CENTRAL LIMIT THEOREM FOR SEQUENTIAL DYNAMICAL SYSTEMS

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Abstract. We present a general approach to establish the Central Limit Theorem with error bounds for sequential dynamical systems. The main tool we develop is the application to this setting of a projective metric on complex cones, following the ideas introduced by Rugh and Dubois. To demonstrate the power of the proposed setting, we apply it to both sequential expanding maps, where similar results are known, and to sequential dispersing billiards, for which no such results are currently known.

Contents

1. Introduction 2
2. Setting and Main Results 3
2.1. Sequential Smooth Expanding Maps: A Preview 3
2.2. Sequential Dispersing Billiards: A Preview 4
2.3. Abstract Framework and Central Limit Theorems 4
3. Proof of Theorem 2.6 9
4. Proof of Theorem 2.7 14
5. Cones 18
5.1. Real Cones 18
5.2. Complex Cones 19
5.3. Cone Contraction Implies Loss of Memory 28
5.4. A Useful Lower Bound 30
6. Application: Smooth Expanding Maps 30
6.1. Sequential Expanding Maps 30
6.2. Conditions (O-1), (O-2) (Applicability of Theorem 2.6) 31
6.3. Complex Cones for Expanding Maps 32
6.4. Verification of Condition (O-3) 33
6.5. Central Limit Theorem and Variance for Sequential Expanding Maps 33
7. Application: Sequential Billiard Maps 37
7.1. Definition and Contraction of the Real Cone $C_{\mathbb{R}} = \mathcal{C}_{c,A,L}(\delta)$ 38
7.2. Conditions (C-1)-(C-4) 39
7.3. Alternative Description of the Cone $C_{\mathbb{R}}$ 40
7.4. Introduction of Observables and Some Basic Properties 41
7.5. Verification of Hypothesis (O-1) 41
7.6. Contraction of the Complex Cone $\mathcal{C}_{\mathbb{C}}$ 42
7.7. Verification of (O-2) and (O-3) 44
7.8. Central Limit Theorem for Sequential Dispersing Billiards 45
7.9. Variance Growth in the Random Case 46
7.10. Proof of Lemma 7.5 47
7.11. Proof of Proposition 7.9 (Bounding the Norm of $\mathcal{L}_T$) 52
Appendix A. Complexification 59
References 60

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1. Introduction

This paper is concerned with obtaining a self-normed Central Limit Theorem with error terms for sequential dynamical systems. Such results are relevant for many open problems; let us just mention random Lorentz gasses of the type initially presented in [28], in which a particle moves between cells formed by randomly placed obstacles in an approximately $\mathbb{Z}^2$ lattice. In each cell, the particle finds a different arrangement of scatterers. From the particle's point of view, it undergoes a sequence of different billiard maps corresponding to the cells it visits. Such a model was considered in several works [28, 29, 1] and most recently in [9], which introduced the application of Birkhoff cones to dispersing billiards. More broadly, many results regarding the Central Limit Theorem for sequential systems already exist, for example, [5, 37, 36, 23, 24, 30, 25, 35, 41, 16] in which are investigated either random dynamical systems or sequential expanding maps, possibly non-uniformly expanding.

In the present paper, we present a general setting to establish a self-normed Central Limit Theorem with error terms for sequential dynamical systems, which allows us to treat both observables that are not bounded and discontinuous hyperbolic dynamical systems, including dispersing billiards. The results are obtained via the study of the properties of the corresponding transfer operators. As we are interested in presenting the general strategy, we do not push for optimal results. Since some estimates rest on a Taylor expansion, better results can be obtained by computing more terms. Note, however, that to obtain optimal results, it is necessary to have an estimate of the higher momenta in terms of the second momenta, and it is not clear how to achieve this in the present generality (see Remark 2.10 for a more detailed explanation of this issue).

In Section 2, we present two sets of abstract hypotheses: the first hypothesis assumes only that the real transfer operator is contracting in a projective metric, together with a uniform bound on the complex twisted transfer operator; the second hypothesis assumes that the complex transfer operator is also contracting. The second possibility yields much sharper results. We explain in Section 5 under which conditions the latter extra information on the transfer operator can be obtained via the theory of complex cones introduced by Rugh [38] and further developed by Dubois [17, 18]. Since this theory is of independent interest and is spread among several articles, for the reader's convenience, we provide a short, self-contained account of the theory in this section. Our presentation of complex cones is not as general as the one that can be found in [38, 17, 18], but it is more than sufficient for all the applications to the Central Limit Theorem we can think of.

To illustrate the broad applicability of our result, we apply it to two emblematic classes of examples; indeed more than half of the paper is devoted to these examples. The first application is to smooth expanding maps (the simplest possibility and one for which the most results already exist); the second application is to dispersing billiards (a technically difficult case motivated by the Lorentz gas and for which no sequential results of this type currently exist). We remark that other cases, such as Anosov and Axiom A diffeomorphisms lie in some sense between the applications contained in this paper. The analysis of hyperbolic systems in dimensions greater than two is not an obstruction to the present technique. Indeed, anisotropic spaces for Anosov and Axiom A diffeomorphisms (including for those with singular SRB measures) in any dimension have been constructed by a variety of authors [20, 4, 21] and the connection between anisotropic spaces and projective cones is described in [8]. While the details of such applications would make the present article excessively long, yet the technique presented here is applicable to such systems as well. Indeed, the expanding maps example shows that it is not necessary that the map have a common invariant measure. On the other hand, the billiards examples are studied using Banach spaces of distributions, which implicitly shows that the existence of an absolutely continuous invariant measure is also not necessary.

The application to expanding maps in Section 6 is done to compare our present Theorems 2.6 and 2.7 with existing results. We show that our technique reproduces all existing results apart from the ones in [15], which just appeared. In addition, in Proposition 6.1 and Corollary 6.4 we provide a new criterion to verify the growth of the variance in the sequential case, while Theorem 7.14 provides a sharper criterion for the random case. This is an issue often not addressed in the literature but fundamental in applications. The strategy used in [15] is similar to ours when applied to the case in which one can use complex cones but takes advantage of the fact that it deals only with bounded observables and expanding maps. In particular, it uses a martingale decomposition trick, adapted from [34, Chapter 3], in order to estimate higher momenta via the second momenta. It is unclear how to extend such a trick to the level of generality we are addressing in the present work. Possibly, something can be done using ideas in [32, Theorem 1.2] applied to the stable

foliation of the sequential maps, but as the present paper is already rather long, we refrain from exploring this issue. The application to billiards in Section 7 is brand new and shows the generality and power of the theory put forward in this paper. We expect this application to be relevant for the study of random Lorentz gasses, a long-standing open problem.

2. Setting and Main Results

Before stating the abstract framework in which we are able to study the Central Limit Theorem for sequential systems, we briefly describe two applications: smooth expanding maps and dispersing billiards. This gives the reader an indication of the broad applicability of the present approach.

2.1. Sequential Smooth Expanding Maps: A Preview.

The case of expanding maps has been extensively studied, and our general theory does not provide any new results, apart from the discussion of the growth of the variance. We discuss this case in Section 6 only so the reader can compare the present results with the literature. However, even in this case, we are able to present new results for the problem of studying the variance.

Let M be a smooth compact connected Riemannian manifold. Let $(f_k)_{k\in\mathbb{N}} \subset C^2(M,M)$ be a uniform family of expanding maps in the following sense: $\exists A>0$, $\vartheta>1$ such that for all $k\in\mathbb{N}$,

- $\|Df_k^{-1}\|_\infty \le \vartheta^{-1}$;
- $\|D^2 f_k\|_\infty \le A$.

Let $g_k \in C^1(M,\mathbb{R})$ with $\sup_{k\in\mathbb{N}} \|g_k\|_{C^1} \le C$ for some $C>0$. Given an initial density $\rho \in L^1$, we have that $g_k \circ f_{k-1} \circ \cdot \circ f_0$ is a random variable. We designate its expectation by

$$\mathbb{E}(g_k \circ f_{k-1} \circ \cdots \circ f_0) = \int_M g_k \circ f_{k-1} \circ \cdots \circ f_0(x)\rho(x)dx,$$

and call $\mathbb{P}$ the associated probability distribution. Define the centered observables,

$$\begin{aligned} \hat{g}_k &= g_k - \int_M g_k \circ f_{k-1} \circ \cdots \circ f_0 \cdot \rho \\ \hat{S}_n &= \sum_{k=0}^{n-1} \hat{g}_k \circ f_{k-1} \circ \cdots \circ f_0 \ ; \qquad \sigma_n^2 = \mathbb{E}(\hat{S}_n^2)\,, \end{aligned} \tag{2.1}$$

and the cumulative distribution function

$$F_n(x) = \mathbb{P}\left(\left\{\frac{\hat{S}_n}{\sigma_n} \le x\right\}\right). \tag{2.2}$$

The above, apart from some minor quirks, is the same model discussed in [5, 37, 24]. In Section 6.1 we will prove the following.

Theorem 2.1. *Let $\rho \in \mathcal{C}^1(M)$. Then for all $c_\star \in (0,1)$ and all $n\in\mathbb{N}$ such that $\sigma_n \ge \max\{1, c_\star n^{\frac{1}{3}} \ln(n+1)\}$, there exists $\bar{C}>0$ such that*

$$\left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \le \bar{C}\sigma_n^{-3} (\ln \sigma_n)^2 n.$$

The above Theorem can be applied only if σ_n grows fast enough. As mentioned, in the special case of expanding maps, it is possible to strengthen the result, see [15]. However, it is always necessary to meet some requirements on the growth of σ_n.

Unfortunately, contrary to the case of a single map, we are not aware of an existing result that provides general checkable criteria to ensure some variance growth. By checkable, we mean that one can ensure the wanted variance growth by computing only averages on a finite number of finite time trajectories.

In Proposition 6.1, we provide such a criterion for the case of smooth expanding maps. Our criterion naturally generalizes the usual criterion for a single map. While the argument is rather general, some technical problems do not allow, at the moment, to extend it to the general setting in which we obtain the CLT.

2.2. **Sequential Dispersing Billiards: A Preview.**
We consider a class of dispersing billiard tables formed by a fixed number ℓ of closed, convex sets, which we call scatterers, on the torus $\mathbb{T}^2$. The boundaries of the scatterers are assumed to have strictly positive curvature. The dynamics of the billiard map are defined by the motion of a point particle traveling at unit speed between collisions and reflecting elastically at collisions with the scatterers. We consider the family of billiard maps $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ corresponding to configurations of ℓ scatterers which satisfy

- The minimum and maximum free flight times between collisions are bounded by τ_* and τ_*^{-1}, respectively.
- The minimum and maximum curvature of the boundary of the scatterers are bounded by $\mathcal{K}_*$ and $\mathcal{K}_*^{-1}$, respectively.
- The maximum of the C^3 norm of the scatterers is bounded by E_*.

