{c,A,L}(\delta)$ and ψ as in the statement of Lemma 7.1. This can be extended to all $\psi \in C^1(M)$ by defining C_ψ as in (7.8) below to conclude

$$\int_M f\psi\, d\mu_{SRB} \le |||f|||_+ C|\psi|_{C^1}.$$

Convergence to equilibrium, including equidistribution, and decay of correlations readily follow from the contraction in the projective metric $\rho_{\mathcal{C}}(\cdot,\cdot)$ of the cone $\mathcal{C}_{c,A,L}(\delta)$. Set $\mu_{\text{SRB}}(f) = \int_M f\, d\mu_{\text{SRB}}$.

Theorem 7.3. *Let $\delta > 0$ satisfy the assumption of Lemma 7.1. There exists $C > 0$ and $\vartheta < 1$ such that for all $n \ge 0$, $f, g \in \mathcal{C}_{c,A,L}(\delta)$ with $\int_M f\, d\mu_{SRB} = \int_M g\, d\mu_{SRB}$, all $W_1, W_2 \in \mathcal{W}^s(\delta)$ and all $\psi_i \in C^1(W_i)$ with $\fint_{W_1}\psi_1 = \fint_{W_2}\psi_2$, we have*

$$\left|\fint_{W_1} \mathcal{L}^n f\,\psi_1\, dm_{W_1} - \fint_{W_2}\mathcal{L}^n g\,\psi_2\, dm_{W_2}\right| \le C\vartheta^n\,(|\psi_1|_{C^1} + |\psi_2|_{C^1})\mu_{SRB}(f).$$

In particular, for all $W \in \mathcal{W}^s(\delta)$ and $\psi \in C^1(W)$,

$$\left|\fint_W \mathcal{L}^n f\,\psi\, dm_W - \mu_{SRB}(f)\fint_W \psi\, dm_W\right| \le C\vartheta^n\,|\psi|_{C^1}\mu_{SRB}(f). \tag{7.5}$$

Proof. It is convenient to extend the definition of $|||\cdot|||_+$ to all of $\mathcal{A}$ by

$$|||f|||_+ = \sup_{\substack{W\in\mathcal{W}^s(\delta)\\ \psi\in\mathcal{D}_{a,\beta}(W)}} \frac{\left|\int_W f\psi\, dm_W\right|}{\int_W \psi\, dm_W}.$$

Note that, with this definition, $|||\cdot|||_+$ is an order-preserving semi-norm in $\mathcal{A}$.[7] Also $\mu_{\text{SRB}}(f) := \int_M f\, d\mu_{\text{SRB}}$ is homogeneous and order preserving in $\mathcal{C}_{c,A,L}(\delta)$ by Lemma 7.1 applied to $\psi \equiv 1$. Then [LSV, Lemma 2.2] implies that, for all $f, g \in \mathcal{C}_{c,A,L}(\delta)$ with $\mu_{\text{SRB}}(f) = \mu_{\text{SRB}}(g)$,[8]

$$|||\mathcal{L}^n f - \mathcal{L}^n g|||_+ \le \left(e^{\rho_{\mathcal{C}}(\mathcal{L}^n f, \mathcal{L}^n g)} - 1\right) \min\{|||\mathcal{L}^n f|||_+, |||\mathcal{L}^n g|||_+\}. \tag{7.6}$$

Hence by Theorem 6.10, Proposition 6.11 and [L95a, Theorem 2.1], there exists $C > 0$ such that for all $n \ge n(\delta) := N(\delta)^- + k_* n_*$,

$$|||\mathcal{L}^n f - \mathcal{L}^n g|||_+ \le C\vartheta^n \min\{|||f|||_+, |||g|||_+\}, \tag{7.7}$$

where $\vartheta = [\tanh(\Delta/4)]^{1/n(\delta)}$. Hence, applying (7.7) with $g = \mu_{\text{SRB}}(f)$ implies,

$$\begin{aligned}\left|\fint_W \mathcal{L}^n f\,\psi\, dm_W - \mu_{\text{SRB}}(f)\fint_W \psi\right| &= \fint_W \psi \left|\frac{\int_W \mathcal{L}^n f\,\psi\, dm_W}{\int_W \psi} - \frac{\int_W \mathcal{L}^n(\mu_{\text{SRB}}(f))\,\psi}{\int_W \psi}\right| \\ &\le C\vartheta^n\, |\psi|_{C^0}\mu_{\text{SRB}}(f)\,.\end{aligned}$$

Since $\mathcal{L}^n 1 = 1$ and $|||\mu_{\text{SRB}}(f)|||_+ = \mu_{\text{SRB}}(f)$, the above proves (7.5) for $\psi \in \mathcal{D}_{a,\beta}(W)$. To extend this estimate to more general $\psi \in C^1(W)$, define $\tilde\psi = \psi + C_\psi$, where

$$C_\psi = |\psi_{\min}| + \tfrac{2}{a}|\psi'|_{C^0}(2\delta)^{1-\beta}\,. \tag{7.8}$$

Then $\tilde\psi' = \psi'$ and $\min_W \tilde\psi \ge \frac{2}{a}|\tilde\psi'|_{C^0}(2\delta)^{1-\beta}$, so that $\tilde\psi \in \mathcal{D}_{\frac{a}{2},\beta}(W)$ by the proof of Lemma 7.1. Then since also $C_\psi \in \mathcal{D}_{a,\beta}(W)$, the estimate for (7.5) follows by writing $\psi = \tilde\psi - C_\psi$ and using the triangle inequality. Finally, the first assertion of the theorem follows from another application of the triangle inequality. □

Theorem 7.4. *Let $\delta > 0$ satisfy the assumption of Lemma 7.1. There exists $C > 0$ such that for all $n \ge 0$, $\psi \in C^1(M)$ and $f, g \in \mathcal{C}_{c,A,L}(\delta)$, with $\mu_{SRB}(f) = \mu_{SRB}(g)$,*

$$\left|\int_M \mathcal{L}^n f\,\psi\, d\mu_{SRB} - \int_M \mathcal{L}^n g\,\psi\, d\mu_{SRB}\right| \le C\vartheta^n |\psi|_{C^1(M)} \min\{|||f|||_+, |||g|||_+\}\,. \tag{7.9}$$

In particular,

$$\left|\int_M f\,\psi\circ T^n\, d\mu_{SRB} - \int_M f\, d\mu_{SRB}\int_M \psi\, d\mu_{SRB}\right| \le C\vartheta^n |\psi|_{C^1(M)} \int_M f\, d\mu_{SRB}\,.$$

Proof. Following the strategy of Theorem 7.3, given $\psi \in C^1(M)$ satisfying the assumption of Lemma 7.1, we define a pseudo-norm for $f \in \mathcal{A}$ by

$$\|f\|_\psi = \left|\int_M f\,\psi\, d\mu_{\text{SRB}}\right|\,. \tag{7.10}$$

[7] A semi-norm $\|\cdot\|$ is order preserving if $-g \preceq f \preceq g$ implies $\|f\| \le \|g\|$. The space $\mathcal{A}$ is defined just before (4.5).

[8] [LSV, Lemma 2.2] is stated for order preserving norms but its proof holds verbatim for order preserving semi-norms.

By Lemma 7.1, $\|\cdot\|_\psi$ is an order-preserving semi-norm, and as in (7.6), invoking again [LSV, Lemma 2.2], Theorem 6.10, Proposition 6.11 and [L95a, Theorem 2.1], we have for $f, g \in \mathcal{C}_{c,A,L}(\delta)$ with $\mu_{\text{SRB}}(f) = \mu_{\text{SRB}}(g)$ and $n \geq n(\delta)$,

$$\|\mathcal{L}^n f - \mathcal{L}^n g\|_\psi \leq C\vartheta^n \min\{\|\mathcal{L}^n f\|_\psi, \|\mathcal{L}^n g\|_\psi\} \leq C\vartheta^n |\psi|_{C^0} \min\{\vert\vert\vert f\vert\vert\vert_+, \vert\vert\vert g\vert\vert\vert_+\},$$

where we applied (7.7) and Remark 7.2. This proves (7.9) for ψ satisfying the assumption of Lemma 7.1. We extend to more general $\psi \in C^1(M)$ by defining $\tilde{\psi} = \psi + C_\psi$, where C_ψ is given by (7.8), and arguing as in the proof of Theorem 7.3.

Next, by definition of $\mathcal{L}$ and using that $\mathcal{L}^n 1 = 1$, we have

$$\int_M f\, \psi \circ T^n \, d\mu_{\text{SRB}} - \int_M f \, d\mu_{\text{SRB}} \int_M \psi \, d\mu_{\text{SRB}} = \int_M \mathcal{L}^n(f - \mu_{\text{SRB}}(f))\, \psi \, d\mu_{\text{SRB}} .$$

Thus applying (7.9) to $g = \mu_{\text{SRB}}(f)$ yields the second claim of the Theorem since $\vert\vert\vert\mu_{\text{SRB}}(f)\vert\vert\vert_+ = \mu_{\text{SRB}}(f)$. □

Corollary 7.5. *The convergence in Theorems 7.3 and 7.4 extend to all $f, g \in C^1(M)$, with $|f|_{C^1(M)}, |g|_{C^1(M)}$ on the right hand side.*

The proof of this corollary relies on the following lemma.

Lemma 7.6. *If $f \in C^1(M)$, then $\lambda + f \in \mathcal{C}_{c,A,L}(\delta)$ for any*

$$\lambda \geq \max\left\{ \frac{L+1}{L-1}|f|_{C^0}, \frac{A + 2^{1-q}}{A - 2^{1-q}}|f|_{C^0}, \frac{cA + 8C_s}{cA - 2C_s}|f|_{C^1} \right\} .$$

Proof of Corollary 7.5. Let $f, g \in C^1(M)$ with $\mu_{\text{SRB}}(f) = \mu_{\text{SRB}}(g)$ and let $\psi \in C^1(M)$. Let λ_f, λ_g be the constants from Lemma 7.6 corresponding to f and g, respectively, and set $\lambda = \max\{\lambda_f, \lambda_g\}$. Then $f + \lambda, g + \lambda \in \mathcal{C}_{c,A,L}(\delta)$ and $\mu_{\text{SRB}}(f + \lambda) = \mu_{\text{SRB}}(g + \lambda)$, so that by Theorem 7.4, for all $n \geq 0$,

$$\left| \int_M \mathcal{L}^n(f - g)\, \psi \, dm \right| = \left| \int_M \mathcal{L}^n(f + \lambda - (g + \lambda))\, \psi \, dm \right| \leq C' \vartheta^n |\psi|_{C^1(M)} \max\{|f|_{C^1(M)}, |g|_{C^1(M)}\},$$

since $\vert\vert\vert f + \lambda\vert\vert\vert_+ \leq \lambda + |f|_{C^0}$, and by Lemma 7.6, $\lambda_f \geq C''|f|_{C^1(M)}$, with analogous estimates for g. This proves (7.9) and the second claim of Theorem 7.4 follows from the first by setting $g = \mu_{\text{SRB}}(f)$ as in the proof of that theorem.

The extension of Theorem 7.3 to $f, g \in C^1(M)$ follows analogously, replacing the integral over M with the integral over W to prove (7.5), and then using the triangle inequality to deduce the first statement of the theorem. □

Proof of Lemma 7.6. We must show that $\lambda + f$ satisfies conditions (4.6) - (4.8) in the definition of $\mathcal{C}_{c,A,L}(\delta)$. Since

$$\vert\vert\vert\lambda + f\vert\vert\vert_+ \leq \lambda + |f|_{C^0}, \qquad \text{and} \qquad \vert\vert\vert\lambda + f\vert\vert\vert_- \geq \lambda - |f|_{C^0}, \tag{7.11}$$

to guarantee (4.6), we need

$$\frac{\lambda + |f|_{C^0}}{\lambda - |f|_{C^0}} \leq L \qquad \Longleftarrow \qquad \lambda \geq |f|_{C^0}\frac{L+1}{L-1}.$$

Next, to guarantee (4.7), for $W \in \mathcal{W}^s_-(\delta)$, $\psi \in \mathcal{D}_{a,\beta}(W)$, we need,

$$|W|^{-q}\frac{\int_W (\lambda + f)\psi}{\fint_W \psi} \leq A\delta^{1-q}(\lambda - |f|_{C^0}) \qquad \Longleftarrow \qquad \lambda \geq |f|_{C^0}\frac{A + 2^{1-q}}{A - 2^{1-q}}.$$

Lastly, we need to show that (4.8) is satisfied. For this, we prove the claim:

$$\left|\frac{\int_{W_1} f\psi_1}{\fint_{W_1}\psi_1} - \frac{\int_{W_2} f\psi_2}{\fint_{W_2}\psi_2}\right| \le 8C_s\delta^{1-\gamma} d_{\mathcal{W}^s}(W_1,W_2)^\gamma |f|_{C^1}\,, \tag{7.12}$$

for W_1, W_2, ψ_1, ψ_2 as in (4.8). Recalling the notation $W_k = \{G_{W_k}(r) = (r, \varphi_{W_k}(r)) : r \in I_{W_k}\}$ for $k = 1, 2$ from Section 4.2, we set $\bar{W}_k = G_{W_k}(I_{W_1} \cap I_{W_2})$ and $W_k^c = W_k \setminus \bar{W}_k$. As in Section 5.2.3, we assume without loss of generality that $|W_2| \ge |W_1|$ and $\fint_{W_1}\psi_1 = 1$. Also, we may assume $|W_2| \ge 2C_s\delta^{1-\gamma} d_{\mathcal{W}^s}(W_1,W_2)^\gamma$; otherwise, (7.12) is trivially bounded by $2|W_2||f|_{C^0} \le 4C_s\delta^{1-\gamma} d_{\mathcal{W}^s}(W_1,W_2)^\gamma |f|_{C^0}$.

Next,

$$\left|\frac{\int_{W_1} f\psi_1}{\fint_{W_1}\psi_1} - \frac{\int_{W_2} f\psi_2}{\fint_{W_2}\psi_2}\right| \le \left|\frac{\int_{\bar{W}_1} f\psi_1 - \int_{\bar{W}_2} f\psi_2}{\fint_{W_1}\psi_1}\right| + \left|\frac{\int_{\bar{W}_2} f\psi_2}{\fint_{W_2}\psi_2}\left(\frac{\fint_{W_2}\psi_2}{\fint_{W_1}\psi_1} - 1\right)\right| + \sum_{k=1}^{2}\frac{|\int_{W_k^c} f\psi_k|}{\fint_{W_k}\psi_k} \tag{7.13}$$

To estimate the first term above, recalling (4.3) and $d_*(\psi_1,\psi_2) = 0$, we have for $r \in I_{W_1} \cap I_{W_2}$,

$$\begin{aligned}
&|(f\psi_1)\circ G_{W_1}(r)\|G'_{W_1}(r)\| - (f\psi_2)\circ G_{W_2}(r)\|G'_{W_2}(r)\||\\
&= \psi_1\circ G_{W_1}(r)\|G'_{W_1}(r)\||f\circ G_{W_1}(r) - f\circ G_{W_2}(r)| \le \psi_1\circ G_{W_1}(r)\|G'_{W_1}(r)\||f'|_{C^0} d_{\mathcal{W}^s}(W_1,W_2)\,,
\end{aligned}$$

and integrating over $I_{W_1} \cap I_{W_2}$ yields,

$$\left|\frac{\int_{\bar{W}_1} f\psi_1 - \int_{\bar{W}_2} f\psi_2}{\fint_{W_1}\psi_1}\right| \le \frac{\int_{\bar{W}_1}\psi_1}{\fint_{W_1}\psi_1}|f'|_{C^0} d_{\mathcal{W}^s}(W_1,W_2) \le 2\delta|f'|_{C^0} d_{\mathcal{W}^s}(W_1,W_2)\,. \tag{7.14}$$

For the second term in (7.13), note that our assumption $|W_2| \ge 2C_s\delta^{1-\gamma} d_{\mathcal{W}^s}(W_1,W_2)^\gamma$ implies as in the estimate following (5.6) that $I_{W_1} \cap I_{W_2} \ne \emptyset$. Thus we may apply (5.9) and (5.10) and use $\fint_{W_1}\psi_1 = 1$ to obtain,

$$\left|\frac{\int_{\bar{W}_2} f\psi_2}{\fint_{W_2}\psi_2}\left(\frac{\fint_{W_2}\psi_2}{\fint_{W_1}\psi_1} - 1\right)\right| \le |f|_{C^0}\left|\int_{W_2}\psi_2 - |W_2|\right| \le |f|_{C^0} 6C_s d_{\mathcal{W}^s}(W_1,W_2)\,. \tag{7.15}$$

Finally, the third term in (7.13) can be estimated by

$$\sum_k \frac{|\int_{W_k^c} f\psi_k|}{\fint_{W_k}\psi_k} \le |f|_{C^0} e^{a(2\delta)^\alpha}\left(|W_1^c| + |W_2^c|\right) \le 2C_s|f|_{C^0} d_{\mathcal{W}^s}(W_1,W_2)\,.$$

Collecting this estimate together with (7.14) and (7.15) in (7.13), we obtain

$$\left|\frac{\int_{W_1} f\psi_1}{\fint_{W_1}\psi_1} - \frac{\int_{W_2} f\psi_2}{\fint_{W_2}\psi_2}\right| \le 8C_s d_{\mathcal{W}^s}(W_1,W_2)|f|_{C^1}\,,$$

proving the bound in (7.12) since $d_{\mathcal{W}^s}(W_1,W_2) \le \delta$.