All billiard maps in $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ act on the same phase space M and preserve the same invariant measure $d\mu_{\text{SRB}} = c_0 \cos\varphi \, dr d\varphi$, where c_0 is the normalizing constant.

Given this family $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$, we prove a sequential Central Limit Theorem along any $N_{\mathcal{F}}$*-admissible sequence* of maps $(T_i)_{i\in\mathbb{N}} \subset \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$. See Definition 7.1 for the definition of $N_{\mathcal{F}}$-admissible sequence.[1]

For the Central Limit Theorem, we consider sequences of observables $g_k \in C^\alpha(M)$ for some $\alpha \in (0, 1/3)$ with $|g_k|_{C^\alpha} \le K$ for some $K > 0$ and all $k \in \mathbb{N}$. Given a strictly positive density $\rho \in C^1$,[2] $d\nu_0 = \rho d\mu_{\text{SRB}}$, and denote by $\mathbb{E}$ the expectation with respect to ν_0. We define the centered variables exactly as in (2.1) (where the integrals are w.r.t. μ_{SRB}) and the cumulative distribution function as in (2.2). Our results can be summarized as follows (see Theorem 7.12 for a precise statement, while the beginning of Section 7 contains a precise definition of the sequential system and Section 7.4 a detailed description of the observables).

Theorem 2.2. *Let $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ be as defined above and let $(T_i)_{i\in\mathbb{N}} \subset \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ be an $N_{\mathcal{F}}$-admissible sequence, and $\ln\rho \in \mathcal{C}^1$, $\int \rho \, d\mu_{SRB} = 1$.*

Then for all $c_\star \in (0, 1)$ there exists $\bar{C} > 0$ such that, for all $n \in \mathbb{N}$ such that $\sigma_n \ge \max\{1, c_\star n^{1/3} \ln(n+1)\}$,

$$\left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \, dy \right| \le \bar{C} \sigma_n^{-3} (\ln \sigma_n)^2 n.$$

Note that in the non-sequential case, when $T_k = T$ and $g_k = g$, if g is not a coboundary, then $\sigma_n^2 \sim n$. Hence the error term is $\mathcal{O}(n^{-\frac{1}{2}} (\ln \sigma_n)^2)$ which is optimal, apart from the logarithm.

In the sequential case we do not have general criteria to check that $\sigma_n \ge \max\{1, c_\star n^{1/3} \ln(n+1)\}$ (apart from the case in which the billiards are extremely hyperbolic, that is the operators $\mathcal{L}_T$ have a large spectral gap[3]). However, sharper results can be obtained in the random case, see Section 7.9 where we improve on the usual condition (cf. Remark 7.16 for details).

We prove Theorem 2.2 in Section 7 (that is, we prove its more precise formulation Theorem 7.12) as an application of the abstract theory described in the next section. However, checking the conditions of the abstract theory entails a non-trivial amount of work, which is carried out in Section 7.

2.3. **Abstract Framework and Central Limit Theorems.** Consider a sequence of dynamical systems $f_k : M_k \to M_{k+1}$, $k \in \mathbb{N}$,[4] where M_k are compact Riemannian manifolds of uniform diameter, and the f_k are measurable functions with respect to the Borel σ-algebra. Let μ_k be a sequence of probability measures such that $(f_k)_* \mu_k \ll \mu_{k+1}$. For all $g \in L^\infty(M_k, \mu_k)$ and $\varphi \in L^1(M_{k+1}, \mu_{k+1})$ we define the Transfer Operator $\mathcal{L}_k$ as

$$\int_{M_k} g\varphi \circ f_k \, d\mu_k = \int_{M_{k+1}} \varphi \mathcal{L}_k g \, d\mu_{k+1}. \tag{2.3}$$

[1]Informally, given $N_{\mathcal{F}} \in \mathbb{N}$, an $N_{\mathcal{F}}$-admissible sequence is a sequence of billiard maps drawn from the family $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ which comprise blocks of length $N_{\mathcal{F}}$ in which all maps in each block are close to one another, but maps in different blocks are not necessarily close.

[2]The results holds more generally for $\rho \in C_{\mathbb{R}}$; see Section 7.1 for the precise definition of $C_{\mathbb{R}}$.

[3]We thank Dmitry Dolgopyat for pointing this out to us.

[4]We use the convention that $0 \in \mathbb{N}$.

A direct computation shows that, μ_{k+1}-a.s., we have

$$\mathcal{L}_k g(x) = \sum_{y \in f_k^{-1}(x)} g(y)/J_k(y),$$

where $J_k(y) = \frac{d\mu_{k+1}(f_k y)}{d\mu_k(y)}$. Note that, by defintion, $\mathcal{L}_k$ is a positive operator and a contraction as an operator from $L^1(M_k, \mu_k)$ to $L^1(M_{k+1}, \mu_{k+1})$. Next, assume that there exist convex cones $C_k \subset L^1(M_k, \mu_k)$, $C_k \cap -C_k = \emptyset$, with the following properties

(C-1) for each $h \in C_k$ we have $\int_{M_k} h \, d\mu_k > 0$.

(C-2) $\mathcal{L}_k(C_k) \subset C_{k+1}$ and the diameter (with repect to the Hilbert metric, see [8, Appendix D]) of $\mathcal{L}_k(C_k)$ in C_{k+1} is uniformly bounded.

(C-3) there exists $\mathbb{e}_k \in C_k$ such that for each $h \in V_k := \mathrm{span}(C_k)$ there exists $\nu \in \mathbb{R}_+$:

$$\nu \mathbb{e}_k + h \in C_k.$$

Without loss of generality, by (C-1), we can choose the $\mathbb{e}_k$ such that

$$\int_{M_k} \mathbb{e}_k d\mu_k = 1. \tag{2.4}$$

Remark 2.3. *Note that the above setting is not the most general possible: e.g. $\mathcal{L}_k$ could be a more general transfer operator with some weight; condition (C-2) could be weakened to some diameter depending on k, provided that it does grow under appropriate control; in the conditions (O-1), (O-2) below one could let K grow, moderately, with k; and so on. We refrain from such endless generalizations, which the reader can easily work out, if needed, to present the basic idea in its more straightforward form.*

Due to condition (C-3) we can associate to each k a Banach space $\mathcal{B}_k$ obtained by the completion of V_k with respect to the norm $\|\cdot\|_k$ associated to the cone C_k:

$$\|h\|_k = \inf\{\lambda \in \mathbb{R}_+ \ : \ -\lambda \mathbb{e}_k \preceq h \preceq \lambda \mathbb{e}_k\} \tag{2.5}$$

where $h \succeq g$ iff $h - g \in C_k$ (see [8, Equation D.2.1] for more details). By [8, Lemma D.5] it follows that $\mathcal{B}_k$ is a Banach lattice with the order structure $\succeq$. In particular, if $-g \preceq h \preceq g$, then $\|g\|_k \geq \|h\|_k$. Also $\|\mathbb{e}_k\|_k = 1$.

Next, note that if $h \succeq g$, i.e. $h - g \in C_k$, by condtion (C-2), $\mathcal{L}_k(h-g) \in C_{k+1}$, so $\mathcal{L}_k h \succeq \mathcal{L}_k g$. That is, $\mathcal{L}_k$ is a positive, order preserving operator. Accordingly, for all $\alpha \in \mathcal{B}_k$ we have $\|\mathcal{L}_k \alpha\|_{k+1} \leq \|\alpha\|_k \|\mathcal{L}_k \mathbb{e}_k\|_{k+1}$.[5] In other words, $\mathcal{L}_k \in L(\mathcal{B}_k, \mathcal{B}_{k+1})$. Moreover, by (C-1) we have

$$\left| \int_{M_k} \alpha d\mu_k \right| \leq \int_{M_k} \mathbb{e}_k \|\alpha\|_k d\mu_k = \|\alpha\|_k. \tag{2.6}$$

In addition, by [8, Lemma D.4] (with the choice $\rho(g) = \int g$), there exists $c > 0$ such that, for each $k \geq j$, $h \in \mathcal{B}_j$, with $\int_{M_j} h d\mu_j = 0$, we have

$$\|\mathcal{L}_k \cdots \mathcal{L}_j h\|_{k+1} \leq 3e^{-c(k-j+1)} \|h\|_j. \tag{2.7}$$

We further assume

(C-4) there exists $C_* > 0$ such that, for each $k \geq j \in \mathbb{N}$, and $h \in \mathcal{B}_j$, we have $\|\mathcal{L}_k \cdots \mathcal{L}_j h\|_{k+1} \leq C_* \|h\|_j$.

Next we introduce the observables for which we shall prove a sequential Central Limit Theorem. Let $\{g_k\}_{k \in \mathbb{N}}$, $g_k \in L^1(M_k, \mu_k)$ be a real function such that

(O-1) there exists $K > 0$, $j_0 \in \mathbb{N}$, $j_0 > 2$, such that, for each $h \in \mathcal{B}_k$, and $j \leq j_0$, we have

$$\|\mathcal{L}_k(g_k^j h)\|_{k+1} \leq K^j \|h\|_k.$$

We define $f_{k,j} = f_{k-1} \circ \cdots \circ f_j$, $f_{j,j}(x) = x$ and consider the sum

$$S_n = \sum_{k=0}^{n-1} g_k \circ f_{k,0}.$$

[5] In fact, by conditions (C-2) and (C-3), $\mathcal{L}_k \mathbb{e}_k \in V_{k+1}$ so $\|\mathcal{L}_k \mathbb{e}_k\|_{k+1} < \infty$.

We let $x_k = g_k \circ f_{k,0}(x_0)$, and assume that x_0 is distributed according to a probability measure $d\nu_0 = \rho d\mu_0$, $\rho \in \mathcal{B}_0$. We will use the notation $\mathbb{E}$ to denote the expectation with respect to the initial measure $d\nu_0 = \rho d\mu_0$.