With the claim proved, we proceed to verify (4.8). Using (5.8) we estimate,

$$\begin{aligned}
\left|\frac{\int_{W_1}(f+\lambda)\psi_1}{\fint_{W_1}\psi_1} - \frac{\int_{W_2}(f+\lambda)\psi_2}{\fint_{W_2}\psi_2}\right| &\le \left|\frac{\int_{W_1} f\psi_1}{\fint_{W_1}\psi_1} - \frac{\int_{W_2} f\psi_2}{\fint_{W_2}\psi_2}\right| + \lambda||W_1| - |W_2||\\
&\le 8C_s\delta^{1-\gamma} d_{\mathcal{W}^s}(W_1,W_2)^\gamma|f|_{C^1} + \lambda 2C_s d_{\mathcal{W}^s}(W_1,W_2)\,.
\end{aligned}$$

Thus (4.8) will be verified if

$$8C_s\delta^{1-\gamma} d_{\mathcal{W}^s}(W_1,W_2)^\gamma|f|_{C^1} + \lambda 2C_s d_{\mathcal{W}^s}(W_1,W_2) \le cA\delta^{1-\gamma} d_{\mathcal{W}^s}(W_1,W_2)^\gamma(\lambda - |f|_{C^0})\,,$$

which is implied by the final condition on λ in the statement of the Lemma since $d_{\mathcal{W}^s}(W_1,W_2) \le \delta$ and $cA > 2C_s$ by (5.34). □

8 Applications

Suppose that we have a billiard table $Q = \mathbb{T}^2 \setminus \cup_i B_i$ and that the particle can escape from the table by entering certain sets $\mathbb{G} \subset Q$, which we call *gates* or *holes*, but only at times kN for some $N \in \mathbb{N}$. One could easily consider also the case of $\mathbb{G} \subset Q \times S^1$ (i.e. some velocity directions are forbidden), however we prefer to keep things simple. On the contrary the limitation that the holes are "open" only at times kN is not very natural and is introduced only since it drastically simplifies the following arguments. To remove such a limitation means that we would have to contend with a limited amount of hyperbolicity from two consecutive visits to a neighborhood of the hole. In turn this would not allow us to use directly the results developed in the previous sections and would force us to redo all the arguments while keeping track of the combinatorics of the trajectories that either fall or do not fall into the holes. To do that is a highly non trivial job (see [LM, AL] for an implementation in simpler situations) which exceeds our current objectives.

A hole $\mathbb{G} \subset Q$ induces a hole $H \subset M$ in the phase space of the billiard map T. We formulate here two abstract conditions on the set H, and then provide examples of concrete, physically relevant situations which induce holes satisfying our conditions.

(H1) (Complexity) There exists $P_0 > 0$ such that any stable curve of length at most δ can be cut into at most P_0 pieces by ∂H, where δ is the length scale of the cone $\mathcal{C}_{c,A,L}(\delta)$.

(H2) (Uniform transversality) There exists $C_t > 0$ such that, for any stable curve $W \in \mathcal{W}^s$ and $\varepsilon > 0$, $m_W(N_\varepsilon(\partial H)) \leq C_t \varepsilon$.

Remark 8.1. *Assumption (H2) can be weakened to, e.g.,* $m_W(N_\varepsilon(\partial H)) \leq C_t \varepsilon^{1/2}$*, but this would then require* $d_{\mathcal{W}^s}(W^1, W^2) \leq \delta^2$ *in our definition of cone condition (4.8). Similar modifications are made to weaken the transversality assumption in the Banach space setting, see for example [DZ3, D2].*

We let $\mathrm{diam}^s(H)$ denote the maximal length of a stable curve in H, which we call the *stable diameter*.

Denoting by $\mathbb{1}_A$ the characteristic function of the set A, the relevant operator is given by $\mathcal{L}_H = \mathcal{L}^N \mathbb{1}_{H^c}$, where H^c denotes the complement of H in M, and $\mathcal{L}$ is the usual transfer operator for the billiard map. The main objective is to control the action of the multiplication operator $\mathbb{1}_{H^c}$ on the cone $\mathcal{C}_{c,A,L}(\delta)$.

8.1 Small holes

Lemma 8.2. *Under assumptions (H1) and (H2), if* $\mathrm{diam}^s(H) \leq \delta \left[\frac{1}{4P_0 A}\right]^{1/q}$*, we have*

$$\mathbb{1}_{H^c}[\mathcal{C}_{c,A,L}(\delta)] \subset \mathcal{C}_{c',A',L'}(\delta),$$

where

$$L' = 2P_0^{1-q} e^{a(2\delta)^\beta} A\,, \quad A' = 2P_0^{1-q} e^{a(2\delta)^\beta} A\,,$$
$$c' = P_0^q e^{a(2\delta)^\alpha} + 2\left(2^q \delta + \tfrac{3}{4} c\right) + 4(P_0 + 2) P_0^{q-1} C_t^q\,.$$

Proof. Letting $f \in \mathcal{C}_{c,A,L}(\delta)$, we must control the cone conditions one by one. We begin with (4.6). Given $W \in \mathcal{W}^s(\delta)$, let $\mathcal{G}_0$ denote the collection of connected curves in $W \setminus H$. Then applying (4.7) to each $W' \in \mathcal{G}_0$, for $\psi \in \mathcal{D}_{a,\beta}(W)$, we estimate

$$\begin{aligned}
\int_W (1_{H^c} f)\psi \, dm_W &= \sum_{W' \in \mathcal{G}_0} \int_{W'} f\psi \, dm_{W'} \\
&\le \sum_{W' \in \mathcal{G}_0} \fint_{W'} \psi dm_{W'} |W'|^q A\delta^{1-q} |||f|||_- \\
&\le P_0^{1-q} e^{a(2\delta)^\beta} A |||f|||_- \int_W \psi \, dm_W .
\end{aligned} \tag{8.1}$$

On the other hand, if the collection of disjoint curves $\{W_i\}$ is such that $\cup_i W_i = W \cap H$,

$$\begin{aligned}
\int_W (1_{H^c} f)\psi \, dm_W &= \int_W f\psi \, dm_W - \int_W (1_H f)\psi \, dm_W \\
&\ge |||f|||_- \int_W \psi \, dm_W - \sum_i |W_i|^q A\delta^{1-q} |||f|||_- \fint_{W_i} \psi \, dm_{W_i} \\
&\ge \left\{ 1 - e^{a(2\delta)^\beta} A P_0 \delta^{-q} \mathrm{diam}^s(H)^q \right\} |||f|||_- \int_W \psi \, dm_W \, .
\end{aligned}$$

Hence, for $\mathrm{diam}^s(H)$ small enough,

$$|||1_{H^c} f|||_- \ge \frac{1}{2} |||f|||_- . \tag{8.2}$$

Accordingly, taking the supremum over W, ψ in (8.1),

$$|||1_{H^c} f|||_+ \le 2P_0^{1-q} e^{a(2\delta)^\beta} A |||1_{H^c} f|||_- =: L' |||1_{H^c} f|||_-$$

Next, to verify (4.7), if $W \in \mathcal{W}_-^s(\delta)$, then estimating as in (8.1),

$$\begin{aligned}
\int_W (1_{H^c} f)\psi dm_W &= \sum_{W' \in \mathcal{G}_0} \int_{W'} f\psi dm_{W'} \\
&\le \sum_{W' \in \mathcal{G}_0} e^{a(2\delta)^\beta} |W'|^q A\delta^{1-q} |||f|||_- \fint_W \psi dm_W \\
&\le P_0^{1-q} |W|^q e^{a(2\delta)^\beta} A\delta^{1-q} |||f|||_- \fint_W \psi dm_W \\
&\le 2P_0^{1-q} |W|^q e^{a(2\delta)^\beta} A\delta^{1-q} |||1_{H^c} f|||_- \fint_W \psi dm_W \\
&=: A' |W|^q \delta^{1-q} |||1_{H^c} f|||_- \fint_W \psi dm_W \, ,
\end{aligned} \tag{8.3}$$

where we have used (8.2) for the third inequaty.

We are left with the last cone condition, (4.8). We take $W^1, W^2 \in \mathcal{W}_-^s(\delta)$ with $d_{\mathcal{W}^s}(W^1, W^2) \le \delta$, and $\psi_i \in \mathcal{D}_{a,\alpha}(W_i)$ with $d_*(\psi_1, \psi_2) = 0$.

As in Section 5.2.3, we may assume w.l.o.g. that $|W^2| \ge |W^1|$ and $\fint_{W^1} \psi_1 = 1$. First of all note that, by condition (4.7) and our estimate above,

$$\frac{\int_{W^k} 1_{H^c} f \psi_k}{\fint_{W^k} \psi_k} \le A' |W^k|^q \delta^{1-q} |||1_{H^c} f|||_- \le \frac{1}{2} d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} c A' |||1_{H^c} f|||_-,$$

for $k=1,2$, provided $|W^2|^q \le \delta^{q-\gamma}\frac{c}{2} d_{\mathcal{W}^s}(W^1,W^2)^\gamma$. Accordingly, it suffices to consider the case $|W^2|^q \ge \delta^{q-\gamma}\frac{c}{2} d_{\mathcal{W}^s}(W^1,W^2)^\gamma$.

It follows from (5.8) that $|W^1|^q \ge \frac{1}{2}\delta^{q-\gamma}\frac{c}{2} d_{\mathcal{W}^s}(W^1,W^2)^\gamma$, recalling that $d_{\mathcal{W}^s}(W^1,W^2)\le\delta$ and (5.7). By (H2), we may decompose $W^k\cap H^c$ into at most P_0 'matched' pieces W^k_j such that $d_{\mathcal{W}^s}(W^1_j,W^2_j)\le d_{\mathcal{W}^s}(W^1,W^2)$ and $I_{W^1_j}=I_{W^2_j}$, and at most P_0+2 'unmatched' pieces $\overline{W}^k_i$, which satisfy,

$$|\overline{W}^k_i| \le C_t d_{\mathcal{W}^s}(W^1,W^2).$$

Then, using condition (4.7) and noticing that $d_*(\psi_1|_{W^1_j},\psi_2|_{W^2_j})=0$,

$$\begin{aligned}
&\left|\frac{\int_{W^1}\mathbb{1}_{H^c}f\psi_1}{\fint_{W^1}\psi_1}-\frac{\int_{W^2}\mathbb{1}_{H^c}f\psi_2}{\fint_{W^2}\psi_2}\right| \le \sum_j\left|\frac{\int_{W^1_j}f\psi_1}{\fint_{W^1}\psi_1}-\frac{\int_{W^2_j}f\psi_2}{\fint_{W^2}\psi_2}\right| + \sum_{i,k}|\overline{W}^k_i|^q\delta^{1-q}A|\!|\!|f|\!|\!|_- e^{a(2\delta)^\alpha}\\
&\le \sum_j \frac{\fint_{W^1_j}\psi_1}{\fint_{W^1}\psi_1} d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}cA|\!|\!|f|\!|\!|_- + \sum_j\left|\frac{\int_{W^2_j}f\psi_2}{\fint_{W^2}\psi_2}\left[1-\frac{\fint_{W^1_j}\psi_1\fint_{W^2}\psi_2}{\fint_{W^2_j}\psi_2\fint_{W^1}\psi_1}\right]\right| \qquad (8.4)\\
&\quad + 8(P_0+2)C_t^q d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}A|\!|\!|\mathbb{1}_{H^c}f|\!|\!|_-,
\end{aligned}$$

using (8.2). Next, since $I_{W^1_j}=I_{W^2_j}$, recalling Remark 4.4 and (5.8) we have $\int_{W^1_j}\psi_1=\int_{W^2_j}\psi_2$ and[9] $||W^1_j|-|W^2_j||\le|W^1_j|d_{\mathcal{W}^s}(W^1,W^2)$. Then applying (4.7) and recalling $\fint_{W_1}\psi_1=1$,

$$\begin{aligned}
&\left|\frac{\int_{W^2_j}f\psi_2}{\fint_{W^2}\psi_2}\left[1-\frac{\fint_{W^1_j}\psi_1\fint_{W^2}\psi_2}{\fint_{W^2_j}\psi_2\fint_{W^1}\psi_1}\right]\right| \le A|\!|\!|f|\!|\!|_-\frac{\fint_{W^2_j}\psi_2}{\fint_{W^2}\psi_2}|W^2_j|^q\delta^{1-q}\left|1-\frac{|W^2_j|}{|W^1_j|}\fint_{W^2}\psi_2\right|\\
&\le A|\!|\!|f|\!|\!|_- e^{a(2\delta)^\alpha}\left(|W^2_j|^q\delta^{1-q}\left|1-\frac{|W^2_j|}{|W^1_j|}\right|+\frac{|W^2_j|^q}{|W^2|^q}\left(\frac{\delta}{|W^2|}\right)^{1-q}\left||W^2|-\int_{W^2}\psi_2\right|\frac{|W^2_j|}{|W^1_j|}\right) \qquad (8.5)\\
&\le A|\!|\!|f|\!|\!|_- 2\left(|W^2_j|^q\delta^{1-q}d_{\mathcal{W}^s}(W^1,W^2)+2\frac{|W^2_j|^q}{|W^2|^q}\left(\frac{\delta}{|W^2|}\right)^{1-q}\left||W^2|-\int_{W^2}\psi_2\right|\right).
\end{aligned}$$

Next, recalling $|W^2|\ge\delta^{1-\frac{\gamma}{q}}[c/2]^{\frac{1}{q}}d_{\mathcal{W}^s}(W^1,W^2)^{\frac{\gamma}{q}}$ and using (5.10) yields,

$$\begin{aligned}
\left(\frac{\delta}{|W^2|}\right)^{1-q}\left||W^2|-\int_{W^2}\psi_2\right| &\le 6C_s[2/c]^{\frac{1}{q}(1-q)}\delta^{\frac{\gamma}{q}-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^{1+\gamma-\frac{\gamma}{q}}\\
&\le 4^{-\frac{1}{q}6c\delta^{1-\gamma}}d_{\mathcal{W}^s}(W^1,W^2)^\gamma,
\end{aligned}$$

where we have again used (5.7) and $d_{\mathcal{W}^s}(W^1,W^2)\le\delta$. Using this estimate and the fact that $q\le 1/2$ in (8.5) and summing over j yields,

$$\begin{aligned}
\sum_j\left|\frac{\int_{W^2_j}f\psi_2}{\fint_{W^2}\psi_2}\left[1-\frac{\fint_{W^1_j}\psi_1\fint_{W^2}\psi_2}{\fint_{W^2_j}\psi_2\fint_{W^1}\psi_1}\right]\right| &\le 2A\delta^{1-\gamma}|\!|\!|f|\!|\!|_- d_{\mathcal{W}^s}(W^1,W^2)^\gamma\sum_j\delta^{1-q}|W^2_j|^q+\frac{3}{4}c\frac{|W^2_j|^q}{|W^2|^q}\\
&\le 2A\delta^{1-\gamma}|\!|\!|f|\!|\!|_- d_{\mathcal{W}^s}(W^1,W^2)^\gamma P_0^{1-q}\left(2^q\delta+\tfrac{3}{4}c\right).
\end{aligned}$$

Finally, using this estimate in (8.4) concludes the proof of the lemma,

$$\begin{aligned}
\left|\frac{\int_{W^1}\mathbb{1}_{H^c}f\psi_1}{\fint_{W^1}\psi_1}-\frac{\int_{W^2}\mathbb{1}_{H^c}f\psi_2}{\fint_{W^2}\psi_2}\right| &\le d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}A2P_0^{1-q}|\!|\!|\mathbb{1}_{H^c}f|\!|\!|_-\Big(P_0^q e^{a(2\delta)^\alpha}+\\
&\quad +2\left(2^q\delta+\tfrac{3}{4}c\right)+4(P_0+2)P_0^{q-1}C_t^q\Big),
\end{aligned}$$

[9]Since $I_{W^1_j}=I_{W^2_j}$, the term on the right side of (5.8) proportional to C_s is absent in this case.

where we have again used (8.2). $\square$

Remark that, by Theorem 6.10, we know that there exists $N_T \in \mathbb{N}$, $N_T \le n_* + C_\star \ln \delta^{-1}$ where n_*, defined in Lemma 6.4, depends only T and δ_0 while $C_\star$ depends only on c, A, L, such that $\mathcal{L}^{N_T}\mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c,\chi A,\chi L}(\delta)$.