Then the x_k are random variables and

$$\mathbb{E}\left(S_n\right) := \sum_{k=0}^{n-1} \int_{M_0} \rho \cdot g_k \circ f_{k,0} d\mu_0 = \sum_{k=0}^{n-1} \int_{M_k} g_k \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho \, d\mu_k.$$

By (2.6), (O-1) and (C-4) we have

$$\sup_{k\in\mathbb{N}} \left| \int_{M_k} g_k \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho \, d\mu_k \right| = \sup_{k\in\mathbb{N}} \left| \int_{M_{k+1}} \mathcal{L}_k g_k \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho d\mu_{k+1} \right| \leq K C_* \|\rho\|_0.$$

It is then natural to define

$$\hat{g}_k = g_k - \int_{M_k} g_k \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho d\mu_k \tag{2.8}$$

so that

$$\int_{M_k} \hat{g}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho d\mu_k = 0, \tag{2.9}$$

and

$$\hat{S}_n = \sum_{k=0}^{n-1} \hat{g}_k \circ f_{k,0}. \tag{2.10}$$

By definition, $\mathbb{E}\left(\hat{S}_n\right) = 0$. It is then natural to define

$$\sigma_n^2 = \mathbb{E}(\hat{S}_n^2) = \sum_{k,j=0}^{n-1} \int_{M_n} \mathcal{L}_{n-1} \cdots \mathcal{L}_k (\hat{g}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_j (\hat{g}_j \mathcal{L}_{j-1} \cdots \mathcal{L}_0 \rho)) \, d\mu_n. \tag{2.11}$$

Lemma 2.4. *There exists $\tilde{C} > 0$ such that, for all $n \in \mathbb{N}$,*

$$\sigma_n^2 \leq \tilde{C} n.$$

Proof. Indeed, using that $\mathcal{L}_{j-1} \cdots \mathcal{L}_0 \rho \in \mathcal{B}_j$, together with (2.6), (C-4) and (O-1), we may estimate

$$\begin{aligned} \sigma_n^2 &= \sum_{k=0}^{n-1} \int_{M_{k+1}} \mathcal{L}_k (\hat{g}_k^2 \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho) \, d\mu_{k+1} \\ &\quad + 2 \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \int_{M_{k+1}} \mathcal{L}_k (\hat{g}_k \mathcal{L}_{k-1} \cdots \mathcal{L}_j (\hat{g}_j \mathcal{L}_{j-1} \cdots \mathcal{L}_0 \rho)) \, d\mu_{k+1} \\ &\leq n C_* K^2 \|\rho\|_0 + 6 C_* K^2 \|\rho\|_0 \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} e^{-c(k-j-1)} \leq \tilde{C} n \,, \end{aligned} \tag{2.12}$$

where, in the last line, we have used (2.7) to estimate $\|\mathcal{L}_{k-1} \cdots \mathcal{L}_j (\hat{g}_j \mathcal{L}_{j-1} \cdots \mathcal{L}_0 \rho)\|_k$, sicne we have that $\int_{M_{j+1}} \mathcal{L}_j (\hat{g}_j \mathcal{L}_{j-1} \cdots \mathcal{L}_0 \rho) \, d\mu_{j+1} = 0$ and $\mathcal{L}_j (\hat{g}_j \mathcal{L}_{j-1} \cdots \mathcal{L}_0 \rho) \in \mathcal{B}_{j+1}$ by condition (O-1). □

To continue, we need to complexify the Banach spaces $\mathcal{B}_k$ and extend the action of $\mathcal{L}_k$ to such an extension (to simplify notation, we still call $\mathcal{L}_k$ the extension).

Lemma 2.5. *There exists a canonical complex exension $\mathbb{B}_k$ of $\mathcal{B}_k$ and the action of $\mathcal{L}_k$ extends to a bounded operator in $L(\mathbb{B}_k, \mathbb{B}_{k+1})$. Moreover, for each $k, m \in \mathbb{N}$, the norm of $\mathcal{L}_{k+m} \cdots \mathcal{L}_k : \mathbb{B}_k \to \mathbb{B}_{k+m+1}$ is bounded by $\sqrt{2} C_*$.*

The above fact is well known but, for the reader's convenience, we provide the proof in Appendix A.

To keep the notation simple from now on we will use $\|\cdot\|_k$ also for the norm on the complex space, since no confusion can arise.

Note that if $f, g \in \mathcal{B}_k$ then, by (2.6),

$$\left|\int_{M_k}(f+ig)d\mu_k\right| \le \left|\int_{M_k} f d\mu_k\right| + \left|\int_{M_k} g d\mu_k\right| \le \|f\|_k + \|g\|_k \le \sqrt{2}\|f+ig\|_k. \tag{2.13}$$

Next, for $k, n \in \mathbb{N}$, $k \le n$, and $\lambda \in \mathbb{R}$, we define the operators $\mathcal{L}_{k,\lambda,n}$: for each $h \in \mathbb{B}_k$,

$$\mathcal{L}_{k,\lambda,n} h = \mathcal{L}_k(e^{i\sigma_n^{-1}\lambda \hat{g}_k} h). \tag{2.14}$$

Note that, for $h \in L^1$ and $\lambda \in \mathbb{R}$ we have $e^{i\sigma_n^{-1}\lambda g_k} h \in L^1$, so $\mathcal{L}_{k,\lambda,n}$ is also well defined, and a contraction, as an operator in $L^1(M_k, \mathbb{C})$. In addition, we assume

(O-2) There exists $\lambda_0 > 0$ and $K > 0$, such that for all $j, k, n \in \mathbb{N}$, $j \le k \le n$, $|\lambda| \le \lambda_0 \sigma_n$, and $h \in \mathcal{B}_j$ we have $\|\mathcal{L}_{k,\lambda,n} \cdots \mathcal{L}_{j,\lambda,n} h\|_{k+1} \le K\|h\|_j$.

Finally, we state a stronger assumption (the composition of twisted transfer operators is of Perron-Frobenius type), that will allow us to obtain stronger results:

(O-3) There exists $\lambda_0, c, K > 0$ and elements $h_{k,j,\lambda} \in \mathcal{B}_k$, $\ell_{k,j,\lambda} \in \mathcal{B}_j'$, $k, j \in \mathbb{N}$, such that, for all $|\lambda| \le \lambda_0 \sigma_n$ and $k, j, l, n \in \mathbb{N}$, $\int_{M_k} h_{k,j,\lambda} d\mu_k = \ell_{k,j,\lambda}(\mathbb{e}_j) = 1$, $\|h_{k,j,\lambda}\|_k + \|\ell_{k,j,\lambda}\|_j' \le K$,[6] $|\ell_{k,j,\lambda}(h_{j,l,\lambda})| \ge K^{-1}$. Morever, there exist $\alpha_{k,j,\lambda} \in \mathbb{C}$, $|\alpha_{k,j,\lambda}| \le K$ such that for all $j, k, n \in \mathbb{N}$, $j \le k \le n$, and $h \in \mathcal{B}_j$ we have

$$\|\mathcal{L}_{k-1,\lambda,n} \cdots \mathcal{L}_{j,\lambda,n} h - \alpha_{k,j,\lambda} h_{k,j,\lambda} \ell_{k,j,\lambda}(h)\|_k \le K|\alpha_{k,j,\lambda}| e^{-c(k-j)} \|h\|_j. \tag{2.15}$$

Notation. *In the following, we will use $c_\#, C_\#$ for a generic constant depending only on the constants in Conditions (C-1)–(C-4) and (O-1)–(O-3). Also, given a Banach space $\mathcal{B}$, we will use the notation $\mathcal{O}(a)$, $a \in \mathbb{R}_+$, to stand for an arbitrary element of $h \in \mathcal{B}$ such that $\|h\| \le C_\# a$. We will not specify explicitly the Banach space (which could be $\mathbb{R}, \mathbb{C}, \mathcal{B}_k, \mathbb{B}_k$ etc ...) since it will always be clear from the context.*

Our main results are the following.

Theorem 2.6. *If the conditions (C-1),(C-2),(C-3), (C-4) and (O-1), with $j_0 = 3$, and (O-2) are satisfied and $\liminf_{n\to\infty} \sigma_n = \infty$, then for each $\varpi > 3$ there exists $C_\varpi, \lambda_1 > 0$ such that, for each $n \in \mathbb{N}$ and $|\lambda| \le \lambda_1 \sigma_n$, we have*

$$\left|\mathbb{E}\left(e^{i\lambda\sigma_n^{-1}\hat{S}_n}\right) - e^{-\frac{\lambda^2}{2}}\right| \le C_\varpi \left(\frac{(\ln \sigma_n)^2 \lambda^4}{\sigma_n^2} + \lambda^2 \sigma_n^{-\varpi} + n\frac{\lambda^3}{\sigma_n^3}\right).$$

We will see that the above result yields helpful information only if for each $C_0 \in \mathbb{R}_+$ there exists $n_0 \in \mathbb{N}$ such that $\sigma_n \ge C_0 n^{\frac{1}{3}}$ for all $n \ge n_0$.

Theorem 2.7. *If the conditions (C-1),(C-2),(C-3), (C-4) and (O-1), with $j_0 = 3$, together with hypothesis (O-3) are satisfied, then for each $\varpi > 3$ there exist constants $C_\varpi, \lambda_1 > 0$ and an analytic functions A_n such that, for each $n \in \mathbb{N}$ and $|\lambda| \le \lambda_1 \sigma_n (\ln \sigma_n)^{-1}$, we have*

$$\begin{aligned}
&\mathbb{E}\left(e^{i\lambda\sigma_n^{-1}\hat{S}_n}\right) = e^{-\frac{\lambda^2}{2} + A_n(\lambda)} \\
&|A_n(\lambda)| \le C_\varpi n \left[\lambda^3 \sigma_n^{-3} (\ln \sigma_n)^2 + \sigma_n^{-\varpi}|\lambda|\right] \\
&|A_n'(\lambda)| \le C_\varpi n \left[\lambda^2 \sigma_n^{-3} (\ln \sigma_n)^2 + \sigma_n^{-\varpi}\right].
\end{aligned}$$

The proof of Theorem 2.6 is the content of Section 3, while Theorem 2.7 is proven in Section 4.

To compare with other results, let us compute what the above results imply for the distribution function

$$F_n(x) = \mathbb{P}\left(\left\{\frac{\hat{S}_n}{\sigma_n} \le x\right\}\right) = \mathbb{E}\left(\mathbb{1}_{\{\hat{S}_n/\sigma_n \le x\}}\right). \tag{2.16}$$

Our first result using (O-2) is nontrivial only if $\sigma_n \gg n^{\frac{1}{3}}$. This is the same restriction in [24]. But there, the error is smaller

[6] Where, for each $\ell \in \mathcal{B}_j'$, we have $\|\ell\|_j' = \sup_{\|h\|_j \le 1} |\ell(h)|$.

Corollary 2.8. *If the conditions (C-1),(C-2),(C-3), (C-4) and (O-1), with $j_0 = 3$, together with hypothesis (O-2) are satisfied, then for all $n \in \mathbb{N}$ we have*

$$\left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq C_{\#} \sigma_n^{-\frac{3}{4}} n^{\frac{1}{4}}, \tag{2.17}$$

Proof. By [19, equation (3.13) of Chapter XVI.3], Theorem 2.6 implies

$$\begin{aligned} \left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| &\leq \frac{C_\varpi}{\pi} \int_{-T_n}^{T_n} \left[\frac{(\ln \sigma_n)^2 \zeta^3}{\sigma_n^2} + n \frac{\zeta^2}{\sigma_n^3} + \sigma_n^{-\varpi} \zeta \right] d\zeta + \frac{24}{\pi T_n} \\ &\leq C_{\#} C_\varpi \left[\frac{(\ln \sigma_n)^2 T_n^4}{\sigma_n^2} + n \frac{T_n^3}{\sigma_n^3} + \sigma_n^{-\varpi} T_n \right] + \frac{24}{\pi T_n}, \end{aligned}$$

for each sequence $\{T_n\}$. Choosing $T_n = \sigma_n^{\frac{3}{4}} n^{-\frac{1}{4}}$, and recalling that (2.12) implies $\sigma_n \leq C_{\#} \sqrt{n}$, the result follows. □

To obtain a stronger result, we must assume (O-3) and apply Theorem 2.7.

Corollary 2.9. *If the conditions (C-1),(C-2),(C-3), (C-4) and (O-1), with $j_0 = 3$, together with hypothesis (O-3) are satisfied, then for each $c_\star \in (0,1)$ and for all $n \in \mathbb{N}$ such that $\sigma_n \geq \max\{1, c_\star n^{\frac{1}{3}} \ln(n+1)\}$, there exists $\bar{C} > 0$ such that*

$$\left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq \bar{C} \sigma_n^{-3} (\ln \sigma_n)^2 n. \tag{2.18}$$

Proof. We use again [19, equation (3.13) of Chapter XVI.3] to write

$$\left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq \frac{1}{\pi} \int_{-T_n}^{T_n} \left| \frac{e^{-\frac{\zeta^2}{2}} - e^{-\frac{\zeta^2}{2} + A_n(\zeta)}}{\zeta} \right| d\zeta + \frac{24}{\pi T_n}.$$

We make the choice $T_n = \frac{\sigma_n^3}{C_\star n (\ln \sigma_n)^2}$, for some constant $C_\star > 0$ large enough, and $\varpi = 4$. By Theorem 2.7 it follows that

$$\begin{aligned} &\frac{\zeta^2}{4} \geq C_\varpi n \left| \zeta^3 \sigma_n^{-3} (\ln \sigma_n)^2 \right| \\ &\left| \frac{d}{d\zeta} e^{-\frac{\zeta^2}{2} + A_n(\zeta)} \right| \leq C_0, \end{aligned}$$

for all $|\zeta| \leq T_n$. Also, setting $S_n = \frac{\sigma_n}{n^{\frac{1}{3}} (\ln \sigma_n)^{\frac{2}{3}}}$, we have that $|A_n(\zeta)|$ is uniformly bounded for all $|\zeta| \leq S_n$. Hence, for all $|\zeta| \leq S_n$ we have, by Theorem 2.7,

$$\begin{aligned} \left| \frac{1 - e^{A_n(\zeta)}}{\zeta} \right| &\leq C_{\#} \left| \frac{\int_0^\zeta A'(z) dz}{\zeta} \right| \leq C_{\#} \sup_{z \in [0,\zeta]} |A'(z)| \\ &\leq C_{\#} C_\varpi n \left| \zeta^2 \sigma_n^{-3} (\ln \sigma_n)^2 + \sigma_n^{-\varpi} \right|. \end{aligned}$$

Accordingly,

$$\begin{aligned} \left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq & \frac{1}{\pi} \int_{-S_n}^{S_n} e^{-\frac{\zeta^2}{2}} C_\varpi n \left[\zeta^2 \sigma_n^{-3} (\ln \sigma_n)^2 + \sigma_n^{-4} \right] d\zeta \\ &+ \frac{2 e^{C_\varpi c_\star^{-\varpi}}}{\pi} \int_{S_n}^{T_n} e^{-\frac{\zeta^2}{4}} + \frac{24}{\pi T_n} \\ &\leq \bar{C} \sigma_n^{-3} (\ln \sigma_n)^2 n, \end{aligned}$$

for some $\bar{C}$ large enough. □

We conclude with a brief discussion on the relation with previous results.

Remark 2.10. *The above results are only slightly weaker than [24]. However, they apply to a much wider range of systems. In particular, they are tailored for possible applications to the problem of a random Lorenz gas where the observable may not the bounded (e.g., see the relevant observables in the recent [16]). For bounded observables, one can take advantage of the fact that* $\mathbb{E}\left(e^{i\lambda\sigma_n^{-1}\hat{S}_n}\right)$ *is an analytic function of* λ*. More generally, our results can be improved by following the same strategy used here and simply computing more terms in the various Taylor expansions. To obtain optimal results, it is however necessary to have bounds on the higher moments in terms of the variance. The latter strategy is used in the recent paper [15], where much stronger results are obtained but limited to bounded observables and expanding maps. Unfortunately, it is not obvious how to extend the martingale approximation argument used to control the higher momenta in [15] to the present general setting.*

3. Proof of Theorem 2.6

In order to compute

$$\Upsilon_n(\lambda) = \mathbb{E}\left(e^{i\lambda\sigma_n^{-1}\hat{S}_n}\right) \tag{3.1}$$

we show that it satisfies a differential inequality.

Proposition 3.1. *For each* $\varpi > 0$*, there exist* $C_\varpi, \lambda_1 > 0$ *such that for all* $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}$*,* $|\lambda| \le \lambda_1\sigma_n$*, we have*

$$\frac{d}{d\lambda}\Upsilon_n(\lambda) = -\lambda\Upsilon_n(\lambda) + \mathcal{E}_n(\lambda)$$
$$|\mathcal{E}_n(\lambda)| \le C_\varpi\left(\frac{(\ln\sigma_n)^2|\lambda|^3}{\sigma_n^2} + |\lambda|\sigma_n^{-\varpi} + n\frac{\lambda^2}{\sigma_n^3}\right).$$

Proposition 3.1 immediately implies Theorem 2.6.

Proof of Theorem 2.6 using Proposition 3.1.
Setting $\theta_n(\lambda) = e^{\frac{\lambda^2}{2}}\Upsilon_n(\lambda)$, we compute

$$\theta_n'(\lambda) = e^{\frac{\lambda^2}{2}}\mathcal{E}_n(\lambda).$$

Since $\Upsilon_n(0) = 1$, we have $\theta_n(0) = 1$, hence

$$\Upsilon_n(\lambda) = e^{-\frac{\lambda^2}{2}} + \int_0^\lambda e^{-\frac{\lambda^2-\xi^2}{2}}\mathcal{E}_n(\xi)d\xi.$$

Moreover, using that $e^{-(\lambda^2-\xi^2)/2} \le 1$, we have

$$\left|\int_0^\lambda e^{-\frac{\lambda^2-\xi^2}{2}}\mathcal{E}_n(\xi)d\xi\right| \le \sup_{|\xi|\le|\lambda|}|\mathcal{E}_n(\xi)|\,|\lambda| \le C_\varpi\left(\frac{(\ln\sigma_n)^2\lambda^4}{\sigma_n^2} + \lambda^2\sigma_n^{-\varpi} + n\frac{|\lambda|^3}{\sigma_n^3}\right).$$

□

For the remainder of this section, we fix $n \in \mathbb{N}$ and prove Proposition 3.1. Since n is fixed, to simplify notation a little, we will drop the last index n in the definition of $\mathcal{L}_{k,\lambda,n}$ from (2.14). That is,

$$\mathcal{L}_{k,\lambda}h := \mathcal{L}_{k,\lambda,n}h = \mathcal{L}_k(e^{i\sigma_n^{-1}\lambda\hat{g}_k}h)\,. \tag{3.2}$$

Proof of Proposition 3.1.
For μ_0-almost every x we have

$$\frac{d}{d\lambda}e^{i\lambda\sigma_n^{-1}\hat{S}_n(x)} = i\sigma_n^{-1}\sum_{k=0}^{n-1}\hat{g}_k\circ f_{k,0}(x)e^{i\lambda\sigma_n^{-1}\hat{S}_n(x)}.$$

By the Schwartz inequality, recalling (2.6) and Conditions (O-1) and (C-4), we have

$$\begin{aligned}\mathbb{E}(|\hat{g}_k\circ f_{k,0}(x)|) &\le \left[\int_{M_{k+1}}\mathcal{L}_k\left[(\hat{g}_k)^2\mathcal{L}_{k-1}\cdots\mathcal{L}_0\rho\right]d\mu_{k+1}\right]^{\frac{1}{2}} \\ &\le KC_*^{\frac{1}{2}}\|\rho\|_0 < \infty.\end{aligned} \tag{3.3}$$

Accordingly,

$$\frac{d}{d\lambda}\Upsilon_n(\lambda) = i\sigma_n^{-1}\sum_{k=0}^{n-1}\mathbb{E}\left(\hat{g}_k\circ f_{k,0}e^{i\lambda\sigma_n^{-1}\hat{S}_n}\right). \tag{3.4}$$

For some constant $C_L > 0$, to be chosen later large enough, we define

$$L_n = C_L \ln \sigma_n. \tag{3.5}$$

Next, for each $k \le n$ let

$$\hat{S}^i_{n,k} = \sum_{\substack{j\in\{0,\dots,n-1\}\\ |j-k|\le L_n}} \hat{g}_j\circ f_{j,0}$$
$$\hat{S}^e_{n,k} = \sum_{\substack{j\in\{0,\dots,n-1\}\\ |j-k|> L_n}} \hat{g}_j\circ f_{j,0}.$$

Also, we define

$$\Theta^0_k(\xi,\lambda) = \mathbb{E}\left(e^{i\lambda\sigma_n^{-1}\hat{S}^e_{n,k}+i\xi\sigma_n^{-1}\hat{S}^i_{n,k}}\right)$$
$$\Theta^1_k(\xi,\lambda) = \mathbb{E}\left(\hat{g}_k\circ f_{k,0}e^{i\lambda\sigma_n^{-1}\hat{S}^e_{n,k}+i\xi\sigma_n^{-1}\hat{S}^i_{n,k}}\right)$$
$$\Theta^2_{k,j}(\xi,\lambda) = \mathbb{E}\left(\hat{g}_k\circ f_{k,0}\hat{g}_j\circ f_{j,0}e^{i\lambda\sigma_n^{-1}\hat{S}^e_{n,k}+i\xi\sigma_n^{-1}\hat{S}^i_{n,k}}\right)$$
$$\Theta^3_{k,j,l}(\xi,\lambda) = \mathbb{E}\left(\hat{g}_k\circ f_{k,0}\hat{g}_j\circ f_{j,0}\hat{g}_l\circ f_{l,0}e^{i\lambda\sigma_n^{-1}\hat{S}^e_{n,k}+i\xi\sigma_n^{-1}\hat{S}^i_{n,k}}\right).$$