Proposition 8.3. *For $n_\star \ge KN_T$, with K depending only on c, A, L, P_0, C_t, if assumptions (H1) and (H2) are satisfied and $\mathrm{diam}^s(H) \le \delta \left[\frac{1}{4P_0 A}\right]^{1/q}$, then, for all $n \ge n_\star$, $[\mathcal{L}^n \mathbb{1}_{H^c}]\mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c,\chi A,\chi L}(\delta)$, where $\mathcal{C}_{c,A,L}(\delta)$ is given in Theorem 6.10.*

Proof. Define $k = N(\delta)^- + k_* n_*$, where $N(\delta)^-$, k_* and n_* are defined in Theorem 6.10. Then for $n = mk$, we may apply both Lemma 8.2 and Theorem 6.10 to obtain,

$$[\mathcal{L}^n \mathbb{1}_{H^c}]\mathcal{C}_{c,A,L}(\delta) \subset \mathcal{L}^{mk}\mathcal{C}_{c',A',L'}(\delta) \le \mathcal{C}_{\chi^m c',\chi^m A',\chi^m L'}(\delta)\,,$$

for as long as $\chi^m c' > c$, $\chi^m A' > A$ and $\chi^m L' > L$. Letting m_1 denote the least m such that $\chi^m c' < c$, $\chi^m A' < A$ and $\chi^m L' < L$, and setting $n_\star = (m_1 + 1)k$ produces the required contraction. $\square$

Remark 8.4. *Once we know the transfer operator for the open system acts as a strict contraction on the cone, it is straight forward to recover the usual full set of results for open systems with exponential escape, including a unique escape rate and limiting conditional invariant measure for all elements of the cone. See Theorem 8.18 for an example.*

8.2 Large holes

The above pertains to relatively small holes. For many applications large holes must be considered. To do so requires either a much closer look at the combinatorics of the trajectories or requiring the holes to open at even longer time intervals than what was needed before. We will pursue the second, much easier, option with the intent to show that large holes are not out of reach. To work with large holes it is convenient to weaken hypothesis (H1):

(H1′) (Complexity) There exists $P_0 > 0$ such that any stable curve of length at most δ_0 can be cut into at most P_0 pieces by ∂H.

When iterating $T^{-n}W$ for $W \in \mathcal{W}^s$, we will need to distinguish between elements of $\mathcal{G}_n(W)$ which intersect H and those that do not. Recall that $\mathcal{G}_n(W)$ subdivides long homogeneous connected components of $T^{-n}W$ into curves of length between δ_0 and $\delta_0/3$. We let $\mathcal{G}_n^H(W)$ denote the connected components of $W_i \cap H^c$, for $W_i \in \mathcal{G}_n(W)$, where $H^c = M \setminus H$. Following the notation of Section 5.2, let $Lo_n^H(W;\delta)$ denote those elements of $\mathcal{G}_n^H(W)$ having length at least δ and let $Sh_n^H(W;\delta)$ denote those elements having length at most δ.

Without the small hole condition, hypotheses (H1′) and (H2) are insufficient to prove Lemma 8.2; however, one can recover the results of Proposition 8.3 and its consequences provided one is willing to wait for a longer time. This is due to the following result.

Lemma 8.5. *If (H1′) and (H2) are satisfied, then for each $\delta > 0$ small enough (depending on $\mu_{\text{SRB}}(H)$) there exists $n_\delta \in \mathbb{N}$, $n_\delta \le C \ln \delta^{-1}$ for some constant $C > 0$, such that for all $W \in \mathcal{W}^s(\delta)$ and $n \ge n_\delta$,*

$$\sum_{W' \in Lo_n^H(W,\delta)} |W|^{-1} \int_{W'} J_{W'} T^n \ge \frac{1}{2}(1 - \mu_{\text{SRB}}(H))\,.$$

Proof. Arguing exactly as in Lemma 8.2 it follows that if (H1′) and (H2) are satisfied, then there exists $c' \geq c, A' \geq A, L' \geq L$ such that $\mathbb{1}_{H^c} + 1 \in \mathcal{C}_{c',A',L'}(\delta)$. Then by Theorem 7.3 applied to this larger cone,

$$\left| \fint_W \mathcal{L}^n(\mathbb{1}_{H^c}) - (1 - \mu_{\text{SRB}}(H)) \right| = \left| \fint_W \mathcal{L}^n(\mathbb{1}_{H^c} + 1) - 2 + \mu_{\text{SRB}}(H) \right| \leq C_H \vartheta^n .$$

On the other hand, recalling Lemma 3.1,

$$\begin{aligned} \left| \fint_W \mathcal{L}^n(\mathbb{1}_{H^c}) - \sum_{W' \in Lo_n^H(W,\delta)} |W|^{-1} \int_{W'} J_{W'}T^n \right| &\leq \sum_{W' \in Sh_n^H(W,\delta)} |W|^{-1} \int_{W'} J_{W'}T^n \\ &\leq P_0(\bar{C}_0 \delta_0^{-1} \delta + C_0 \theta_1^n), \end{aligned}$$

which implies the Lemma. □

We are now able to state the analogue of Proposition 8.3 without the small hole condition. Note, however, that now $n_\star$ has a worse dependence on δ that we refrain from making explicit.

Proposition 8.6. *Under assumptions (H1′) and (H2), for each $\delta > 0$ small enough there exists $c, A, L > 0$, $\chi \in (0,1)$ and $n_\star \in \mathbb{N}$ such that, for all $n \geq n_\star$, $[\mathcal{L}^n \mathbb{1}_{H^c}]\mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$.*

Before proving Proposition 8.6, we state an auxiliary lemma, similar to Lemma 8.2.

Lemma 8.7. *There exists $\bar{n}_\delta > 0$ such that for $n \geq \bar{n}_\delta$, $[\mathcal{L}^n \mathbb{1}_{H^c}]\mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{c',A',L'}(\delta)$, where*

$$c' = cP_0, \quad A' = A\frac{6}{1 - \mu_{SRB}(H)}, \quad \textit{and} \quad L' = L\frac{9}{1 - \mu_{SRB}(H)} .$$

Proof of Proposition 8.6. Letting $n = mk + n_\delta$, with $k = N(\delta)^- + k_* n_*$, we may apply both Lemma 8.7 and Theorem 6.10 to obtain,

$$[\mathcal{L}^n \mathbb{1}_{H^c}]\mathcal{C}_{c,A,L}(\delta) \subset \mathcal{L}^{mk}\mathcal{C}_{c',A',L'}(\delta) \leq \mathcal{C}_{\chi^m c', \chi^m A', \chi^m L'}(\delta) ,$$

for as long as $\chi^m c' > c$, $\chi^m A' > A$ and $\chi^m L' > L$. Letting m_1 denote the least m such that $\chi^m c' < c$, $\chi^m A' < A$ and $\chi^m L' < L$, and setting $n_\star = (m_1 + 1)k + n_\delta$ produces the required contraction. □

Proof of Lemma 8.7. Let $n \geq n_\delta$ (from Lemma 8.5) and $f \in \mathcal{C}_{c,A,L}(\delta)$. For each $W \in \mathcal{W}^s(\delta)$ and $\psi \in \mathcal{D}_{a,\beta}(W)$, we have

$$\int_W \psi \, \mathcal{L}^n(\mathbb{1}_{H^c} f) = \sum_{W_i \in Lo_n^H(W;\delta)} \int_{W_i} \widehat{T}^n_{W_i} \psi \, f + \sum_{W_i \in Sh_n^H(W;\delta)} \int_{W_i} \widehat{T}^n_{W_i} \psi \, f, \tag{8.6}$$

where we are using the notation of Section 5.1 for the test functions. Since any element of $\mathcal{G}_n(W)$ may produce up to P_0 elements of $Sh_n^H(W;\delta)$ according to assumption (H1′), we estimate

$$\begin{aligned} \int_W \psi \, \mathcal{L}^n(\mathbb{1}_{H^c} f) &\leq \sum_{W_i \in Lo_n^H(W;\delta)} |||f|||_+ \int_{T^n W_i} \psi + AP_0 |||f|||_- e^{a(2\delta)^\beta} \int_W \psi \, (\bar{C}_0 \delta \delta_0^{-1} + C_0 \theta_1^n) \\ &\leq |||f|||_+ \int_W \psi \, \left(1 + AP_0 e^{a(2\delta)^\beta} (\bar{C}_0 \delta \delta_0^{-1} + C_0 \theta_1^n)\right) , \end{aligned}$$

where we have used $|W| \ge \delta$ and cone condition (4.7), as well as Lemma 3.1(b) to sum over elements of $Sh_n^H(W;\delta)$.

Analogously, using Lemma 8.5,

$$\begin{aligned}\int_W \psi\, \mathcal{L}^n(\mathbb{1}_{H^c} f) &\ge \sum_{W_i \in Lo_n^H(W;\delta)} |||f|||_- \int_{T^n W_i} \psi - AP_0 |||f|||_- e^{a(2\delta)^\beta} \int_W \psi\, (\bar{C}_0 \delta \delta_0^{-1} + C_0 \theta_1^n) \\ &\ge |||f|||_- \int_W \psi \left(\frac{e^{-a(2\delta)^\beta}}{2} (1 - \mu_{\text{SRB}}(H)) - AP_0 e^{a(2\delta)^\beta} (\bar{C}_0 \delta \delta_0^{-1} + C_0 \theta_1^n) \right) .\end{aligned}$$

Let n_2 be such that $2AP_0 C_0 \theta_1^{n_2} \le \frac{1}{24}(1 - \mu_{\text{SRB}}(H))$, then for $n \ge n_2$ and δ small enough we have

$$|||\mathcal{L}^n(\mathbb{1}_{H^c} f)|||_- \ge |||f|||_- \, \frac{1}{6}(1 - \mu_{\text{SRB}}(H)). \tag{8.7}$$

Accordingly, for $n \ge \max\{n_2, n_\delta\} =: \bar{n}_\delta$ and δ small enough, we obtain

$$\frac{|||\mathcal{L}^n(\mathbb{1}_{H^c} f)|||_+}{|||\mathcal{L}^n(\mathbb{1}_{H^c} f)|||_-} \le \frac{\frac{3}{2} |||f|||_+}{|||f|||_- (\frac{1}{6}(1 - \mu_{\text{SRB}}(H))} \le \frac{9L}{1 - \mu_{\text{SRB}}(H)} =: L' . \tag{8.8}$$

The contraction of A follows step-by-step from our estimates in Section 5.2.2. Taking $W \in \mathcal{W}_-^s(\delta)$ and grouping terms as in (8.6) we treat both long and short pieces precisely as in Section 5.2.2 with the additional observation that each element of $\mathcal{G}_n(W)$ produces at most P_0 elements of $Sh_n^H(W;\delta)$ by assumption (H1′). Thus (5.4) becomes,

$$\begin{aligned}\frac{|\int_W \psi\, \mathcal{L}^n(\mathbb{1}_{H^c} f)|}{\fint_W \psi} &\le A\delta^{1-q} |W|^q |||f|||_- \left(2LA^{-1} + P_0 e^{a(2\delta)^\beta} (\bar{C}_0 \delta_0^{-1} |W| + C_0 \theta_1^n)^{1-q} \right) \\ &\le A\delta^{1-q} |W|^q |||\mathcal{L}^n(\mathbb{1}_{H^c} f)|||_- \frac{6}{1 - \mu_{\text{SRB}}(H)} =: A' \delta^{1-q} |W|^q |||\mathcal{L}^n(\mathbb{1}_{H^c} f)|||_- \, ,\end{aligned} \tag{8.9}$$

where we have applied (8.7) and assumed $n \ge \max\{n_2, n_\delta\}$.

Finally, we show how the parameter c contracts from cone condition (4.8). Following Section 5.2.3, we take $W^1, W^2 \in \mathcal{W}_-^s(\delta)$ with $d_{\mathcal{W}^s}(W^1, W^2) \le \delta$, and $\psi_k \in \mathcal{D}_{a,\alpha}(W^k)$ with $d_*(\psi_1, \psi_2) = 0$. As before, we assume w.l.o.g. that $|W^2| \ge |W^1|$ and $\fint_{W^1} \psi_1 = 1$.

We begin by recording that, by (8.9),

$$\frac{\int_{W^k} \psi_k\, \mathcal{L}^n(\mathbb{1}_{H^c} f)}{\fint_{W^k} \psi_k} \le A' |W^k|^q \delta^{1-q} |||\mathcal{L}^n(\mathbb{1}_{H^c} f)|||_- \le \frac{1}{2} d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} c A' |||\mathbb{1}_{H^c} f|||_-,$$

for $k = 1, 2$, provided $|W^2|^q \le \delta^{q-\gamma} \frac{c}{2} d_{\mathcal{W}^s}(W^1, W^2)^\gamma$. Accordingly, it suffices to consider the case $|W^2|^q \ge \delta^{\gamma - q} \frac{c}{2} d_{\mathcal{W}^s}(W^1, W^2)^\gamma$.

It follows from (5.8) that $|W^1|^q \ge \frac{1}{2} \delta^{q-\gamma} \frac{c}{2} d_{\mathcal{W}^s}(W^1, W^2)^\gamma$, recalling that $d_{\mathcal{W}^s}(W^1, W^2) \le \delta$ and (5.7).

Next, following (5.11), we decompose elements of $\mathcal{G}_n^H(W^k)$ into matched and unmatched pieces, as in (5.12). We estimate the unmatched pieces precisely as in (5.14), noting that by (H1′) and the transversality condition $(H2)$, each previously unmatched element of $\mathcal{G}_n(W^k)$ may be subdivided into at most P_0 additional unmatched pieces V_j^k, while each matched element may produce up to P_0 additional unmatched pieces each having length at most,

$$|V_j^k| \le C_t C_5 n \Lambda^n d_{\mathcal{W}^s}(W^1, W^2)\, ,$$

by Lemma 5.5(a). Thus,

$$\sum_{j,k} \left| \int_{V_j^k} f\, \widehat{T}^n_{V_j^k} \psi_k \right| \le \frac{9P_0}{1-\mu_{\mathrm{SRB}}(H)} C_4 A L \delta^{1-\gamma} d_{\mathcal{W}^s}(W^1, W^2)^\gamma |||\mathcal{L}^n(1\!\!1_{H^c} f)|||_- \,, \tag{8.10}$$

where we have used (8.7) in (5.13) to estimate

$$|||\mathcal{L}^n f|||_- \le |||\mathcal{L}^n f|||_+ \le \tfrac{3}{2} |||f|||_+ \le \tfrac{3}{2} L |||f|||_- \le \tfrac{9L}{1-\mu_{\mathrm{SRB}}(H)} |||\mathcal{L}^n(1\!\!1_{H^c} f)|||_- \,. \tag{8.11}$$

The estimate on matched pieces proceeds precisely as in (5.18), and with an additional factor of P_0 in (5.19), we arrive at (5.23), again applying (8.7),

$$\sum_j \left| \int_{U_j^1} f\, \widehat{T}_{U_j^1} \psi_1 - \int_{U_j^2} f\, \widehat{T}_{U_j^2} \psi_2 \right|$$
$$\le \tfrac{6P_0}{1-\mu_{\mathrm{SRB}}(H)} 24 \bar{C}_0 C_s A \delta^{1-\gamma} d_{\mathcal{W}^s}(W^1, W^2)^\gamma |||\mathcal{L}^n(1\!\!1_{H^c} f)|||_- \left(2^q 40 C_5 \delta^{q-\gamma} + c C_5 n^\gamma \Lambda^{-n\gamma} + 2^q C_5 n \Lambda^{-n} \delta \right).$$

Combining this estimate together with (8.10) in (5.11) (with A' in place of A in (5.11)), and recalling (5.12), yields by (5.24),

$$\left| \frac{\int_{W^1} \mathcal{L}^n f\, \psi_1}{\int_{W^1} \psi_1} - \frac{\int_{W^2} \mathcal{L}^n f\, \psi_2}{\int_{W^2} \psi_2} \right| \le \frac{6P_0}{1-\mu(H)} c A \delta^{1-\gamma} d_{\mathcal{W}^s}(W^1, W^2)^\gamma |||\mathcal{L}^n(1\!\!1_{H^c} f)|||_- \,,$$

where we have applied (5.25) to simplify the expression. Setting $c' = P_0 c$ and recalling the definition of A' from (8.9) completes the proof of the lemma. □

8.3 Sequential open systems

We conclude the section by illustrating several physically relevant models to which our results apply. Admittedly, we cannot treat the most general cases, yet we believe the following shows convincingly that the techniques developed here can be the basis of a general theory.