Lemma 3.2. *For each $\xi\in\mathbb{R}$, the functions $\Theta^i(\xi,\cdot)$ are $\mathcal{C}^{j_*-i}(\mathbb{R},\mathbb{C})$.*

Proof. The computation is the same as for equation (3.3), using (2.6) and Conditions (O-1) and (C-4). □

By (3.4) and Lemma 3.2 we can Taylor expand with respect to ξ and write

$$\begin{aligned}\frac{d}{d\lambda}\Upsilon_n(\lambda) =& i\sigma_n^{-1}\sum_{k=0}^{n-1}\Theta^1_k(\lambda,\lambda)\\ =& i\sigma_n^{-1}\sum_{k=0}^{n-1}\Theta^1_k(0,\lambda) - \sigma_n^{-2}\lambda\sum_{k=0}^{n-1}\sum_{\substack{j\in\{0,\dots,n-1\}\\ |j-k|\le L_n}}\Theta^2_{k,j}(0,\lambda)\\ &- i\sigma_n^{-3}\int_0^\lambda d\xi\int_0^\xi dz\sum_{k=0}^{n-1}\sum_{\substack{j\in\{0,\dots,n-1\}\\ |j-k|\le L_n}}\sum_{\substack{l\in\{0,\dots,n-1\}\\ |l-k|\le L_n}}\Theta^3_{k,j,l}(z,\lambda).\end{aligned} \tag{3.6}$$

Remark 3.3. *Note that nothing prevents us from expanding Θ^1_k to higher orders. Doing so yields better estimates, but at the price of a much lengthier computation. We refrain from doing so as our goal is to present clearly the idea rather than to state the optimal results.*

Next, we must compute the terms in the above equation. To this end, we use the twisted transfer operators $\mathcal{L}_{k,\lambda}$ (recall (3.2)). Let us define, for all $k,j\in\mathbb{Z}$,

$$\mathcal{L}_{k,j,\lambda} = \begin{cases}\mathbb{1} & \text{if } k<0\\ \mathcal{L}_{k,\lambda}\cdots\mathcal{L}_{\max\{j,0\},\lambda} & \text{if } k\ge j\\ \mathbb{1} & \text{if } k<j.\end{cases} \tag{3.7}$$

Note that, for $h\in L^1$,

$$\int_{M_{k+1}}|\mathcal{L}_{k,\lambda}h| \le \int_{M_{k+1}}\mathcal{L}_k|h| = \int_{M_k}|h|.$$

Hence, by (2.13), (O-1) and (C-4), we have

$$\left|\int_{M_k}\mathcal{L}_{k,\lambda}\cdots\mathcal{L}_{j,\lambda}\hat{g}_j\mathcal{L}_{j-1,\lambda}\cdots\mathcal{L}_{0,\lambda}\rho\right| \le \sqrt{2}KC_*^2\|\rho\|_0.$$

Note that, for all $L_n \le k \le n - L_n$ and $k \le j \le k + L_n$

$$\begin{aligned}
\Theta_k^0(\xi,\lambda) &= \int_{M_n} \mathcal{L}_{n,k+L_n+1,\lambda}\mathcal{L}_{k+L_n,k-L_n+1,\xi}\,\mathcal{L}_{k-L_n,0,\lambda}\rho \\
\Theta_k^1(\xi,\lambda) &= \int_{M_n} \mathcal{L}_{n,k+L_n+1,\lambda}\mathcal{L}_{k+L_n,k,\xi}\,\hat{g}_k\mathcal{L}_{k-1,k-L_n+1,\xi}\,\mathcal{L}_{k-L_n,0,\lambda}\rho \\
\Theta_{k,j}^2(\xi,\lambda) &= \int_{M_n} \mathcal{L}_{n,k+L_n+1,\lambda}\mathcal{L}_{k+L_n,j,\xi}\,\hat{g}_j\mathcal{L}_{j-1,k,\xi}\hat{g}_k\mathcal{L}_{k-1,k-L_n+1,\xi}\,\mathcal{L}_{k-L_n,0,\lambda}\rho.
\end{aligned} \tag{3.8}$$

The above formulae correspond to the case $j \ge k$, the definition in the other cases being obvious. To estimate the above terms, the following lemmata will be instrumental.

Lemma 3.4. *For each $k, m_1, m_2 \in \mathbb{N}$ and $h \in \mathbb{B}_k$, we have*

$$\begin{aligned}
&\|\mathcal{L}_{k+m_1+m_2,k+m_2,0}\,\hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,k,0}h\|_{k+m_1+m_2+1} \le C_\# e^{-c\min\{m_1,m_2\}}\|h\|_k \\
&\left|\int_{M_{k+m_1+m_2}} \mathcal{L}_{k+m_1+m_2,k+m_2,0}\,\hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,k,0}h\right| \le C_\# e^{-cm_2}\|h\|_k \\
&\|\mathcal{L}_{k+m_1,k,0}\,\hat{g}_k\mathcal{L}_{k-1,0,0}\rho\|_{k+m_1+1} \le C_\# e^{-cm_1}
\end{aligned}$$

Proof. Let us define

$$\phi_{k,j} := \mathcal{L}_{k-1,0}\cdots\mathcal{L}_{j,0}\mathbb{e}_j. \tag{3.9}$$

By (C-4), $\|\phi_{k,j}\|_k \le C_*$ while (2.3) implies $\int_{M_k}\phi_{k,j}d\mu_k = 1$. Equation (2.7) yields

$$\begin{aligned}
\left\|\mathcal{L}_{k-1,0}\cdots\mathcal{L}_{j,0}h - \phi_{k,j}\int_{M_j} hd\mu_j\right\|_k &= \left\|\mathcal{L}_{k-1,0}\cdots\mathcal{L}_{j,0}\left[h - \mathbb{e}_j\int_{M_j} hd\mu_j\right]\right\|_k \\
&\le C_\# e^{-c(k-j)}\|h\|_j.
\end{aligned} \tag{3.10}$$

In particular, choosing $h = \mathcal{L}_{j-1,0}\cdots\mathcal{L}_{0,0}\mathbb{e}_0$,

$$\|\phi_{k,0} - \phi_{k,j}\|_k \le C_\# e^{-c(k-j)}. \tag{3.11}$$

Then, using (3.10) multiple times, conditions (O-1), (C-4) and equation (3.11) yields

$$\begin{aligned}
&\mathcal{L}_{k+m_1+m_2,k+m_2,0}\,\hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,k,0}h = \phi_{k+m_1+m_2+1,k+m_2+1} \\
&\times\int \mathcal{L}_{k+m_2,0}\hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,k,0}h + \mathcal{O}(e^{-cm_1})\|\mathcal{L}_{k+m_2,0}\hat{g}_{k+m_2}\phi_{k+m_2,k}\int h\|_{k+m_2} \\
&= \phi_{k+m_1+m_2+1,k+m_2+1}\int \mathcal{L}_{k+m_2,0}\hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,k,0}h + \mathcal{O}(e^{-cm_1} + e^{-cm_2})\|h\|_k \\
&= \phi_{k+m_1+m_2+1,0}\int \mathcal{L}_{k+m_2,0}\hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,k,0}h + \mathcal{O}(e^{-cm_1} + e^{-cm_2})\|h\|_k \\
&= \phi_{k+m_1+m_2,0}\int \mathcal{L}_{k+m_2,0}\hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,0,0}\rho\int_{M_k} h + \mathcal{O}(e^{-cm_1} + e^{-cm_2})\|h\|_k \\
&= \phi_{k+m_1+m_2,0}\int \hat{g}_{k+m_2}\mathcal{L}_{k+m_2-1,0,0}\rho\int_{M_k} h + \mathcal{O}(e^{-cm_1} + e^{-cm_2})\|h\|_k \\
&= \mathcal{O}(e^{-c\min\{m_1,m_2\}})\|h\|_k,
\end{aligned}$$

where, in the third equality, we have used again (3.10) to estimate

$$\begin{aligned}
&\left\|\mathcal{L}_{k+m_2-1,k,0}h - \mathcal{L}_{k+m_2-1,0,0}\rho\int_{M_k} h\right\|_{k+m_2} = \left\|\mathcal{L}_{k+m_2-1,k,0}\left(h - \mathcal{L}_{k-1,0,0}\rho\int_{M_k} h\right)\right\|_{k+m_2} \\
&\le \left\|\phi_{k+m_2,k}\int\left(h - \mathcal{L}_{k-1,0,0}\rho\int_{M_k} h\right)\right\|_{k+m_2} + C_\# e^{-cm_2}\left\|h - \mathcal{L}_{k-1,0,0}\rho\int_{M_k} h\right\|_k \\
&\le C_\# e^{-cm_2}\|h\|_k.
\end{aligned}$$

This proves the first inequality. The other two are proven similarly. □

Let

$$\begin{aligned}\Gamma_k^0 &= \mathcal{L}_{k+L_n,k-L_n+1,0}\\ \Gamma_k^1 &= \mathcal{L}_{k+L_n,k,0}\,\hat{g}_k\mathcal{L}_{k-1,k-L_n+1,0}\\ \Gamma_{k,j}^2 &= \mathcal{L}_{k+L_n,j,0}\,\hat{g}_j\mathcal{L}_{j-1,k,0}\hat{g}_k\mathcal{L}_{k-1,k-L_n+1,0}.\end{aligned} \tag{3.12}$$

Lemma 3.5. *For each $\varpi > 0$ there exists $C_L = C_L(\varpi) > 0$ such that for all $n - L_n \geq k \geq L_n$ and $h \in \mathcal{B}_{k-L_n}$, we have*

$$\begin{aligned}&\|\Gamma_k^1 h\|_{k+L_n} \leq C_\# \sigma^{-\varpi}\|h\|_{k-L_n}\\ &\|\Gamma_{k,j}^2 h - \Sigma_{k,j}\Gamma_k^0 h\|_{k+L_n} \leq C_\# \sigma_n^{-\varpi}\|h\|_{k-L_n},\end{aligned}$$

where

$$\Sigma_{k,j} = \int \hat{g}_k\mathcal{L}_{k-1,j,0}\hat{g}_j\mathcal{L}_{j-1,0,0}\rho. \tag{3.13}$$

Proof. Equations (2.7), (2.6) and conditions (C-4), (O-1) imply

$$\begin{aligned}\left\|\Gamma_k^1 h\right\|_{k+L_n} &\leq \left\|\phi_{k+L_n,k+1}\int_{M_{k+1}} \mathcal{L}_k\hat{g}_k\mathcal{L}_{k-1,\mathcal{L}_{k-L_n+1},0}h d\mu_k\right\|_{k+1} + C_\# e^{-cL_n}\|h\|_{k-L_n}\\ &\leq C_\#\left|\int_{M_k}\hat{g}_k\mathcal{L}_{k-1,0,0}\rho d\mu_0\right|\cdot\left|\int_{M_{k-L_n}} h d\mu_{k-L_n}\right| + C_\# K\sigma_n^{-\varpi}\|h\|_{k-L_n}\\ &\leq C_\#\sigma_n^{-\varpi}\|h\|_{k-L_n}\end{aligned}$$

by (2.9) and provided we have chosen C_L large enough. Next, we study $\Gamma_{k,j}^2$ for $j \geq k$, the case $j < k$ being identical. If $|k-j| \geq L_n/2$, then

$$\begin{aligned}\Gamma_{k,j}^2 h &= \mathcal{L}_{k+L_n,j,0}\,\hat{g}_j\mathcal{L}_{j-1,k,0}\hat{g}_k\mathcal{L}_{k-1,k-L_n+1,0}h\\ &= \mathcal{L}_{k+L_n,j,0}\,\hat{g}_j\phi_{j,k+1}\int_{M_{k+1}}\mathcal{L}_k\hat{g}_k\mathcal{L}_{k-1,k-L_n+1,0}h + \mathcal{O}(e^{-c|k-j|}\|h\|_{k-L_n})\\ &= \mathcal{O}(\sigma_n^{-\varpi}\|h\|_{k-L_n})\end{aligned} \tag{3.14}$$

provided we choose C_L large enough. On the other hand, if $|k-j| \leq L_n/2$, then $k + L_n - j \geq L_n/2$ and $j - k + L_n \geq L_n/2$, hence, recalling (3.10),

$$\begin{aligned}\Gamma_{k,j}^2 h =&\phi_{k+L_n,j+1}\int \mathcal{L}_j\hat{g}_j\mathcal{L}_{j-1,k,0}\hat{g}_k\mathcal{L}_{k-1,k-L_n+1,0}h\\ &+ \mathcal{O}(e^{-c|k+L_n-j|}\|h\|_{k-L_n})\\ =&\ \phi_{k+L_n,j+1}\int\mathcal{L}_j\hat{g}_j\mathcal{L}_{j-1,k,0}\hat{g}_k\mathcal{L}_{k-1,0,0}\rho\int h + \mathcal{O}(\sigma_n^{-\varpi}\|h\|_{k-L_n})\\ =&\ \Sigma_{k,j}\Gamma_k^0 h + \mathcal{O}(\sigma_n^{-\varpi}\|h\|_{k-L_n}).\end{aligned} \tag{3.15}$$

The Lemma follows noting that, for $|k-j| \geq L_n/2$, $|\Sigma_{k,j}| \leq C_\#\sigma^{-\varpi}$. □

We can now estimate the terms in (3.6) one at a time.
If $L_n \leq k \leq n - L_n$, Lemma (3.5) implies

$$\left|\Theta_k^1(0,\lambda)\right| \leq C_\#\sigma_n^{-\varpi}$$

If $k \geq n - L_n$, then by the second inequality of Lemma 3.4 and condition (O-2)

$$\left|\Theta_k^1(0,\lambda)\right| \leq C_\#\sigma_n^{-\varpi}\|\mathcal{L}_{k-L_n,0,\lambda}\rho\|_{k-L_n} \leq C_\#\sigma_n^{-\varpi}.$$

While, if $k \leq L_n$, then by the equation (2.6), condition (O-2) and third inequality of Lemma 3.4

$$\left|\Theta_k^1(0,\lambda)\right| \leq C_\#\|\mathcal{L}_{k+L_n,k,0}\,\hat{g}_k\mathcal{L}_{k-1,0,0}\rho\|_{k+L_n+1} \leq C_\#\sigma_n^{-\varpi}.$$

It follows that for all $k \in \mathbb{N}$,

$$|\Theta_k^1(0,\lambda)| \leq C_\#\sigma_n^{-\varpi}. \tag{3.16}$$

Next, let us compute $\Theta^2_{k,j}(0,\lambda)$. As before we treat only the case $j \geq k$ since the case $j < k$ is identical and is left to the reader.
For $n - L_n \geq k \geq L_n$ Lemma 3.5 implies

$$\Theta^2_{k,j}(0,\lambda) = \Sigma_{k,j}\Theta^0_k(0,\lambda) + \mathcal{O}(\sigma_n^{-\varpi}). \tag{3.17}$$

Note that, by equations (2.11) and (3.10),

$$\sigma_n^2 = \sum_{k=0}^{n-1} \sum_{|j-k|\leq L_n} \Sigma_{k,j} + \mathcal{O}(\sigma_n^{-\varpi}). \tag{3.18}$$

The other possibilities can be treated as we did for Θ^1_k and yield a contribution of order $\sigma_n^{-\varpi}$. Finally, we compute[7]

$$\begin{aligned}
\left|\sum_{k=0}^{n-1} \sum_{\substack{j,l\in\{0,\dots,n-1\}\\ |j-k|\leq L_n\\ |l-k|\leq L_n}} \Theta^3_{k,j,l}(z,\lambda)\right| &\leq \sum_{k=0}^{n-1} K\mathbb{E}\left(\left[\sum_{\substack{l\in\{0,\dots,n-1\}\\ |l-k|\leq L_n}} \hat{g}_l \circ f_{l,0}\right]^2\right)\\
&\leq K\sum_{k=0}^{n-1} \sum_{\substack{j\in\{0,\dots,n-1\}\\ |j-k|\leq L_n}} \sum_{\substack{l\in\{0,\dots,n-1\}\\ |l-k|\leq L_n}} \mathbb{E}\left(\hat{g}_j \circ f_{j,0}\hat{g}_l \circ f_{l,0}\right)\\
&\leq K \sum_{\substack{j\in\{0,\dots,n-1\}\\ |j-l|\leq 2L_n}} (2L_n - |l-j|)\mathbb{E}\left(\hat{g}_j \circ f_{j,0}\hat{g}_l \circ f_{l,0}\right)\\
&\leq 2K\sigma_n^2 L_n - K \sum_{\substack{j\in\{0,\dots,n-1\}\\ |s|\leq 2L_n}} |s|\mathbb{E}\left(\hat{g}_j \circ f_{j,0}\hat{g}_{j-s} \circ f_{j-s,0}\right)\\
&\leq 2K\sigma_n^2 L_n + C_\# n
\end{aligned} \tag{3.19}$$

since the sum over s is bounded by (3.10).

To conclude, note that

$$\Upsilon_n(\lambda) = \Theta^0_k(\lambda,\lambda) = \Theta^0_k(0,\lambda) + \partial_\xi\Theta^0_k(0,\lambda)\lambda + \int_0^\lambda d\eta \int_0^\eta dz \partial_\xi^2\Theta^0_k(z,\lambda). \tag{3.20}$$

Arguing as before, we have

$$\begin{aligned}
|\partial_\xi\Theta^0_k(0,\lambda)| &\leq C\sigma_n^{-\varpi}\\
|\partial_\xi^2\Theta^0_k(0,\lambda)| &\leq C_\# C_L^2\sigma_n^{-2}(\ln\sigma_n)^2.
\end{aligned} \tag{3.21}$$

Hence, we can rewrite (3.17) as

$$\Theta^2_{k,j}(0,\lambda) = \Sigma_{k,j}\Upsilon_n + \mathcal{O}\left(\Sigma_{k,j}\sigma_n^{-2}C_L^2\lambda^2(\ln\sigma_n)^2 + \sigma_n^{-\varpi}\right). \tag{3.22}$$

Collecting the above computations, we have

$$\begin{aligned}
\frac{d}{d\lambda}\Upsilon_n(\lambda) &= -\sigma_n^{-1} \sum_{\substack{k\in\{0,\cdots,n-1\}\\ |k-j|\leq L_n}} \left[\Sigma_{k,j}\lambda\Upsilon(\lambda) + \mathcal{O}\left(\Sigma_{k,j}\sigma_n^{-2}\lambda^3 C_L^2(\ln\sigma_n)^2 + \sigma_n^{-\varpi}\lambda\right)\right]\\
&\quad + \mathcal{O}\left(n\lambda^2\sigma_n^{-3} + \lambda^2\sigma_n^{-2}C_L\ln\sigma_n\right)\\
&= -\lambda\Upsilon_n(\lambda) + \mathcal{O}\left(\frac{C_L^2(\ln\sigma_n)^2\lambda^3}{\sigma_n^2} + \lambda\sigma_n^{-\varpi} + n\frac{\lambda^2}{\sigma_n^3}\right),
\end{aligned}$$

[7]Note that $\int_{M_0}|\hat{g}_l|^2 \circ f_{l,0}\rho\, d\mu_0 = \int_{M_l}|\hat{g}_l|^2\mathcal{L}_{l,0}\rho\, d\mu_l = \int_{M_{l+1}}\mathcal{L}_{l,0}|\hat{g}_l|^2\mathcal{L}_{l,0,0}\rho\, d\mu_{l+1}$, so equation (2.6) and conditions (O-1), (C-4) imply that $\hat{g}_l \circ f_{l,0} \in L^2(M_0,\nu_0)$.

and this concludes the proof recalling that C_L is a constant depending on ϖ, see Lemma 3.5. □

4. Proof of Theorem 2.7

We define Υ_n as in equation (3.1).

Proposition 4.1. *For each $\varpi > 0$, there exists $\lambda_1 > 0$ such that, for each $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, $|\lambda| \leq \lambda_1 \sigma_n (\ln \sigma_n)^{-1}$, we have*

$$\begin{aligned}
&\frac{d}{d\lambda}\Upsilon_n(\lambda) = -\left(\lambda + \mathcal{E}_n(\lambda)\right)\Upsilon_n(\lambda)\\
&\Upsilon(0) = 1\\
&|\mathcal{E}_n(\lambda)| \leq C_\varpi \left(\lambda^2 \sigma_n^{-3} (\ln \sigma_n)^2 n + n\sigma_n^{-\varpi}\right).
\end{aligned}$$

Proposition 4.1 immediately implies Theorem 2.7.