Dispersing billiards with small holes have been studied in [DWY, D1, D2], and results obtained regarding the existence and uniqueness of limiting distributions in the form of SRB-like conditionally invariant measures, and singular invariant measures supported on the survivor set. In the present context, we are interested in generalizing these results to the non-stationary setting. Analogous results for sequences of expanding maps with holes have been proved in [MO, GO].

We consider a family of billiard tables on $\mathbb{T}^2$ with uniform constants. In order that all the maps and transfer operators act on the same space M, we first choose a number d of scatterers B_i and a set of d arclengths $(\ell_1, \dots, \ell_d)$ for the perimeters of the scatterers. Next fix $\tau_*, \mathcal{K}_* > 0$ and $E_* < \infty$. Let $\mathcal{Q}_d(\tau_*, \mathcal{K}_*, E_*)$ denote the set of billiard tables Q on $\mathbb{T}^2$ with precisely d pairwise disjoint convex scatterers with C^3 boundaries having perimeters $\ell_1, \dots, \ell_d$, and satisfying,

$$\tau_* \le \tau_{\min} \le \tau_{\max} \le \tau_*^{-1}, \quad \mathcal{K}_* \le \mathcal{K}_{\min} \le \mathcal{K}_{\max} \le \mathcal{K}_*^{-1}, \quad \text{and} \quad \|\partial Q\|_{C^3} \le E_* \,.$$

Let $\mathcal{T}_d(\tau_*, \mathcal{K}_*, E_*)$ denote the associated set of billiard maps. Since we have fixed the number and perimeter of each scatterer, all the maps in $\mathcal{T}_d(\tau_*, \mathcal{K}_*, E_*)$ act on the same phase space M, although maps pertaining to different arrangements of scatterers may be very different.

It is shown in [DZ2] that one can choose k_0 in the definition of homoegeneity strips, and $\delta_0 > 0$ such that all $T \in \mathcal{T}_d(\tau_*, \mathcal{K}_*, E_*)$ satisfy the distortion bounds, one-step expansion and growth lemmas of Section 2 with uniform constants depending only on τ_*, $\mathcal{K}_*$ and E_*. Indeed, this family of maps preserves a common set of stable curves $\mathcal{W}^s$ [DZ2, Section 6.1]. In addition, an inspection of the proof of Lemma 6.4 shows that n_* is continuous in $\mathcal{Q}_d(\tau_*, \mathcal{K}_*, E_*)$. Accordingly, a direct application of Theorem 6.10 yields:

Proposition 8.8. *Fix $\tau_*, \mathcal{K}_* > 0$ and $E_* < \infty$, and let a, c, A, L, δ and δ_0 satisfy the conditions of Theorem 6.10. There exist $\chi < 1$ and $N_{\mathcal{T}} > 0$, such that for all $T \in \mathcal{T}_d(\tau_*, \mathcal{K}_*, E_*)$ if $n \geq N_{\mathcal{T}}$, then $\mathcal{L}_T^n \mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$, where $\mathcal{L}_T$ is the transfer operator corresponding to the map T.*

Next, we introduce holes into a billiard table $Q \in \mathcal{Q}_d(\tau_*, \mathcal{K}_*, E_*)$. Let $\mathcal{H}(P_0, C_t)$ define the set of holes $H \subset M$ which satisfy (H1), (H1′) and (H2) with constants P_0 and C_t, respectively.

For concreteness, we give two example of physical holes that satisfy our hypotheses, following [DWY, D2].

Holes of Type I. Let $\mathbb{G} \subset \partial Q$ be an arc in the boundary of one of the scatterers. Trajectories of the billiard flow are absorbed when they collide with $\mathbb{G}$. This induces a hole H in the phase space M of the billiard map of the form $(a, b) \times [-\pi/2, \pi/2]$. Note that ∂H consists of two vertical lines, so that H satisfies assumption (H2) since the vertical direction is uniformly transverse to the stable cone, as well as assumptions (H1) and (H1′) with $P_0 = 3$.

Holes of Type II. Let $\mathbb{G} \subset Q$ be an open convex set bounded away from ∂Q and having a C^3 boundary. Such a hole induces a hole H in M via its 'forward shadow.'

We define H to be the set of $(r, \varphi) \in M$ whose backward trajectory under the billiard flow enters $\mathbb{G}$ before it collides with ∂Q. Thus points in M which are about to enter $\mathbb{G}$ before their next collision under the forward billiard flow are considered still in the open system, while those points in M which would have passed through $\mathbb{G}$ on the way to their current collision are considered to have been absorbed by the hole.

With this definition, the geometry of H is simple to state: if we view $\mathbb{G}$ as an additional scatterer in Q, then H is simply the image of $\mathbb{G}$ under the billiard map. Thus H will have connected components on each scatterer that has a line of sight to $\mathbb{G}$, and ∂H will comprise curves of the form $\mathcal{S}_0 \cup T(\mathcal{S}_0)$, which are positively sloped curves, all uniformly transverse to the stable cone. Thus holes of Type II satisfy (H2) as well as (H1) and (H1′) with $P_0 = 3$. (See the discussion in [D2, Section 2.2].)

Still other holes are presented in [D2] such as side pockets, or holes that depend on both position and angle, which satisfy (H1), (H1′) and (H2), but for the sake of brevity, we do not repeat those definitions here.

As noted, both holes of Type I and Type II satisfy (H1) and (H1′) with $P_0 = 3$. Moreover, holes of Type I satisfy (H2) with C_t depending only on the maximum slope of curves in the stable cone. This slope is bounded by $\mathcal{K}_{\max} + \frac{1}{\tau_{\min}}$, so choosing $C_t \geq \mathcal{K}_* + \tau_*^{-1}$ suffices. Since ∂H for holes of Type II have positive slope, the same choice of C_t will suffice for such holes to satisfy (H2).

Fixing $\mathcal{T}_d(\tau_*, \mathcal{K}_*, E_*)$ and $\mathcal{H}(P_0, C_t)$, we define a non-stationary open billiard by choosing a sequence of maps and holes $((T_i, H_i))_{i=1}^\infty$ such that $T_i \in \mathcal{T}_d(\tau_*, \mathcal{K}_*, E_*)$ and $H_i \in \mathcal{H}(P_0, C_t)$. For each i, the corresponding open system is defined by

$$\mathring{T}_i : T_i^{-M_{\mathcal{T}}}(M \setminus H_i) \to M \setminus H_i, \quad \mathring{T}_i(x) = T_i^{M_{\mathcal{T}}}(x) \text{ for } x \in T_i^{-M_{\mathcal{T}}}(M \setminus H_i)\,,$$

where $M_{\mathcal{T}} = N_{\mathcal{T}} + n_\star$, and $N_{\mathcal{T}}$ is defined in Proposition 8.8, while $n_\star$ is as in Proposition 8.3 or Proposition 8.6 depending on which hypotheses are satisfied. To concatenate these into a sequential system, define

$$\mathring{T}_{j,i}(x) = \mathring{T}_j \circ \cdots \circ \mathring{T}_i(x) \text{ for } x \in \cap_{l=1}^j \mathring{T}_i^{-1} \circ \cdots \circ \mathring{T}_l^{-1}(M \setminus H_l)\,.$$

The transfer operator for the sequential system is defined by

$$\mathring{\mathcal{L}}_{j,i} f = \mathcal{L}_{T_j}^{M_{\mathcal{T}}} \mathbb{1}_{H_j^c} \cdots \mathcal{L}_{T_i}^{M_{\mathcal{T}}} \mathbb{1}_{H_i^c} f\,. \tag{8.12}$$

Remark 8.9. *The definition of open system we adopt here permits trajectories to escape only if they enter the hole at multiples of $M_{\mathcal{T}}$ iterates. This contrasts with the usual situation in the literature in which escape is possible at each iterate of the map. Such results rely on the assumption that the holes are sufficiently small in an appropriate sense. Yet here we are interested instead in situations in which the holes are large, and so we use the iterates of the map to acquire sufficient hyperbolicity to overcome this difficulty. To extend our results for large holes to the more natural case in which the particles can escape at each time requires an analysis of the combinatorics of the trajectories which exceeds our present goals.*

We will be interested in the evolution of probability densities under the sequential system, given by $\frac{\mathring{\mathcal{L}}_{n,k} f}{\int_M \mathring{\mathcal{L}}_{n,k} f \, d\mu_{\mathrm{SRB}}}$. Note that if $f \in \mathcal{C}_{c,A,L}(\delta)$ then $\int_M \mathring{\mathcal{L}}_{n,k} f > 0$ for each n (thus the normalization is well defined). When $f \geq 0$, this normalization coincides with the $L^1(\mu_{\mathrm{SRB}})$ norm; however, we use the integral rather than the L^1 norm as the normalization since the integral is order preserving with respect to our cone, while the L^1 norm is not. We conclude the section with a result regarding exponential loss of memory for the sequence of open billiards.

Theorem 8.10. *Fix $\tau_*, \mathcal{K}_* > 0$ and $E_* < \infty$, and let a, c, A, L, δ and δ_0 satisfy the conditions of Theorem 6.10 and Lemma 7.1. There exist $C > 0$ and $\vartheta < 1$ such that for all sequences $((T_i, H_i))_{i=1}^{\infty} \in \mathcal{T}_d(\tau_*, \mathcal{K}_*, E_*)^{\infty} \times \mathcal{H}(P_0, C_t)^{\infty}$, satisfying either (H1), (H2) or (H1′), (H2), for all $\psi \in C^1(M)$, for all $f, g \in \mathcal{C}_{c,A,L}(\delta)$ and all $1 \leq k \leq n$,*

$$\left| \int_M \frac{\mathring{\mathcal{L}}_{n,k} f}{\mu_{SRB}(\mathring{\mathcal{L}}_{n,k} f)} \psi \, d\mu_{SRB} - \int_M \frac{\mathring{\mathcal{L}}_{n,k} g}{\mu_{SRB}(\mathring{\mathcal{L}}_{n,k} g)} \psi \, d\mu_{SRB} \right| \leq C L \vartheta^{n-k} |\psi|_{C^1(M)} .$$

Proof. Proposition 8.8 implies that the constants appearing in Propositions 8.3 and 8.6 are uniform. Hence, if $f, g \in \mathcal{C}_{c,A,L}(\delta)$, then for each $k \leq n \in \mathbb{N}$, $\mathring{\mathcal{L}}_{n,k} f$, $\mathring{\mathcal{L}}_{n,k} g \in \mathcal{C}_{c,A,L}(\delta)$. Since $\int_M \frac{\mathring{\mathcal{L}}_{n,k} f}{\mu_{\mathrm{SRB}}(\mathring{\mathcal{L}}_{n,k} f)} d\mu_{\mathrm{SRB}} = \int_M \frac{\mathring{\mathcal{L}}_{n,k} g}{\mu_{\mathrm{SRB}}(\mathring{\mathcal{L}}_{n,k} g)} d\mu_{\mathrm{SRB}} = 1$, the theorem follows arguing exactly as in the proof of Theorem 7.4, using again the order preserving semi-norm $\| \cdot \|_{\psi}$, as well as the fact that by Remark 7.2,

$$\frac{\|\mathring{\mathcal{L}}_{n,k} f\|_{\psi}}{\mu_{\mathrm{SRB}}(\mathring{\mathcal{L}}_{n,k} f)} \leq \bar{C} |\psi|_{C^1} \frac{|||\mathring{\mathcal{L}}_{n,k} f|||_+}{|||\mathring{\mathcal{L}}_{n,k} f|||_-} \leq \bar{C} L |\psi|_{C^1} .$$

When invoking (7.6), it holds that $\rho_{\mathcal{C}}(\mathring{\mathcal{L}}_{n,k} f / \mu_{\mathrm{SRB}}(\mathring{\mathcal{L}}_{n,k} f), \mathring{\mathcal{L}}_{n,k} g / \mu_{\mathrm{SRB}}(\mathring{\mathcal{L}}_{n,k} g)) = \rho_{\mathcal{C}}(\mathring{\mathcal{L}}_{n,k} f, \mathring{\mathcal{L}}_{n,k} g)$ due to the projective nature of the metric. □

Note that, by changing variables, $\int_M \mathring{\mathcal{L}}_{n,k} f \, \psi \, d\mu_{\mathrm{SRB}} = \int_{\mathring{M}_{n,k}} f \, \psi \circ \mathring{T}_{n,k} \, d\mu_{\mathrm{SRB}}$, where $\mathring{M}_{n,k} = \cap_{i=1}^{n} \mathring{T}_k^{-1} \circ \cdots \circ \mathring{T}_i^{-1}(M \setminus H_i)$. Thus the conclusion of the theorem is equivalent to the expression,

$$\left| \frac{\int_{\mathring{M}_{n,k}} f \, \psi \circ \mathring{T}_{n,k} \, d\mu_{\mathrm{SRB}}}{\int_{\mathring{M}_{n,k}} f \, d\mu_{\mathrm{SRB}}} - \frac{\int_{\mathring{M}_{n,k}} g \, \psi \circ \mathring{T}_{n,k} \, d\mu_{\mathrm{SRB}}}{\int_{\mathring{M}_{n,k}} g \, d\mu_{\mathrm{SRB}}} \right| \leq C L \vartheta^{n-k} |\psi|_{C^1(M)} .$$

Remark 8.11. *Taking $H_i = \emptyset$ for each i yields an exponential loss of memory for sequential billiards without holes. Such systems have been studied and similar results obtained in [SYZ]. Note however that we allow for drastic, occasional, changes in the billiard sequence while [SYZ] deals only with slowly changing billiard tables.*

Next we show that sequential systems with holes allow us to begin investigating some physical problems that have attracted much attention: chaotic scattering and random Lorentz gasses.

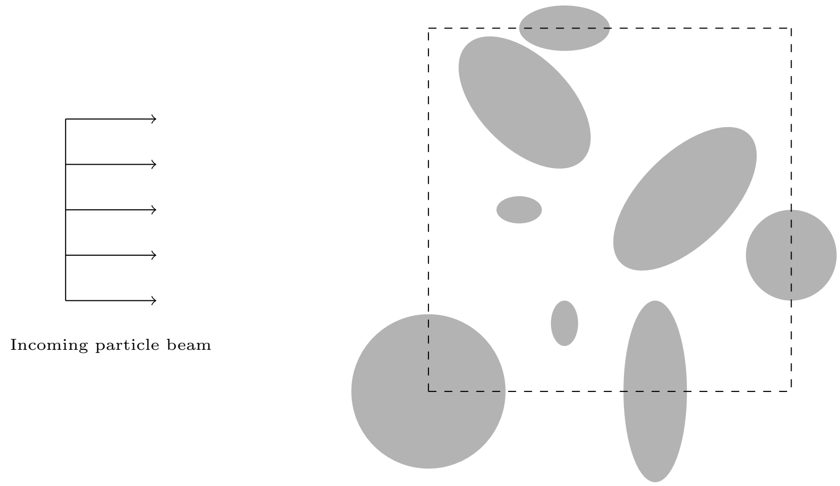


Figure 1: Obstacle configuration for which the non-eclipse condition fails and the box R (dashed line).

8.4 Chaotic scattering (boxed)

Consider a collection of strictly convex pairwise disjoint obstacles $\{B_i\}$ in $\mathbb{R}^2$ *for which the non-eclipse condition may fail.*[10] Assume that there exists a closed rectangular box $R = [a, b] \times [c, d]$ such that if an obstacle does not intersect its boundary, then it is contained in the box. In addition, if an obstacle intersects the boundary of R, then it is symmetrical with respect to a reflection across all the linear pieces of the boundary which the obstacle intersects (see Figure 1 for a picture). Finally, we will assume a finite horizon condition on the cover $\widetilde{Q}$ defined after Remark 8.14.

Remark 8.12. *The restriction regarding symmetrical reflections on the configuration of obstacles is necessary only because we did not develop the theory in the case of billiards in a polygonal box (see Remark 8.14 and the following text to see why this is relevant). Such an extension is not particularly difficult and should eventually be done. Other extensions that should be within reach of our technology are more general types of holes and billiards with corner points. Here, however, we are interested in presenting the basic ideas; addressing all possible situations would make our message harder to understand.*

Lemma 8.13. *If a particle exits R at time $t_0 \in \mathbb{R}$, then, in the time interval (t_0, ∞), it will experience only a finite number of collisions and it will never enter R again.*

Proof. Recall that $R = [a, b] \times [c, d]$. Of course, the lemma is trivially true if, after exiting R, the particle has no collisions. Let us immagine that the particle, after exiting from the vertical side $(b, c) - (b, d)$, collides instead with the obstacle B_i at the point $p = (p_1, p_2)$. Note that B_i must then intersect the same boundary, otherwise it would be situated to the left of the line $x = b$ and the particle could not collide since necessarily $p_1 > b$. Our hypothesis that B_i be symmetric with respect to reflection across $x = b$ implies that also $(2b - p_1, p_2) \in \partial B_i$. Thus, by the convexity of B_i, the horizontal segment joining p and $(2b - p_1, p_2)$ is contained in B_i. This implies that, calling $\eta = (\eta_1, \eta_2)$ the normal to ∂B_i in p, it must be $\eta_1 \geq 0$. In addition, if $v = (v_1, v_2)$ denotes the particle's velocity just before collision, it must be that $v_1 > 0$ since the particle has crosses a

[10]Remember that the *non-eclipse condition* is the requirement that the convex hull of any two obstacles does not intersect any other obstacle.

vertical line to exit B. Finally, $\langle v,\eta\rangle \le 0$, otherwise the particle would not collide with B_i. But since the velocity after collision is given by $v^+ = v - \langle v,\eta\rangle\eta$, it follows $v_1^+ = v_1 - \langle v,\eta\rangle\eta_1 \ge v_1$. That is, the particle cannot come back to the box B. Since all the obstacles are contained in a larger box B_1 and since there is a minimal distance between obstacles, the above also implies that the particle can have only finitely many collisions in the future. The other cases can be treated exactly in the same manner. □

Remark 8.14. *We want to consider a scattering problem: the particles enter the box coming from far away and with random position and/or velocity, interact and, eventually, leave the box. The basic question is how long they stay in the box or, better, what is the probability that they stay in the box longer than some time t. This is nothing other than an open billiard with holes. Unfortunately, the holes are large and our current theory allows us to deal with large holes only if enough hyperbolicity is present. To extend the result to systems with small hyperbolicity is a very important (and hard) problem as one needs to understand the combinatorics of the trajectories for long times.*

Given the above remark we modify the system in order to have the needed hyperbolicity. This is not completely satisfactory, yet it shows that our machinery can deal with large holes and illustrates exactly what further work is necessary to address the general case.

Fixing N sufficiently large, we suppose that when a particle enters the box, the boundaries of the box become reflecting and are transparent again only between the collisions kN and $kN+1$, $k \in \mathbb{N}$, counting only collisions with the convex obstacles.

More precisely, consider the billiard in R with elastic reflection at ∂R. We call such a billiard Q. Let $M = \big(\cup_i \partial B_i \cap R\big) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ be the Poincaré section,[11] and consider the Poincaré map $T : M \to M$ describing the dynamics from one collision with a convex body to the next. Unfortunately, this is not a type of billiard that fits our hypothesis. Yet, when the particle collides with ∂R we can reflect the box and imagine that the particle continues in a straight line. Note that, by our hypothesis, the image of the obstacles that intersect the boundary are the obstacle themselves, this is the reason why we restricted the obstacle configuration. We can then reflect the box three times, say across its right and top sides and then once more to make a full rectangle with twice the width and height of R, and identify the opposite sides of this larger rectangle. In this way we obtain a torus $\mathbb{T}^2$ containing pairwise disjoint convex obstacles. Such a torus is covered by four copies of R, let us call them $\{R_i\}_{i=1}^4$. We call such a billiard $\widetilde{Q}$, and we consider the Poincaré map $\widetilde{T}$ which maps from one collision with a convex body to the next, and denote its phase space by $\widetilde{M} = \cup_{i=1}^4 \widetilde{M}_i$.

Our final assumption on the obstacle configuration is that $\widetilde{Q}$ is a Sinai billiard with finite horizon. Hence $\widetilde{T} : \widetilde{M} \circlearrowleft$ falls within the scope of our theory. By construction there is a map $\pi : \widetilde{M} \to M$ which sends the motion on the torus to the motion in the box. Indeed, if $\tilde{x} \in \widetilde{M}$ and $x = \pi(\tilde{x})$, then $T^n(x) = \pi(\widetilde{T}^n(\tilde{x}))$, for all $n \in \mathbb{N}$.

We then consider the maps $\tilde{S} = \widetilde{T}^N$ and $S = T^N$, again $\pi(\widetilde{S}(\tilde{x})) = S(\pi(\tilde{x}))$. Define also the projections $\tilde{\pi}_1 : \widetilde{M} \to \widetilde{Q}$ and $\pi_1 : M \to Q$, which map a point in the Poincaré section to its position on the billiard table. For $\tilde{x} \in \widetilde{M}$, let us call $\widetilde{O}(\tilde{x})$ the straight trajectory in $\mathbb{T}^2$ between $\tilde{\pi}_1(\tilde{x})$ and $\tilde{\pi}_1(\widetilde{T}(\tilde{x}))$, and setting $x = \pi(\tilde{x})$, $O(x)$ the trajectory between $\pi_1(x)$ and $\pi_1(T((x))$. Note that the latter trajectory can consist of several straight segments joined at the boundary of R, where a reflection takes place. By construction, if $\widetilde{O}(\tilde{x})$ intersects m of the sets ∂R_i, then the trajectory

[11] Recall that $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the angle made by the post-collision velocity vector and the outward pointing normal to the boundary

$O(x)$ experiences m reflections with ∂R. Accordingly, we introduce, in our billiard system $(\widetilde{M}, \widetilde{S})$, the following holes : $\widetilde{H} = \widetilde{T}\{\tilde{x} \in \widetilde{M} \ : \ \widetilde{O}(\tilde{x}) \cap (\cup_i \partial R_i) \neq \emptyset\}$ and set $H = \pi(\widetilde{H})$.

The above makes precise the previous informal statement: the system (M, S) with hole H, describes the dynamics of the billiard (M, T) in which the particle can exit R only at the times kN, $k \in \mathbb{Z}$. The transfer operator associated with the open system $(M, S; H)$ is $\mathbb{1}_{H^c}\mathcal{L}_S\mathbb{1}_{H^c}$, yet since $(\mathbb{1}_{H^c}\mathcal{L}_S\mathbb{1}_{H^c})^n = \mathbb{1}_{H^c}(\mathcal{L}_S\mathbb{1}_{H^c})^n$, it is equivalent to study the asymptotic properties of $\mathring{\mathcal{L}}_S := \mathcal{L}_S\mathbb{1}_{H^c}$.

For a function $f : M \to \mathbb{C}$, we define its lift $\tilde{f} : \widetilde{M} \to \mathbb{C}$ by $\tilde{f} = f \circ \pi$. The pointwise identity then follows,

$$\mathring{\mathcal{L}}_{\tilde{S}}\tilde{f} := \mathcal{L}_{\tilde{S}}(\mathbb{1}_{\widetilde{H}^c}\tilde{f}) = \mathcal{L}_{\tilde{S}}((\mathbb{1}_{H^c}f) \circ \pi) = (\mathring{\mathcal{L}}_S f) \circ \pi \, . \tag{8.13}$$

While $\widetilde{H}$ is not exactly a hole of Type II, its boundary nevertheless comprises increasing curves since it is a forward image under the flow of a wave front with zero curvature (a segment of ∂R_i). Hence condition (H1′) of Section 8.2 holds with $P_0 = 3$ and condition (H2) holds with C_t depending only on the uniform angle between stable and unstable curves in $\widetilde{M}$. Thus Proposition 8.6 applies to $\mathring{\mathcal{L}}_{\tilde{S}}$ with $n_\star$ depending on C_t and $P_0 = 3$. In fact, our next result shows that also $\mathring{\mathcal{L}}_S$ contracts $\mathcal{C}_{c,A,L}(\delta)$ on M.

Proposition 8.15. *Let $n_\star \in \mathbb{N}$ be from Proposition 8.6 corresponding to $P_0 = 3$ and $C_t > 0$. Then for each small enough $\delta > 0$, there exist $c, A, L > 0$, $\chi \in (0, 1)$ such that choosing $N \geq n_\star$, $\mathring{\mathcal{L}}_S(\mathcal{C}_{c,A,L}(\delta)) \subset \mathcal{C}_{\chi c,\chi A,\chi L}(\delta)$, where $S = T^N$.*

Proof. As already noted above, Proposition 8.6 implies the existence of δ, c, A, L and χ such that $\mathring{\mathcal{L}}_{\tilde{S}}(\widetilde{\mathcal{C}}_{c,A,L}(\delta)) \subset \widetilde{\mathcal{C}}_{\chi c,\chi A,\chi L}(\delta)$ if we choose $N \geq n_\star$. Note that the constant C_t is the same on $\widetilde{M}$ and M. In fact the same choice of parameters for the cone works for $\mathring{\mathcal{L}}_S$.

For any stable curve W, $\pi^{-1}W = \cup_{i=1}^4 \widetilde{W}_i$ where each $\widetilde{W}_i$ is a stable curve satisfying $\pi(\widetilde{W}_i) = W$. Since π is invertible on each $\widetilde{M}_i$, we may define the restriction $\pi_i = \pi|_{\widetilde{M}_i}$ such that $\pi_i^{-1}(W) = \widetilde{W}_i$. Conversely, the projection of any stable curve $\widetilde{W}$ in $\widetilde{M}$ is also a stable curve in M.

Since each π_i is an isometry, and recalling (8.13), for any stable curve $W \subset M$, each $f \in \mathcal{C}_{c,A,L}(\delta)$, and all $n \geq 0$,

$$\int_{\widetilde{W}_i} \psi \circ \pi \, \mathring{\mathcal{L}}^n_{\tilde{S}}\tilde{f} \, dm_{\widetilde{W}} = \int_W \psi \, \mathring{\mathcal{L}}^n_S f \, dm_W, \quad \forall \, \psi \in C^0(\widetilde{W}),$$

where $\tilde{f} = f \circ \pi$. Moreover, if $\psi \in \mathcal{D}_{a,\beta}(W)$, then $\psi \circ \pi \in \mathcal{D}_{a,\beta}(\widetilde{W}_i)$, for each $i = 1, \ldots, 4$. This implies in particular that $|\!|\!|\mathring{\mathcal{L}}^n_S f|\!|\!|_\pm = |\!|\!|\mathring{\mathcal{L}}^n_{\tilde{S}}\tilde{f}|\!|\!|_\pm$ for all $n \geq 0$, and that $f \in \mathcal{C}_{c,A,L}(\delta)$ if and only if $\tilde{f} = f \circ \pi \in \widetilde{\mathcal{C}}_{c,A,L}(\delta)$. Consequently, $\mathring{\mathcal{L}}_S f \in \mathcal{C}_{\chi c,\chi A,\chi L}(\delta)$ if and only if $\mathring{\mathcal{L}}_{\tilde{S}}\tilde{f} \in \widetilde{\mathcal{C}}_{\chi c,\chi A,\chi L}(\delta)$, which proves the proposition. □

In contrast to the sequential systems studied in Section 8.3, the open billiard in this section corresponds to a fixed billiard map T (and its lift $\widetilde{T}$). Thus we can expect the (normalized) iterates of $\mathring{\mathcal{L}}_S$ to converge to a type of equilibrium for the open system. Such an equilibrium is termed a limiting or physical conditionally invariant measure in the literature, and often corresponds to a maximal eigenvalue for $\mathring{\mathcal{L}}_S$ on a suitable function space. Unfortunately, conditionally invariant measures for open ergodic invertible systems are necessarily singular with respect to the invariant measure and so will not be contained in our cone $\mathcal{C}_{c,A,L}(\delta)$. However, we will show that for our open billiard, the limiting conditional invariant measure is contained in the completion of $\mathcal{C}_{c,A,L}(\delta)$ with respect to the following norm.

Definition 8.16. *Let $\mathbb{V} = span\big(\mathcal{C}_{c,A,L}(\delta)\big)$ For all $f \in \mathbb{V}$ we define*

$$\|f\|_\star = \inf\{\lambda \geq 0 : -\lambda \preceq f \preceq \lambda\} \, .$$

Lemma 8.17. *The function $\|\cdot\|_\star$ has the following properties:*

a) The function $\|\cdot\|_\star$ is an order-preserving norm, that is: $-g \preceq f \preceq g$ implies $\|f\|_\star \le \|g\|_\star$.

b) There exists $C > 0$ such that for all $f \in \mathcal{C}_{c,A,L}(\delta)$ and $\psi \in C^1(M)$,

$$\left|\int_M f\,\psi\, d\mu_{SRB}\right| \le C|\!|\!|f|\!|\!|_+ |\psi|_{C^1(M)} \le C\|f\|_\star |\psi|_{C^1(M)}\,.$$

Proof. In this proof, for brevity we write $\mathcal{C}$ in place of $\mathcal{C}_{c,A,L}(\delta)$.
a) First note that $\|f\|_\star < \infty$ for any $f \in \mathbb{V}$ by the proof of Proposition 6.11 since there for any $f \in \mathcal{C}$, we find $\lambda, \mu > 0$ such that $f - \lambda$ and $\mu - f$ belong to $\mathcal{C}$.

Next, if $\|f\|_\star = 0$, then there exists a sequence $\lambda_n \to 0$ such that $-\lambda_n \preceq f \preceq \lambda_n$, and so $\lambda_n + f, \lambda_n - f \in \mathcal{C})$ for each n. Since $\mathcal{C}$ is closed (see footnote 1), this yields $f, -f \in \mathcal{C} \cup \{0\}$ and so $f = 0$ since $\mathcal{C} \cap -\mathcal{C} = \emptyset$ by construction.

Since $f \preceq g$ is equivalent to $\nu f \preceq \nu g$ for $\nu \in \mathbb{R}_+$, it follows immediately that $\|\nu f\|_\star = \nu\|f\|_\star$.

To prove the triangle inequality, let $f, g \in \mathbb{V}$. For each $\varepsilon > 0$, there exists a, b, $a \le \varepsilon + \|f\|_\star$, $b \le \varepsilon + \|g\|_\star$ such that $-a \preceq f \preceq a$ and $-b \preceq g \preceq b$. Then

$$-(\|f\|_\star + \|g\|_\star + 2\varepsilon) \preceq -(a+b) \preceq f + g \preceq a + b \le \|f\|_\star + \|g\|_\star + 2\varepsilon\,,$$

implies the triangle inequality by the arbitrariness of ε. We have thus proven that $\|\cdot\|_\star$ is a norm.