Proof of Theorem 2.7.
Let $A_n(\lambda) = \int_0^\lambda \mathcal{E}_n(\xi) d\xi$ then Proposition 4.1 implies $\Upsilon_n(\lambda) = e^{-\frac{\lambda^2}{2} - A_n(\lambda)}$. □

To conclude we must thus prove Proposition 4.1.

Proof of Proposition 4.1.
We start again our proof by

$$\frac{d}{d\lambda}\Upsilon_n(\lambda) = i\sigma_n^{-1} \sum_{k=0}^{n-1} \int_{M_n} \mathcal{L}_{n-1,k,\lambda} \hat{g}_k \mathcal{L}_{k-1,0,\lambda} \rho d\mu_n, \tag{4.1}$$

where we have used the same notation as in (3.8). If $2L_n \leq k \leq n - 2L_n$, then, recalling conditions (O-2) and (O-3) we can write

$$\begin{aligned}
\mathcal{L}_{n-1,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,0,\lambda}\rho = {} & \alpha_{n,k+L_n+1,\lambda}\alpha_{k-L_n-1,0,\lambda} h_{n,k+L_n+1,\lambda}\\
& \times \ell_{n,k+L_n+1,\lambda}(\mathcal{L}_{k+L_n,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,k-L_n,\lambda}h_{k-L_n-1,0,\lambda})\ell_{k-L_n-1,0,\lambda}(\rho)\\
& + \mathcal{O}(\sigma^{-\varpi}\alpha_{n,k+L_n+1,\lambda}\alpha_{k-L_n-1,0,\lambda})
\end{aligned} \tag{4.2}$$

provided we have chosen C_L large enough.

Lemma 4.2. *There exists $n_1 \in \mathbb{N}$ each $n \geq k > j > l$, $k - l \geq n_1$, we have*

$$\alpha_{k,l,\lambda} = \alpha_{k,j,\lambda}\alpha_{j-1,l,\lambda}\ell_{k,j,\lambda}\left(h_{j,l,\lambda}\right) + \mathcal{O}\left(\alpha_{k,l,\lambda}\alpha_{j-1,l,\lambda}e^{-c\min\{|k-j|,|j-l|\}}\right).$$

Proof. By assumption (O-3) we have, on the one hand

$$\left|\int_{M_k} \mathcal{L}_{k-1,l,\lambda}(\mathbb{e}_l) d\mu_k - \alpha_{k,l,\lambda}\right| \leq K|\alpha_{k,l,\lambda}|e^{-c|k-l|}.$$

On the other hand

$$\begin{aligned}
\int_{M_k} \mathcal{L}_{k-1,l,\lambda}(\mathbb{e}_l) d\mu_k &= \int_{M_k} \mathcal{L}_{k-1,j,\lambda}\left(\mathcal{L}_{j-1,l,\lambda}(\mathbb{e}_l)\right) d\mu_k\\
&= \alpha_{k,j,\lambda}\ell_{k,j,\lambda}\left(\mathcal{L}_{j-1,l,\lambda}(\mathbb{e}_l)\right) + \mathcal{O}\left(\alpha_{k,j,\lambda}e^{-c|k-j|}\left\|\mathcal{L}_{j-1,l,\lambda}(\mathbb{e}_l)\right\|_j\right)\\
&= \alpha_{k,j,\lambda}\alpha_{j-1,l,\lambda}\ell_{k,j,\lambda}\left(h_{j,l,\lambda}\right) + \mathcal{O}\left(\alpha_{k,j,\lambda}\alpha_{j-1,l,\lambda}e^{-c\min\{|k-j|,|j-l|\}}\right).
\end{aligned}$$

Hence, If $Ke^{-cn_1} \leq 1/2$, the lemma follows. □

Since

$$\begin{aligned}
\mathcal{L}_{l,j,\lambda}h = {} & \mathcal{L}_{l,j,0}h + i\sigma_n^{-1}\lambda\sum_{m=j}^{l}\mathcal{L}_{j,m,0}\,\hat{g}_m\mathcal{L}_{m-1,j,0}h\\
& - \sigma_n^{-2}\sum_{m,s=j}^{l}\int_0^\lambda(\lambda-\xi)\mathcal{L}_{l,m,\xi}\,\hat{g}_m\mathcal{L}_{m-1,s,\xi}\hat{g}_s\mathcal{L}_{s-1,l,\xi}h d\xi,
\end{aligned} \tag{4.3}$$

we can compute

$$
\begin{aligned}
\mathcal{L}_{k+L_n,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,k-L_n,\lambda}h &= \mathcal{L}_{k+L_n,k,0}\hat{g}_k\mathcal{L}_{k-1,k-L_n,0}h \\
&+ i\sigma_n^{-1}\lambda\sum_{j=k-L_n}^{k+L_n}\mathcal{L}_{k+L_n,j,0}\hat{g}_j\mathcal{L}_{j-1,k,0}\,\hat{g}_k\mathcal{L}_{k-1,k-L_n,0}h \\
&- \frac{\lambda^2}{2\sigma_n^2}\sum_{m,s=k-L_n}^{k+L_n}\mathcal{L}_{k+L_n,m,0}\,\hat{g}_m\mathcal{L}_{m-1,k,0}\hat{g}_k\mathcal{L}_{k-1,s,0}\hat{g}_s\mathcal{L}_{s-1,k-L_n,0}h \\
&+ \mathcal{O}(\lambda^3\sigma_n^{-3}L_n^3\|h\|_{k-L_n}).
\end{aligned}
\tag{4.4}
$$

Lemma 4.3. *For $\lambda \le \lambda_0\sigma_n$, we have*[8]

$$
\begin{aligned}
\left\|\mathcal{L}_{k+L_n,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,k-L_n,\lambda}h_{k-L_n-1,0,\lambda} - i\sigma_n^{-1}\lambda\sum_{j=k-L_n}^{k+L_n}\Sigma_{k,j}\mathcal{L}_{k+L_n,k-L_n,\lambda}h_{k-L_n-1,0,\lambda}\right\| \le \\
\le C_{\#}\left[\lambda^2\sigma_n^{-2}L_n^2 + \sigma_n^{-\varpi}\right].
\end{aligned}
$$

Proof. Equation (4.4)

$$
\begin{aligned}
\mathcal{L}_{k+L_n,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,k-L_n,\lambda}h_{k-L_n-1,0,\lambda} &= \mathcal{L}_{k+L_n,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,k-L_n,0}h_{k-L_n-1,0,\lambda} \\
&+ i\sigma_n^{-1}\lambda\sum_{j=k}^{k+L_n}\mathcal{L}_{k+L_n,j,0}\hat{g}_j\mathcal{L}_{j-1,k,0}\hat{g}_k\mathcal{L}_{k-1,k-L_n,0}h_{k-L_n-1,0,\lambda} \\
&+ i\sigma_n^{-1}\lambda\sum_{j=k-L_n}^{k-1}\mathcal{L}_{k+L_n,k,0}\hat{g}_k\mathcal{L}_{k-1,j,0}\hat{g}_j\mathcal{L}_{j-1,k-L_n,0}h_{k-L_n-1,0,\lambda} \\
&+ \mathcal{O}(\left[\lambda^2\sigma_n^{-2}L_n^2\right]\|h_{k-L_n-1,0,\lambda}\|_{k-L_n}).
\end{aligned}
$$

By Lemma 3.5 and equation (4.3) it follows

$$
\begin{aligned}
\mathcal{L}_{k+L_n,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,k-L_n,\lambda}h_{k-L_n-1,0,\lambda} =& i\sigma_n^{-1}\lambda\sum_{j=k-L_n}^{k+L_n}\Sigma_{k,j}\mathcal{L}_{k+L_n,k-L_n,0}h_{k-L_n-1,0,\lambda} \\
&+ \mathcal{O}(\left[\lambda^2\sigma_n^{-2}L_n^2 + \sigma_n^{-\varpi}\right]\|h_{k-L_n-1,0,\lambda}\|_{k-L_n}) \\
=& i\sigma_n^{-1}\lambda\sum_{j=k-L_n}^{k+L_n}\Sigma_{k,j}\mathcal{L}_{k+L_n,k-L_n,\lambda}h_{k-L_n-1,0,\lambda} \\
&+ \mathcal{O}(\left[\lambda^2\sigma_n^{-2}L_n^2 + \sigma_n^{-\varpi}\right]\|h_{k-L_n-1,0,\lambda}\|_{k-L_n}).
\end{aligned}
$$

The Lemma follows recalling that, by condition (O-3), $\|h_{k-L_n-1,0,\lambda}\|_{k-L_n} \le K$. □

To continue, we need some rough estimates.

Lemma 4.4. *There exists $\lambda_1 \in (0,\lambda_0)$ such that, for all $|\lambda| \le \lambda_1\sigma_n L_n^{-1}$ and $j \le k \le n$, $k-j \le L_n$, we have, for each $h \in \mathcal{B}_j$,*

a) $|\alpha_{k,j,\lambda}| \ge \dfrac{1}{2(1+K)}$

b) $|\ell_{k,j,\lambda}(h)| \ge c_{\#}\left|\displaystyle\int_{M_j} h\right| - C_{\#}\left[\lambda\sigma_n^{-1}|k-j| + e^{-c|k-j|}\right]\|h\|_j$

c) $\|h_{k,j,\lambda} - \phi_{k,j}\|_k \le C_{\#}\left[\lambda\sigma_n^{-1}|k-j| + e^{-c|k-j|}\right].$

Proof. By condition (O-3) we have

$$
\left|\alpha_{k,j,\lambda} - \int_{M_k}\mathcal{L}_{k-1,j,\lambda}\mathbb{e}_j d\mu_k\right| \le Ke^{-c|k-j|}|\alpha_{k,j,\lambda}|.
$$

[8]Recall that $\Sigma_{k,j}$ is defiend in (3.13), while λ_0 refers to condition (O-3).