Next, suppose that $-g \preceq f \preceq g$ and let b be as above. Then

$$-\|g\|_\star - \varepsilon \preceq -b \preceq -g \preceq f \preceq g \preceq b \preceq \|g\|_\star + \varepsilon$$

which implies $\|f\|_\star \le \|g\|_\star$, again by the arbitrariness of ε. Hence, the norm is order preserving.
b) The first inequality is contained in Remark 7.2. For the second inequality, we will prove that

$$|\!|\!|f|\!|\!|_+ \le \|f\|_\star \qquad \text{for all } f \in \mathcal{C}. \tag{8.14}$$

To see this, note that if $-\lambda \preceq f \preceq \lambda$, then $|\!|\!|\lambda - f|\!|\!|_- \ge 0$ by Remark 4.6. Thus for any $\widetilde{W} \in \widetilde{\mathcal{W}}^s$ and $\psi \in \mathcal{D}_{a,\beta}(\widetilde{W})$,

$$0 \le \frac{\int_W (\lambda - f)\,\psi}{\int_W \psi} \implies \frac{\int_W f\,\psi}{\int_W \psi} \le \lambda\,,$$

and taking suprema over W and ψ yields $|\!|\!|f|\!|\!|_+ \le \lambda$, which implies (8.14). □

We now define $\mathcal{C}_\star$ to be the completion of $\mathcal{C}_{c,A,L}(\delta)$ in the $\|\cdot\|_\star$ norm. We remark that by Lemma 8.17(b), $\mathcal{C}_\star$ embeds naturally into $(C^1(M))'$, where $(C^1(M))'$ is the closure of $C^0(M)$ with respect to the norm $\|f\|_{-1} = \sup_{|\psi|_{C^1} \le 1} \int_M f\psi\, d\mu_{\text{SRB}}$. We shall show that the conditionally invariant measure for the open system $(M, T; H)$ belongs to $\mathcal{C}_\star$.

Theorem 8.18. *Let $(M, S; H)$ be as defined above, where $S = T^N$. If $N \ge n_\star$, where $n_\star$ is from Proposition 8.6, then:*

a) $h := \lim_{n\to\infty} \dfrac{\mathring{\mathcal{L}}_S^n 1}{\mu_{SRB}(\mathring{\mathcal{L}}_S^n 1)}$ is an element of $\mathcal{C}_\star$. Moreover, h is a nonnegative probability measure satisfying $\mathring{\mathcal{L}}_S h = \nu h$ for some $\nu \in (0,1)$ such that

$$\log \nu = \lim_{n\to\infty} \frac{1}{n} \log \mu_{SRB}(\cap_{i=0}^n S^{-i}(M \setminus H))\,,$$

i.e. $-\log\nu$ is the escape rate of the open system.

b) There exists $C > 0$ and $\vartheta \in (0,1)$ such that for all $f \in \mathcal{C}_{c,A,L}(\delta)$ and $n \geq 0$,

$$\left\| \frac{\mathring{\mathcal{L}}_S^n f}{\mu_{SRB}(\mathring{\mathcal{L}}_S^n f)} - h \right\|_\star \leq C\vartheta^n .$$

In addition, there exists a linear functional $\ell : \mathcal{C}_{c,A,L}(\delta) \to \mathbb{R}$ such that for all $f \in \mathcal{C}_{c,A,L}(\delta)$, $\ell(f) > 0$ and

$$\|\nu^{-n}\mathring{\mathcal{L}}_S^n f - \ell(f)h\|_\star \leq C\vartheta^n \ell(f)\|h\|_\star.$$

The constant C depends on $\mathcal{C}_{c,A,L}(\delta)$, but not on f.

Remark 8.19. *(a) The conclusions of Theorem 8.18 apply equally well to the open system $(\widetilde{M}, \tilde{S}; \widetilde{H})$.*

(b) By Lemma 8.17(b), the convergence in the $\|\cdot\|_\star$ norm given by Theorem 8.18(b) implies convergence when integrated against smooth functions $\psi \in C^1(M)$. As usual, by standard approximation arguments, the same holds for Hölder functions.

(c) Also by Lemma 8.17(b), the above convergence in $\|\cdot\|_\star$ implies leafwise convergence as well. First note that for $W \in \mathcal{W}^s(\delta)$, each $f \in \mathcal{C}_{c,A,L}(\delta)$ induces a leafwise distribution on W defined by $f_W(\psi) = \int_W f\,\psi\, dm_W$, for $\psi \in \mathcal{D}_{a,\beta}(W)$. This extends by density to $f \in \mathcal{C}_\star$. Since $h \in \mathcal{C}_\star$ by Theorem 8.18(a), let h_W denote the leafwise measure induced by h on $W \in \mathcal{W}^s(\delta)$. Then by Lemma 8.17(b) and Theorem 8.18(b), there exists $C > 0$ such that for all $n \geq 0$,

$$\left| \frac{\int_W \mathring{\mathcal{L}}_S^n f\,\psi\, dm_W}{\mu_{SRB}(\mathring{\mathcal{L}}_S^n f)} - h_W(\psi) \right| \leq C\delta^{-1}\vartheta^n , \quad \forall f \in \mathcal{C}_{c,A,L}(\delta), \forall \psi \in C^\beta(W) ,$$

and also,

$$\left| \nu^{-n} \int_W \mathring{\mathcal{L}}_S^n f\,\psi\, dm_W - \ell(f)h_W(\psi) \right| \leq C\delta^{-1}\vartheta^n \ell(f) .$$

In particular, the escape rate with respect to each $W \in \mathcal{W}^s(\delta)$ equals the escape rate with respect to μ_{SRB}.

Proof of Theorem 8.18. We argue as in the proof of Theorem 7.3. Recalling that $\|\cdot\|_\star$ is an order-preserving norm, we can apply [LSV, Lemma 2.2], taking the homogeneous function ρ to also be $\|\cdot\|_\star$ and obtain that, as in (7.6), for all $f, g \in \mathcal{C}_{c,A,L}(\delta)$,

$$\left\| \frac{\mathring{\mathcal{L}}_S^n f}{\|\mathring{\mathcal{L}}_S^n f\|_\star} - \frac{\mathring{\mathcal{L}}_S^n g}{\|\mathring{\mathcal{L}}_S^n g\|_\star} \right\|_\star \leq C\vartheta^n , \tag{8.15}$$

since $\left\| \frac{\mathring{\mathcal{L}}_S^n f}{\|\mathring{\mathcal{L}}_S^n f\|_\star} \right\|_* = 1$ and similarly for g. This implies that $\left(\frac{\mathring{\mathcal{L}}_S^n f}{\|\mathring{\mathcal{L}}_S^n f\|_\star} \right)_{n \geq 0}$ is a Cauchy sequence in the $\|\cdot\|_\star$ norm, and in addition, the limit is independent of f. Hence, defining $h_0 = \lim_{n\to\infty} \frac{\mathring{\mathcal{L}}_S^n 1}{\|\mathring{\mathcal{L}}_S^n 1\|_\star}$, we have $h_0 \in \mathcal{C}_\star$ with $\|h_0\|_\star = 1$ such that[12] for all $\psi \in C^1(M)$,

$$\int_M \mathring{\mathcal{L}}_S h_0 \psi = \lim_{n\to\infty} \frac{1}{\|\mathring{\mathcal{L}}_S^n 1\|_\star} \int_{\widetilde{M}} \mathring{\mathcal{L}}_S^{n+1} 1\psi = \lim_{n\to\infty} \frac{\|\mathring{\mathcal{L}}_S^{n+1} 1\|_\star}{\|\mathring{\mathcal{L}}_S^n 1\|_\star} \int_M h_0\psi = \|\mathring{\mathcal{L}}_S h_0\|_\star \int_M h_0\psi =: \nu \int_M h_0\psi ,$$

[12] Note that $\mathring{\mathcal{L}}_S$ extends naturally to $(\mathcal{C}^1(M))'$ and therefore to $\mathcal{C}_\star$.

where all integrals are taken with respect to μ_{SRB}. Thus, $\mathring{\mathcal{L}}_S h_0 = \nu h_0$. Moreover, the definition of h_0 implies that,

$$|h_0(\psi)| \le |\psi|_{C^0} \lim_{n\to\infty} \frac{\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n 1)}{\|\mathring{\mathcal{L}}_S^n 1\|_\star} = |\psi|_{C^0} h_0(1)\,, \quad \forall\, \psi \in C^1(M)\,, \tag{8.16}$$

thus h_0 is a measure. Addition, by the positivity of $\mathring{\mathcal{L}}_S$, h_0 is a nonnegative measure and since $\|h_0\|_\star = 1$, it must be that $h_0(1) \neq 0$. Thus we may renormalize and define

$$h := \frac{1}{h_0(1)} h_0\,.$$

Then $\frac{1\!\!1_{H^c} h}{h(H^c)}$ represents the limiting conditionally invariant probability measure for the open system $(M, S; H)$. However, we will work with h rather than its restriction to H^c because h contains information about entry into H, which we will exploit in Proposition 8.20 below.

Due to the equality in (8.16), h has the alternative characterization,

$$h = \lim_{n\to\infty} \frac{\mathring{\mathcal{L}}_S^n 1}{\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n 1)} = \lim_{n\to\infty} \frac{\mathring{\mathcal{L}}_S^n 1}{\mu_{\text{SRB}}(\mathring{M}^n)}\,,$$

as required for item (a) of the Theorem, where $\mathring{M}^n = \cap_{i=0}^n S^{-i}(M \setminus H)$ and convergence is in the $\|\cdot\|_\star$ norm.

Remark that (8.15) implies $\frac{\mathring{\mathcal{L}}_S^n f}{\|\mathring{\mathcal{L}}_S^n f\|_\star}$ converges to h_0 at the exponential rate ϑ^n. Integrating this relation and using Lemma 8.17(b), we conclude that in addition the normalization ratio $\frac{\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n f)}{\|\mathring{\mathcal{L}}_S^n f\|_\star}$ converges to $h_0(1)$ at the same exponential rate. Putting these two estimates together and using the triangle inequality yields for all $n \ge 0$,

$$\left\| \frac{\mathring{\mathcal{L}}_S^n f}{\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n f)} - h \right\|_\star \le C\vartheta^n h_0(1)^{-1}\,, \quad \forall\, f \in \mathcal{C}_{c,A,L}(\delta)\,,$$

proving the first inequality of item (b).

Next, for each, $f \in \mathcal{C}_{c,A,L}(\delta)$ let

$$\ell(f) = \limsup_{n\to\infty} \nu^{-n} \mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n f)\,. \tag{8.17}$$

Note that ℓ is bounded, homogeneous of degree one and order preserving. By Lemma 8.17(b), ℓ can be extended to $\mathcal{C}_\star$. Since $\ell(h) = 1$, $\nu^{-n}\mathring{\mathcal{L}}_S^n h = h$ and $\ell(\nu^{-n}\mathring{\mathcal{L}}_S^n f) = \ell(f)$ we can apply, again, [LSV, Lemma 2.2] as in (7.6) to f and $\ell(f)h$ and obtain

$$\|\nu^{-n}\mathring{\mathcal{L}}_S^n f - h\ell(f)\|_\star = \nu^{-n}\|\mathring{\mathcal{L}}_S^n f - \ell(f)\mathring{\mathcal{L}}_S^n h\|_\star \le C\vartheta^n \ell(f)\|h\|_\star\,, \tag{8.18}$$

proving the second inequality of item (b) of the Theorem. Note that (8.18) also implies (integrating and applying Lemma 8.17(b)) that the limsup in (8.17) is, in fact, a limit, and hence ℓ is linear. Remark that ℓ is also nonnegative for $f \in \mathcal{C}_{c,A,L}(\delta)$ by Remark 7.2.

By definition, if $f \in \mathcal{C}_{c,A,L}(\delta)$ and $\lambda > \|f\|_\star$ then $\lambda + f, \lambda - f \in \mathcal{C}_{c,A,L}(\delta)$, so that using the linearity and nonnegativity of ℓ yields,

$$-\lambda\ell(1) \le \ell(f) \le \lambda\ell(1)\,, \quad \forall\, f \in \mathcal{C}_{c,A,L}(\delta),\ \lambda > \|f\|_\star\,. \tag{8.19}$$

Thus either $\ell(f) = 0$ for all $f \in \mathcal{C}_{c,A,L}(\delta)$ or $\ell(f) \neq 0$ for all $f \in \mathcal{C}_{c,AL}(\delta)$. But if the first alternative holds, then by the continuity of ℓ with respect to the $\|\cdot\|_\star$ norm (Lemma 8.17(b)), ℓ is identically 0 on $\mathcal{C}_\star$, which is a contradiction since $\ell(h) = 1$. Thus $\ell(f) > 0$ for all $f \in \mathcal{C}_{c,A,L}(\delta)$.

Finally, applying (8.18) to $f \equiv 1$ integrated with respect to μ_{SRB} and using again Lemma 8.17(b), we obtain

$$|\nu^{-n}\mu_{\text{SRB}}(\mathring{M}^n) - \ell(1)| \leq C\vartheta^n \ell(1)\|h\|_\star\,,$$

which in turn implies that $\log \nu = \lim_{n\to\infty} \frac{1}{n} \log \mu_{\text{SRB}}(\mathring{M}^n)$ since $\ell(1) \neq 0$, as required for the remaining item of part (a) of the Theorem. Note that $\nu \neq 0$ by Remark 7.2 and (8.7), while $\nu \neq 1$ by monotonicity since the escape rate for this class of billiards is known to be exponential for arbitrarily small holes [DWY, D2]. □

We can use Theorem 8.18 to obtain exit statistics from the open billiard in the plane. As an example, for $\theta \in [0, 2\pi)$ let us define H_θ to be the set of $x \in H$ such that the first intersection of $O(T^{-1}x)$ with ∂R has velocity making an angle of θ with the positive horizontal axis. Note that H_θ is a finite union of unstable curves since it is the image of a wave front with zero curvature moving with parallel velocities. The fact that H_θ comprises unstable curves is not altered by the fact that the flow in R may reflect off of ∂R several times before arriving at a scatterer because such collisions are neutral; also, since the corners of R are right angles, the flow remains continuous at these corner points.

If the incoming particles at time zero are distributed according to a measure with density $f \in \mathcal{C}_{c,A,L}(\delta)$, then the probability that a particle leaves the box at time nN with a direction in the interval $\Theta = [\theta_1, \theta_2]$, call it $\mathbb{P}_f(x_n \in [\theta_1, \theta_2])$, can be expressed as

$$\mathbb{P}_f(x_n \in [\theta_1, \theta_2]) = \int_M 1\!\!1_{H_\Theta} \mathring{\mathcal{L}}_S^n f \, d\mu_{\text{SRB}}\,, \tag{8.20}$$

where $H_\Theta := \cup_{\theta\in\Theta} H_\theta$. Although the boundary of H_Θ comprises increasing (unstable) curves as already mentioned, the restriction on the angle may prevent ∂H_Θ from enjoying the property of continuation of singularities common to billiards. See Figure 2 (see also [D2, Sect. 8.2.2] for other examples of holes without the continuation of singularities property).

Similarly, for $p \in \partial R$, define H_p to be the set of $x \in H$ such that the last intersection of $O(T^{-1}x)$ with ∂R is p. Then for an interval $P \subset \partial R$, we define $H_P = \cup_{p\in P} H_p$, and $\int_M 1\!\!1_{H_P} \mathring{\mathcal{L}}_S^n f$ denotes the probability that a particle leaves the box at time nN through the boundary interval P.

Proposition 8.20. *For any intervals of the form $\Theta = [\theta_1, \theta_2]$, or $P = [p_1, p_2]$, any $f \in C^1(M)$ with $f \geq 0$ and $\int f \, d\mu_{SRB} = 1$, and all $n \geq 0$, we have*[13]

$$\mathbb{P}_f(x_n \in \Theta) = \nu^n h(1\!\!1_{H_\Theta})\ell(f) + \|f\|_{C^1}\mathcal{O}\big(\nu^n \vartheta^{\frac{q}{q+1}n}\big)\,, \quad \textit{and}$$
$$\mathbb{P}_f(x_n \in P) = \nu^n h(1\!\!1_{H_P})\ell(f) + \|f\|_{C^1}\mathcal{O}\big(\nu^n \vartheta^{\frac{q}{q+1}n}\big)\,.$$

Remark 8.21. *If $f \in \mathcal{C}_{c,A,L}(\delta)$, then $\ell(f) > 0$ by Theorem 8.18(b), and Proposition 8.20 provides a precise asymptotic for the escape of particles through H_Θ and H_P. For more general $f \in C^1(M)$, it may be that $\ell(f) = 0$, in which case Proposition 8.20 merely gives an upper bound on the exit statistic compared to the rate of escape given by ν.*

Proof. We prove the statement for $1\!\!1_\Theta$. The statement for $1\!\!1_P$ is similar.

To start with we assume $f \in \mathcal{C}_{c,A,L}(\delta)$, $f \geq 0$ with $\int f \, d\mu_{\text{SRB}} = 1$. As already mentioned, ∂H_Θ comprises finitely many unstable (increasing) curves in M and so H_Θ satisfies (H1′) and (H2) with

[13] If instead $f \in \mathcal{C}_{c,A,L}(\delta)$, $f \geq 0$ and $\int f \, d\mu_{\text{SRB}} = 1$, then $\|f\|_{C^1}$ can be dropped from the right hand side.

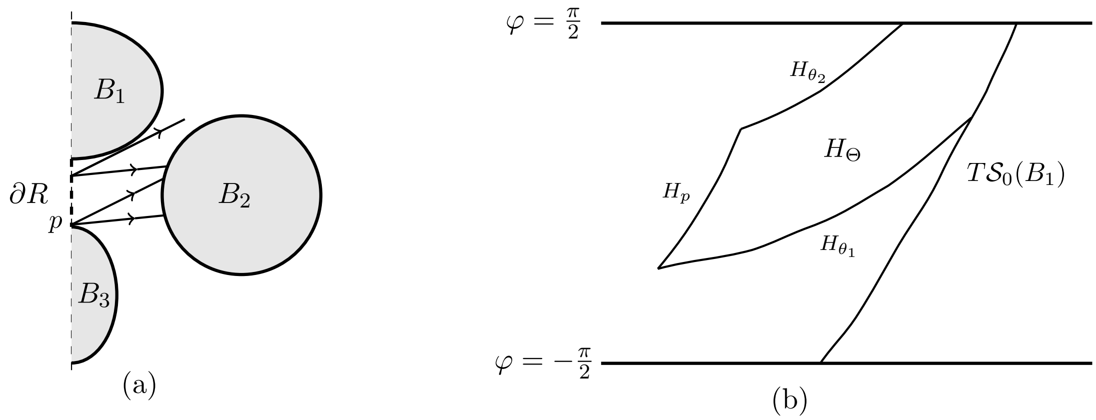


Figure 2: a) Sample rays with $\theta = \theta_1$ and $\theta = \theta_2$ striking the scatterer B_2. The point p is the topmost point of ∂B_3. b) Component of H_Θ on the scatterer B_2. In this configuration, H_{θ_1} intersects the singularity curve $T\mathcal{S}_0$ coming from B_1 while H_{θ_2} reaches $\mathcal{S}_0$ directly; however, the left boundary of H_Θ is an arc of H_p and the continuation of singularities properties fails for a hole of this type since $\theta_1 > 0$.

$P_0 = 3$ and C_t depending only on the uniform angle between the stable and unstable curves. Since $1\!\!1_{H_\Theta}$ is not in $C^1(M)$, we cannot apply Lemma 8.17(b) directly; we will use a mollification to bypass this problem.

Let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonnegative, C^∞ function supported in the unit disk with $\int \rho = 1$, and define $\rho_\varepsilon(\cdot) = \varepsilon^{-2}\rho(\cdot\,\varepsilon^{-1})$. For $\varepsilon > 0$, define the mollification,

$$\psi_\varepsilon(x) = \int 1\!\!1_{H_\Theta}(y)\rho_\varepsilon(x-y)\,dy \qquad x \in M\,.$$

We have $|\psi_\varepsilon|_{C_0} \le 1$ and $|\psi_\varepsilon'|_{C^0} \le C\varepsilon^{-1}$. Note that $\psi_\varepsilon = 1\!\!1_{H_\Theta}$ outside an ε-neighborhood of ∂H_θ (including $\mathcal{S}_0$). Letting $\tilde{\psi}_\varepsilon$ denote a C^1 function with $|\tilde{\psi}_\varepsilon|_{C^0} \le 1$, which is 1 on $N_\varepsilon(\partial H_\Theta)$ and 0 on $M \setminus N_{2\varepsilon}(\partial H_\Theta)$, we have $|1\!\!1_{H_\Theta} - \psi_\varepsilon| \le \tilde{\psi}_\varepsilon$. Due to (H2), for any $W \in \mathcal{W}^s$ such that $W \cap N_\varepsilon(\partial H_\Theta) \neq \emptyset$, using first the fact that $f \ge 0$ and then applying cone condition (4.7),

$$\begin{aligned}\int_W |1\!\!1_{H_\Theta} - \psi_\varepsilon|\, \mathring{\mathcal{L}}_S^n f\, dm_W &\le \int_W \tilde{\psi}_\varepsilon\, \mathring{\mathcal{L}}_S^n f\, dm_W \le \int_{W\cap N_{2\varepsilon}(\partial H_\Theta)} \mathring{\mathcal{L}}_S^n f\, dm_W \\ &\le 2^{1+q} A \delta^{1-q} C_t^q \varepsilon^q |||\mathring{\mathcal{L}}_S^n f|||_-\,,\end{aligned} \tag{8.21}$$

where we have used the fact that $W \cap N_{2\varepsilon}(\partial H_\Theta)$ has at most 2 connected components of length $2C_t\varepsilon$. Then integrating over M and disintegrating μ_{SRB} as in the proof of Lemma 7.1, we obtain,

$$\int_M |1\!\!1_{H_\Theta} - \psi_\varepsilon|\, \frac{\mathring{\mathcal{L}}_S^n f}{\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n f)}\, d\mu_{\text{SRB}} \le \int_M \tilde{\psi}_\varepsilon\, \frac{\mathring{\mathcal{L}}_S^n f}{\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n f)}\, d\mu_{\text{SRB}} \le C\varepsilon^q \frac{|||\mathring{\mathcal{L}}_S^n f|||_-}{\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n f)}\,. \tag{8.22}$$

By Remark 7.2, $\mu_{\text{SRB}}(\mathring{\mathcal{L}}_S^n f) \ge \bar{C}^{-1} |||\mathring{\mathcal{L}}_S^n f|||_-$, so the bound is uniform in n. Since $\tilde{\psi}_\varepsilon \in C^1(M)$ the bound carries over to $h(\tilde{\psi}_\varepsilon)$, and since h is a nonnegative measure, to $h(1\!\!1_{H_\Theta} - \psi_\varepsilon)$. Thus for each $n \ge 0$ and $\varepsilon > 0$,

$$\begin{aligned}\int 1\!\!1_{H_\Theta}\, \mathring{\mathcal{L}}_S^n f\, d\mu_{\text{SRB}} &= \int (1\!\!1_{H_\Theta} - \psi_\varepsilon)\, \mathring{\mathcal{L}}_S^n f\, d\mu_{\text{SRB}} + \left(\int \psi_\varepsilon\, \mathring{\mathcal{L}}_S^n f\, d\mu_{\text{SRB}} - \nu^n \ell(f) h(\psi_\varepsilon)\right) \\ &\quad + \nu^n \ell(f) h(\psi_\varepsilon - 1\!\!1_{H_\Theta}) + \nu^n \ell(f) h(1\!\!1_{H_\theta}) \\ &= \mathcal{O}\big(\varepsilon^q \nu^n \ell(f)\big) + \mathcal{O}\big(|\psi_\varepsilon|_{C^1} \nu^n \vartheta^n \ell(f)\big) + \nu^n \ell(f) h(1\!\!1_{H_\Theta})\,,\end{aligned} \tag{8.23}$$

where we have applied (8.22) to the first and third terms and Theorem 8.18(b) and Lemma 8.17(b) to the second term. Since $|\psi_\varepsilon|_{C^1} \leq \varepsilon^{-1}$, choosing $\varepsilon = \vartheta^{n/(q+1)}$ yields the required estimate for $f \in \mathcal{C}_{c,A,L}(\delta)$.

To conclude, note that by Lemma 7.6, there exists $C_\flat > 0$ such that, if $f \in C^1(M)$, then, for each $\lambda \geq C_\flat \|f\|_{\mathcal{C}^1}$, $\lambda + f \in \mathcal{C}_{c,A,L}(\delta)$. Hence, by the linearity of the integral, $\ell(f)$ as defined in (8.17) can be extended to $f \in C^1$ by $\ell(f) = \ell(\lambda + f) - \ell(\lambda)$, and the limsup is in fact a limit since since the limit exists for $\lambda + f, \lambda \in \mathcal{C}_{c,A,L}(\delta)$ (see (8.18) and following).

Now take $f \in C^1$ with $\int f \, d\mu_{\text{SRB}} = 1$ and $\lambda \geq C_\flat \|f\|_{C^1}$ as above. Then, necessarily $\lambda + f \geq 0$, and so recalling (8.20), we have

$$\begin{aligned}\mathbb{P}_{\frac{\lambda+f}{1+\lambda}}(x_n \in \Theta) &= \int_M \mathbb{1}_{H_\Theta} \mathring{\mathcal{L}}_S^n \Big(\tfrac{\lambda+f}{1+\lambda}\Big) = \frac{\lambda}{1+\lambda} \int_M \mathbb{1}_{H_\Theta} \mathring{\mathcal{L}}_S^n 1 + \frac{1}{1+\lambda} \int_M \mathbb{1}_{H_\Theta} \mathring{\mathcal{L}}_S^n f \\ &= \frac{\lambda}{1+\lambda} \mathbb{P}_1(x_n \in \Theta) + \frac{1}{1+\lambda} \mathbb{P}_f(x_n \in \Theta).\end{aligned}$$

Hence by (8.23),

$$\begin{aligned}\mathbb{P}_f(x_n \in \Theta) &= (1+\lambda)\mathbb{P}_{\frac{\lambda+f}{1+\lambda}}(x_n \in \Theta) - \lambda \mathbb{P}_1(x_n \in \Theta) \\ &= \nu^n h(\mathbb{1}_{H_\Theta})(\lambda \ell(1) + \ell(f)) - \nu^n h(\mathbb{1}_{H_\Theta}) \lambda \ell(1) + \lambda \mathcal{O}\big(\nu^n \vartheta^{\frac{q}{q+1} n}\big) \\ &= \nu^n h(\mathbb{1}_{H_\Theta}) \ell(f) + \|f\|_{\mathcal{C}^1} \mathcal{O}\big(\nu^n \vartheta^{\frac{q}{q+1} n}\big).\end{aligned}$$

□

8.5 Random Lorentz gas (lazy gates)

Consider a Lorentz gas described in [AL, Section2]. That is, we have a lattice of cells of size one with circular obstacles of fixed radius r at their corners and a random obstacle $B(z)$ of fixed radius ρ and center in a set O at their interior.[14] The central obstacle is small enough not to intersect with the other obstacles but large enough to prevent trajectories from crossing the cell without colliding with an obstacle. We call the openings between different cells *gates*, see Figure 3b, and require that no trajectory can cross two gates without making at least one collision with the obstacles. Thus we fix r and ρ satisfying[15] the following conditions:

$$\tfrac{1}{3} \leq r < \tfrac{1}{2}, \quad \text{and} \quad 1 - 2r < \rho < \tfrac{\sqrt{2}}{2} - r. \tag{8.24}$$

With r and ρ fixed, the set of possible configurations of the central obstacle are described by $\omega \in \Omega = O^{\mathbb{Z}^2}$. In order to ensure that particles cannot cross directly from $\hat{R}_1$ to $\hat{R}_3$ or from $\hat{R}_2$ to $\hat{R}_4$ without colliding with an obstacle, and to ensure a minimum distance between scatterers, we fix $\varepsilon_* > 0$ and require the center $c = (c_1, c_2)$ of the random obstacle B_ω, $\omega \in \Omega$, (the central obstacle C_5 in Figure 3b) to satisfy,

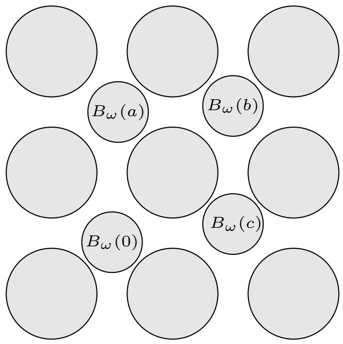
$$1 - (r + \rho - \varepsilon_*) \leq c_1, c_2 \leq r + \rho - \varepsilon_*. \tag{8.25}$$

Note that (8.24) and (8.25) imply that all possible positions of the central scatterer B_ω result in a billiard table with $\tau_{\min} \geq \tau_* := \min\{\varepsilon_*, 1 - 2r\} > 0$.

On Ω the space of translations ξ_z, $z \in \mathbb{Z}^2$, acts naturally as $[\xi_z(\omega)]_x = \omega_{z+x}$, see Figure 3a. We assume that the obstacle configurations are described by a measure $\mathbb{P}_e$ which is ergodic with respect to the translations.

[14] The assumption that all obstacles are circular is not essential and can be relaxed by requiring that the obstacles at the corners are symmetric with respect to reflections as described in Section 8.4.

[15] Finite horizon requires $r \geq \frac{1}{1+\sqrt{2}}$, yet our added condition that a particle cannot cross diagonally from, say, $\hat{R}_1$ to $\hat{R}_2$ without making a collision requires further that $r \geq \frac{1}{3}$.



$a = (1,0); b = (1,1); c = (1,0)$

Fig 3a: *Configuration of random obstacles* $B_\omega(z)$

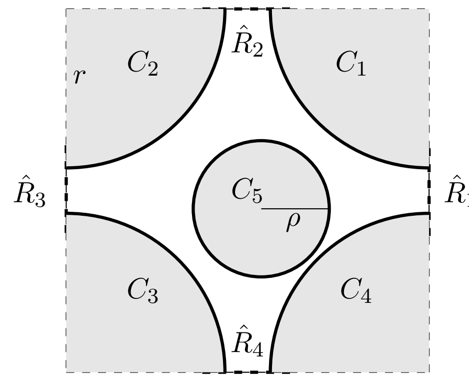


Fig 3b: *Poincaré section* C_i *and gates* $\hat{R}_i$

Exactly as in the section 8.4, we assume that the gates are reflecting and become transparent only after N collisions with the obstacles. Thus when the particle enters a cell it will stay in that cell for at least N collisions with the obstacles, hence the *lazy* adjective.

As described in section 8.4, when the particle reflects against a gate one can reflect the table three times and see the flow (for the times at which the gates are closed) as a flow in a finite horizon Sinai billiard on the two torus. Note that the Poincaré section $M = \cup_{i=1}^5 C_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ in each cell is exactly the same for each ω and z since the arclength of the boundary is always the same, while the Poincaré map T_z changes depending on the position of the central obstacle, see Figure 3b. As in Section 8.3 let us call $\mathcal{T}(\tau_*)$ the collection of the different resulting billiard maps corresponding to tables that maintain a minimum distance $\tau_* > 0$ between obstacles, as required by (8.24) and (8.25). (Note that the parameters $\mathcal{K}_*$ and E_* of Section 8.3 are fixed in this class once r and ρ are fixed.) The only difference with Section 8.4, as far as the dynamics in a cell is concerned, consists in the fact that we have to be more specific about which cell the particle enters, as now exiting from one cell means entering into another.

Recalling the notation of Section 8.4, if we call $R(z)$ the cell at the position $z \in \mathbb{Z}^2$, then the gates $\hat{R}_i$ are subsets of $\partial R(z)$. We denote by $\widetilde{R}(z)$ the lifted cell (viewed as a subset of $\mathbb{T}^2$) after reflecting $R(z)$ three times, and by $(\widetilde{M}, \widetilde{T}_z)$ the corresponding billiard map. As before, the projection $\pi : \widetilde{M} \to M$ satisfies $\pi \circ \widetilde{T} = T \circ \pi$. Then the hole $\widetilde{H}(z)$ can be written as $\widetilde{H}(z) = \cup_{i=1}^4 \widetilde{H}_i(z)$, where $\pi(\widetilde{H}_i(z)) =: H_i(z)$ are the points $x \in M$ such that $O(T^{-1}x) \cap \partial R(z) \in \hat{R}_i$.[16] Due to our assumption (8.24), this point of intersection is unique for each x since consecutive collisions with ∂R cannot occur. Then $H(z) = \pi(\widetilde{H}(z)) = \cup_{i=1}^4 H_i(z)$.
As discussed in Section 8.4, the holes, are neither of Type I nor of Type II, yet they satisfy (H1′) and (H2) with $P_0 = 3$ and C_t depending only on the uniform angle between stable and unstable cones for the induced billiard map.

Yet for our dynamics, when a particle changes cell at the Nth collision, it is because after $N-1$ collisions, that particle is in $G_i(z) := T_z^{-1} H_i(z)$, and in fact it will never reach $H_i(z)$. Unfortunately, the geometry of $G(z) := \cup_{i=1}^4 G_i(z)$ is not convenient for our machinery, yet we will be able reconcile this difficulty after defining the dynamics precisely as follows.

The phase space is $\mathbb{Z}^2 \times M$. For $x \in M$, denote by $p(x)$ the position of x in $R(z)$ and by $\theta(x)$

[16] The hole depends on the trajectory of x, which is different in different cells and hence depends on z, while the gates $\hat{R}_i$ are independent of z.

the angle of its velocity with respect to the positive horizontal axis in $R(z)$. We define

$$w(z,x) = \begin{cases} 0 =: w_0 & \text{if } x \notin G(z) \\ e_1 =: w_1 & \text{if } x \in G_1(z) \\ e_2 =: w_2 & \text{if } x \in G_2(z) \\ -e_1 =: w_3 & \text{if } x \in G_3(z) \\ -e_2 =: w_4 & \text{if } x \in G_4(z). \end{cases}$$

Also we set $\mathfrak{W} = \{w_0, \dots, w_4\}$. If $x \in G_i(z)$, then we call $\bar{q}(x) = (q, \theta) \in \hat{R}_i \times [0, 2\pi)$ the point $\bar{q}$ such that $q = O(x) \cap \hat{R}_i$ and $\theta = \theta(x)$, i.e. without reflection at $\hat{R}_i$. We then consider $\bar{q}$ as a point in the cell $z + w(z,x) = z + w_i$ and call $T_{z,i}(x)$ the post collisional velocity at the next collision with an obstacle under the flow starting at $\bar{q}$. Note that in the cell $R(z + w_i)$, $\bar{q} \in \hat{R}_{\bar{i}}$, where $\bar{i} = i + 2 \pmod{4^*}$.[17] Thus if Φ_t^z denotes the flow in $R(z)$, then with this notation, $G_i(z)$ is the projection on M of $\hat{R}_i$ under the inverse flow Φ_{-t}^z while $H_{\bar{i}}(z + w(z,x))$ is the projection on M of $\hat{R}_{\bar{i}}$ under the forward flow $\Phi_t^{z+w_i}$. Thus,

$$H_{\bar{i}}(z + w_i) = T_{z,i} G_i(z) \implies \mathbb{1}_{G_i(z)} \circ T_{z,i}^{-1} = \mathbb{1}_{H_{\bar{i}}(z+w_i)}\,, \tag{8.26}$$

which is a relation we shall use to control the action of the relevant transfer operators below.

Differing slightly from the previous section, here is convenient to set $S_z = T_z^{N-1}$, and define

$$F(z,x) = \begin{cases} (z, S_z \circ T_z(x)) =: (z, \widehat{S}_z(x)) & \text{if } x \notin G(z) \\ (z + w(z,x), S_{z+w(z,x)} \circ T_{z,i}(p)) =: (z + w(z,x), \widehat{S}_z(x)) & \text{if } x \in G_i(z). \end{cases}$$

We set $(z_n, x_n) = F^n(z,x)$ and we call n the *macroscopic time*, which corresponds to Nn collisions with the obstacles. The above corresponds to a dynamics in which when the particle enters a cell it is trapped in the cell for N collisions with the obstacles; then the gates open and until the next collision the particle can change cell, after which it is trapped again for N collisions and so on.

We want to compute the probability that a particle visits the sets $G_{k_0}(z_0), \cdots G_{k_{n-1}}(z_{n-1})$, in this order, where we have set $G_0(z) = M \setminus \cup_{i=1}^4 G_i(z)$. Similarly, we define $H_0(z) = M \setminus \cup_{i=1}^4 H_i(z)$. This itinerary corresponds to a particle that at time i changes its position in the lattice by w_{k_i}. Following the notation of [AL], we call $\mathbb{P}_\omega$ the probability distribution in the path space $\mathfrak{W}^{\mathbb{N}}$ conditioned on the central obstacles being in the positions specified by $\omega \in \Omega$. Hence, if the particle starts from the cell $z_0 = (0,0)$ with x distributed according to a probability distribution with smooth density $f \in \mathcal{C}_{c,A,L}(\delta)$, then we have[18] $z_n = \sum_{k=0}^{n-1} w_{k_i}$ and, for each obstacle distribution $\omega \in \Omega$,

$$\begin{aligned} \mathbb{P}_\omega(z_0, z_1, \dots, z_n) &= \int_M f(x) \mathbb{1}_{G_{k_0}(z_0)}(x) \mathbb{1}_{G_{k_1}(z_1)}(\widehat{S}_0(x)) \cdots \\ &\qquad \cdots \mathbb{1}_{G_{k_{n-1}}(z_{n-1})}(\widehat{S}_{z_{n-2}} \circ \cdots \circ \widehat{S}_0(x))\, d\mu_{\text{SRB}}(x) \\ &= \int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})} \cdots \mathring{\mathcal{L}}_{G_{k_0}(z_0)} f \, d\mu_{\text{SRB}} \end{aligned} \tag{8.27}$$

where $\mathring{\mathcal{L}}_{G_{k_j}(z_j)} := \mathcal{L}_{T_{z_{j+1}}}^{N-1} \mathcal{L}_{T_{z_j,k_j}} \mathbb{1}_{G_{k_j}(z_j)}$, and we have set $T_{z,0} := T_z$. See [AL] for more details. We will prove below that if N is sufficiently large, then Theorem 8.10 applies to each operator $\mathring{\mathcal{L}}_{G_k}$.

[17] By (mod 4*) we mean cyclic addition on 1, 2, 3, 4 rather than 0, 1, 2, 3.

[18] Since $z_0 = (0,0)$, it is equivalent to specify $z_1, \dots z_n$ or $w_{k_0}, \dots w_{k_{n-1}}$ since w_{k_j} can be recovered as $w_{k_j} = z_{j+1} - z_j$.

This suffices to obtain an exponential loss of memory property (the analogue of the result obtained for piecewise expanding maps in [AL, Theorem 6.1]), that is property **Exp** in [AL, Section 4.1]. This is the content of the following theorem.

Theorem 8.22. *There exist $C_* > 0$ and $\vartheta \in (0,1)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, if x is distributed according to $f \in \mathcal{C}_{c,A,L}(\delta)$ with $\int_M f = 1$ and $z_0 = (0,0)$, for all $n > m \ge 0$ and all $w \in \mathfrak{W}^{\mathbb{N}}$,*

$$\left|\mathbb{P}_\omega(w_{k_n} \mid w_{k_0} \dots w_{k_{n-1}}) - \mathbb{P}_{\xi_{z_m}\omega}(w_{k_n} \mid w_{k_m} \dots w_{k_{n-1}})\right| \le C_* \vartheta^{n-m}. \tag{8.28}$$

Proof. Note that for $m \ge 0$, $\xi_{z_m}\omega$ sends the cell at z_m to (0,0). Thus according to equation 8.27, for x distributed according to $f \in \mathcal{C}_{c,A,L}(\delta)$ with $z_0 = (0,0)$, we have

$$\mathbb{P}_{\xi_{z_m}\omega}(w_{k_m}, \dots w_{k_n}) = \int_M \mathring{\mathcal{L}}_{G_{k_n}(z_n)} \cdots \mathring{\mathcal{L}}_{G_{k_m}(z_m)} f \, d\mu_{\text{SRB}} \, .$$

As remarked earlier, the sets $G_i(z)$ do not satisfy assumption (H2) so that Proposition 8.6 does not apply directly. Yet, it follows from (8.26) that for $g \in \mathcal{C}_{c,A,L}(\delta)$,

$$\mathring{\mathcal{L}}_{G_{k_j}(z_j)} g = \mathcal{L}^{N-1}_{T_{z_{j+1}}} \mathcal{L}_{T_{z_j,k_j}} (\mathbb{1}_{G_{k_j}(z_j)} g) = \mathcal{L}^{N-1}_{T_{z_{j+1}}} \left(\mathbb{1}_{H_{\bar{k}_j}(z_{j+1})} \mathcal{L}_{T_{z_j,k_j}} g\right),$$

where, as before, $\bar{k}_j = k_j + 2 \pmod{4^*}$. Then, just as in the proof of Proposition 8.6, it may be the case that $\mathcal{L}_{T_{z_j,k_j}} g$ is not in $\mathcal{C}_{c,A,L}(\delta)$. Yet, it is immediate from our estimates in Section 5 that $\mathcal{L}_{T_{z_j,k_j}} g \in \mathcal{C}_{c',A',3L}(\delta)$ for any billiard map $T_{z_j,k_j} \in \mathcal{T}(\tau_*)$ for some constants c', A' depending only on $\mathcal{T}(\tau_*)$. Since the sets $H_i(z)$ do satisfy (H1$'$) and (H2) with $P_0 = 3$ and C_t depending only on the angle between stable and unstable cones, which has a uniform minimum in the family $\mathcal{T}(\tau_*)$, there exists $\chi < 1$ and N sufficiently large as in Proposition 8.6 so that[19] $\left[\mathcal{L}^{N-1}_{T_{z_{j+1}}} \mathbb{1}_{H_{\bar{k}_j}(z_{j+1})}\right] \mathcal{C}_{c',A',3L}(\delta) \subset \mathcal{C}_{\chi c, \chi A, \chi L}(\delta)$, and both χ and N are independent of z_{j+1} and k_j. This implies in particular that

$$\mathring{\mathcal{L}}_{G_i(z)} \mathcal{C}_{c,A,L}(\delta) \subset \mathcal{C}_{\chi c,\chi A,\chi L}(\delta) \qquad \text{for each } i \text{ and all } z \in \mathbb{Z}^2.$$

Now as in the proof of Theorem 7.3, using the fact that $\mu_{\text{SRB}}(\cdot)$ is homogeneous and order preserving on $\mathcal{C}_{c,A,L}(\delta)$ and that $\mu_{\text{SRB}}(\bar{\mathcal{L}}_m f) = \mu_{\text{SRB}}(f) = 1$, where $\bar{\mathcal{L}}_m f = \frac{\mathring{\mathcal{L}}_{G_{k_{m-1}}(z_{m-1})} \cdots \mathring{\mathcal{L}}_{G_{k_0}(z_0)} f}{\int_M \mathring{\mathcal{L}}_{G_{k_{m-1}}(z_{m-1})} \cdots \mathring{\mathcal{L}}_{G_{k_0}(z_0)} f} \in \mathcal{C}_{c,A,L}(\delta)$, we estimate as in (7.6) and (7.7),

$$\begin{aligned}
&\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})} \cdots \mathring{\mathcal{L}}_{G_{k_m}(z_m)} (f - \bar{\mathcal{L}}_m f) \, d\mu_{\text{SRB}} \\
&\quad \le C \vartheta^{n-m} \min \left\{ \int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})} \cdots \mathring{\mathcal{L}}_{G_{k_m}(z_m)} f, \int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})} \cdots \mathring{\mathcal{L}}_{G_{k_m}(z_m)} \bar{\mathcal{L}}_m f \right\},
\end{aligned} \tag{8.29}$$

for some $\vartheta < 1$ depending on the diameter of $\mathcal{C}_{\chi c,\chi A,\chi L}(\delta)$ in $\mathcal{C}_{c,A,L}(\delta)$.

[19]Here in fact our operators are of the form $\mathcal{L}^n \mathbb{1}_H$ while in Proposition 8.6 they have the form $\mathcal{L}^n \mathbb{1}_{H^c}$ for some set H. Yet, this is immaterial since the boundaries of H and H^c in M are the same so that (H1$'$) and (H2), and in particular Lemma 8.5, apply equally well to both sets.

Finally, the left hand side of (8.28) reads

$$
\begin{aligned}
&\left|\frac{\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})}\cdots\mathring{\mathcal{L}}_{G_{k_0}(z_0)}f}{\int_M \mathring{\mathcal{L}}_{G_{k_{n-2}}(z_{n-2})}\cdots\mathring{\mathcal{L}}_{G_{k_0}(z_0)}f}-\frac{\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}f}{\int_M \mathring{\mathcal{L}}_{G_{k_{n-2}}(z_{k_{n-2}})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}f}\right|\\
&\le\left|\frac{\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}\bar{\mathcal{L}}_m f-\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}f}{\int_M \mathring{\mathcal{L}}_{G_{k_{n-2}}(z_{n-2})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}\bar{\mathcal{L}}_m f}\right|\\
&\quad+\left|\frac{\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}f}{\int_M \mathring{\mathcal{L}}_{G_{k_{n-2}}(z_{k_{n-2}})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}\bar{\mathcal{L}}_m f}-\frac{\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}f}{\int_M \mathring{\mathcal{L}}_{G_{k_{n-2}}(z_{k_{n-2}})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}f}\right|\\
&\le C\vartheta^{n-m}+C\vartheta^{n-m-1},
\end{aligned}
$$

where we have applied (8.29) twice and used the fact that $\frac{\int_M \mathring{\mathcal{L}}_{G_{k_{n-1}}(z_{n-1})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}g}{\int_M \mathring{\mathcal{L}}_{G_{k_{n-2}}(z_{k_{n-2}})}\cdots\mathring{\mathcal{L}}_{G_{k_m}(z_m)}g}\le 1$ for any $g\in\mathcal{C}_{c,A,L}(\delta)$. □

In particular, Theorem 8.22, together with[20] [AL, Theorem 6.4], implies that $\lim_{n\to\infty}\frac{1}{n}z_n=0$ for $\mathbb{P}_e$ almost all ω, that is, the walker has, $\mathbb{P}_e$-almost-surely, no drift. See [AL, Section 6] for details.[21] This latter fact could be deduced also from the ergodicity result in [Le06, Theorem 5.4], however Theorem 8.22 is much stronger (indeed, by [AL, Theorem 6.4], it implies [Le06, Theorem 5.4]) since it proves some form of memory loss that is certainly not implied by ergodicity alone. It is therefore sensible to expect that more information on the random walk will follow from Theorem 8.22, although this will require further work.

We conclude with a corollary of Theorem 8.22 which implies the same exponential loss of memory for particles distributed according to two different initial distributions. For $f\in\mathcal{C}_{c,A,L}(\delta)$, let $\mathbb{P}_{\omega,f}(\cdot)$ denote the probability in the path space $\mathfrak{W}^{\mathbb{N}}$ conditioned on the central obstacles being in position $\omega\in\Omega$ and with x initially distributed according to $fd\mu_{\text{SRB}}$.

Corollary 8.23. *There exist $C>0$ and $\vartheta\in(0,1)$ such that for all $f,g\in\mathcal{C}_{c,A,L}(\delta)$ with $\int_M f=\int_M g=1$ and $\mathbb{P}$-a.e. $\omega\in\Omega$, if $z_0=(0,0)$, then for all $n\ge 0$ and all $w\in\mathfrak{W}^{\mathbb{N}}$,*

$$\left|\mathbb{P}_{\omega,f}(w_{k_n}\mid w_{k_0}\dots w_{k_{n-1}})-\mathbb{P}_{\omega,g}(w_{k_n}\mid w_{k_0}\dots w_{k_{n-1}})\right|\le C\vartheta^n.$$

Proof. The proof is the same as that of Theorem 8.22 since (8.29) holds as well with $\bar{\mathcal{L}}_m f$ replaced by g. □

[20] Remark that [AL, Theorem 6.4] requires $\mu_{\text{SRB}}(G_i(z))$ to be the same for each i and z, independently of ω. This is precisely the case here since $G_i(z)$ is defined as the projection of $\hat{R}_i$ under the inverse flow Φ^z_{-t}, and $\text{Leb}(\hat{R}_i\times[0,2\pi))$ in the phase space of the flow is independent of i, while μ_{SRB} is the projection onto M of Lebesgue measure, which is invariant under the flow.

[21] The arguments in [AL, Section 6] are developed for expanding maps, but the relevant parts apply verbatim to the present context.

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