While (4.3) and (2.4) and Condition (O-1) imply

$$\begin{aligned}\int_{M_k} \mathcal{L}_{k-1,j,\lambda}\mathfrak{e}_j =& \int_{M_k} \mathcal{L}_{k-1,j,0}\mathfrak{e}_j + i\sigma_n^{-1}\sum_{m=j}^{k}\int_0^\lambda d\xi \int_{M_k} \mathcal{L}_{k-1,m,\xi}\,\hat{g}_m \mathcal{L}_{m-1,j,\xi}\mathfrak{e}_j\\ =&1+\mathcal{O}(\sigma_n^{-1}\lambda|k-j|).\end{aligned}$$

Which yields

$$|\alpha_{k,j,\lambda}| \geq \frac{1-C_\#\sigma_n^{-1}\lambda|k-j|}{1+Ke^{-c|k-j|}}. \tag{4.5}$$

The above implies $|\alpha_{k,j,\lambda}| \geq (2+2K)^{-1}$, provided $|k-j| \leq L_n$ and λ_1 is chosen small enough. Next, for each $h \in \mathcal{B}_j$,

$$\int_{M_k} \mathcal{L}_{k-1,j,\lambda} h = \alpha_{k,j,\lambda}\ell_{k,j,\lambda}(h) + \mathcal{O}(e^{-c|k-j|}\alpha_{k,l,\lambda}\|h\|_j).$$

Thus, for $|k-j| \leq L_n$, by the first statement of the Lemma and equation (4.3)

$$\begin{aligned}|\ell_{k,j,\lambda}(h)| &\geq c_\# \left|\int_{M_k} \mathcal{L}_{k-1,j,\lambda} h\right| - C_\# e^{-c|k-j|}\|h\|_j\\ &\geq c_\# \left|\int_{M_j} h\right| - C_\#\left[\lambda\sigma_n^{-1}|k-j| + e^{-c|k-j|}\right]\|h\|_j,\end{aligned} \tag{4.6}$$

proving item (b).

Finally, by Conditions (O-3), (O-2) and equation (4.3) we have

$$\begin{aligned}&\alpha_{k,j,\lambda}\left(h_{k,j,\lambda}-\phi_{k,j}\right) = \mathcal{L}_{k-1,j,\lambda}\mathfrak{e}_j - \mathcal{L}_{k-1,j,0}\mathfrak{e}_j\int \mathcal{L}_{k-1,j,\lambda}\mathfrak{e}_j + \mathcal{O}\left(\alpha_{k,j,\lambda}e^{-c|k-j|}\right)\\ &= \sum_{l=j}^{k}\mathcal{L}_{k-1,l+1,0}\mathcal{L}_{l,j,\lambda}\mathfrak{e}_j\int \mathcal{L}_{k-1,l+1,\lambda}\mathfrak{e}_{l+1} - \mathcal{L}_{k-1,l,0}\mathcal{L}_{l-1,j,\lambda}\mathfrak{e}_j\int \mathcal{L}_{k-1,l,\lambda}\mathfrak{e}_l\\ &\quad + \mathcal{O}\left(\alpha_{k,j,\lambda}e^{-c|k-j|}\right) = \mathcal{O}\left(\lambda\sigma_n^{-1}|k-j| + \alpha_{k,j,\lambda}e^{-c|k-j|}\right).\end{aligned}$$

The Lemma follows remembering (4.5). □

Using Lemma 4.3 in equation (4.2) and recalling Lemma 4.2 we obtain, for $2L_n \leq k \leq n-2L_n$, choosing C_L large enough,

$$\begin{aligned}&\mathcal{L}_{n-1,k,\lambda}\hat{g}_k\mathcal{L}_{k-1,0,\lambda}\rho = i\sigma_n^{-1}\lambda\sum_{j=k-L_n}^{k+L_n}\Sigma_{k,j}\alpha_{n,k+L_n+1}\alpha_{k-L_n-1,0,\lambda}h_{n,k+L_n+1,\lambda}\\ &\times \ell_{n,k+L_n+1}\left(\mathcal{L}_{k+L_n,k-L_n,\lambda}h_{k-L_n-1,0,\lambda}\right)\left[1+\mathcal{O}(\left[\lambda\sigma_n^{-1}L_n+\sigma_n^{-\varpi}\right])\right]\\ &\times \ell_{k-L_n-1,0,\lambda}(\rho) + \mathcal{O}(\sigma^{-\varpi}\alpha_{n,0,\lambda})\\ &= \left(i\lambda\sigma_n^{-1}\sum_{j=k-L_n}^{k+L_n}\Sigma_{k,j} + \mathcal{O}\left(\lambda^2\sigma_n^{-2}L_n^2+\sigma_n^{-\varpi}\right)\right)\mathcal{L}_{n-1,0,\lambda}\rho\\ &\quad + \mathcal{O}(\sigma_n^{-\varpi}\alpha_{n,0,\lambda}),\end{aligned} \tag{4.7}$$

where, in the second line, we have used the fact that Lemma 4.4 implies, recalling Condition (O-3),

$$\begin{aligned}|\ell_{n,k+L_n,\lambda}(\mathcal{L}_{k+L_n-1,k-L_n,\lambda}h_{k-L_n-1,0,\lambda})| \geq& |\alpha_{k+L_n,k-L_n,\lambda}\ell_{n,k+L_n,\lambda}(h_{k+L_n,0,\lambda})|\\ &\times|\ell_{k-L_n-1,0,\lambda}(h_{k-L_n-1,0,\lambda})| - C_\#(\sigma_n^{-\varpi})\\ \geq& c_\# - C_\#\sigma_n^{-\varpi} \geq c_\#.\end{aligned}$$

To complete the estimate needed for Proposition 4.1, we need the following lemma.

Lemma 4.5. *If C_L is choosen large enough, there exists $c_\# > 0$ such that for all $n \in \mathbb{N}$ and each $\lambda \in \mathbb{R}$, $|\lambda| \le \lambda_1 \sigma_n L_n^{-1}$, we have*

$$\left| \int_{M_n} \mathcal{L}_{n-1,0,\lambda} \rho \right| \ge c_\# |\alpha_{n,0,\lambda}| \,.$$

We postpone the proof of the lemma and use it to complete the proof of Proposition 4.1.

Applying Lemma 4.5 to (4.7), we obtain

$$\begin{aligned} \int_{M_n} \mathcal{L}_{n-1,k,\lambda} \hat{g}_k \mathcal{L}_{k-1,0,\lambda} \rho = &\left(i\lambda \sigma_n^{-1} \sum_{j=k-L_n}^{k+L_n} \Sigma_{k,j} + \mathcal{O}\left(\lambda^2 \sigma_n^{-2} L_n^2 + \sigma_n^{-\varpi}\right) \right) \\ &\times \int_{M_n} \mathcal{L}_{n-1,0,\lambda} \rho \,, \end{aligned} \tag{4.8}$$

which is our main estimate in the case $2L_n \le k \le n - 2L_n$.

It remains to consider the cases $k \le 2L_n$ and $k \ge n - 2L_n$. If $k > n - 2L_n$, then, using (4.3),

$$\begin{aligned} &\int_{M_n} \mathcal{L}_{n-1,k,\lambda} \hat{g}_k \mathcal{L}_{k-1,0,\lambda} \rho d\mu_n = \int_{M_n} \mathcal{L}_{n-1,k,0} \hat{g}_k \mathcal{L}_{k-1,k-L_n,0} \mathcal{L}_{k-L_n-1,0,\lambda} \rho d\mu_n \\ &+ i\sigma_n^{-1} \lambda \sum_{m=k-L_n}^{n} \int_{M_n} \mathcal{L}_{n-1,m,0}\, \hat{g}_m \mathcal{L}_{m-1,k,0} \hat{g}_k \mathcal{L}_{k-1,k-L_n,0} \mathcal{L}_{k-L_n-1,0,\lambda} \rho d\mu_n \\ &+ \mathcal{O}(\sigma_n^{-2} \lambda^2 L_n^2 \alpha_{n,0,\lambda}). \end{aligned}$$

Arguing as in Lemma 4.3 yields

$$\begin{aligned} \int_{M_n} \mathcal{L}_{n-1,k,\lambda} \hat{g}_k \mathcal{L}_{k-1,0,\lambda} \rho = &\sum_{j=k-L_n}^{n} \left(i\lambda \sigma_n^{-1} \Sigma_{k,j} + \mathcal{O}\left(\lambda^2 \sigma_n^{-2} L_n^2\right) \right) \\ &\times \int_{M_n} \mathcal{L}_{n-1,0,\lambda} \rho + \mathcal{O}(\sigma_n^{-\varpi} \alpha_{n,0,\lambda}) \\ = &\sum_{j=k-L_n}^{n} \left(i\lambda \sigma_n^{-1} \Sigma_{k,j} + \mathcal{O}\left(\lambda^2 \sigma_n^{-2} L_n^2 + \sigma_n^{-\varpi}\right) \right) \int_{M_n} \mathcal{L}_{n-1,0,\lambda} \rho, \end{aligned} \tag{4.9}$$

where we have applied Lemma 4.5 in the last line.

Analogously, if $k \le 2L_n$, then

$$\begin{aligned} \int_{M_n} \mathcal{L}_{n-1,k,\lambda} \hat{g}_k \mathcal{L}_{k-1,0,\lambda} \rho = &\sum_{j=0}^{k+L_n} \left(i\lambda \sigma_n^{-1} \Sigma_{k,j} + \mathcal{O}\left(\lambda^2 \sigma_n^{-2} L_n^2 + \sigma_n^{-\varpi}\right) \right) \\ &\times \int_{M_n} \mathcal{L}_{n-1,0,\lambda} \rho. \end{aligned} \tag{4.10}$$

Substituting respectively (4.8), (4.9) or (4.10) in each case in equation (4.1), summing over k, and recalling (3.18) we finally have

$$\begin{aligned} &\frac{d}{d\lambda} \Upsilon_n(\lambda) = -\left(\lambda + \mathcal{E}_n(\lambda)\right) \Upsilon_n(\lambda) \\ &|\mathcal{E}_n(\lambda)| \le C_\# \left(\lambda^2 \sigma_n^{-3} L_n^2 n + n \sigma_n^{-\varpi} \right), \end{aligned}$$

concluding the proof of Proposition 4.1. □

Proof of Lemma 4.5. We may assume that $n \ge \max\{2L_n, n_1\}$, where n_1 is from Lemma 4.2 since otherwise the lemma is trivial using (4.3).

For such n, we write, using (O-3)

$$\begin{aligned} \mathcal{L}_{n-1,0,\lambda} \rho = &\mathcal{L}_{n-1,L_n,\lambda} \mathcal{L}_{L_n-1,0,\lambda} \rho \\ = &\mathcal{L}_{n-1,L_n,\lambda} \left(\alpha_{L_n,0,\lambda} \ell_{L_n,0,\lambda}(\rho) h_{L_n,0,\lambda} + \mathcal{O}(\alpha_{L_n,0,\lambda} e^{-cL_n}) \right) . \end{aligned}$$

Integrating and estimating as in (4.6) for each $\varpi > 0$ we can choose C_L such that

$$\alpha_{L_n,0,\lambda}\ell_{L_n,0,\lambda}(\rho) = \int \mathcal{L}_{L_n,0,\lambda}\rho + \mathcal{O}(\alpha_{L_n,0,\lambda}e^{-cL_n}) = 1 + \mathcal{O}(\alpha_{L_n,0,\lambda}(\lambda\sigma_n^{-1}L_n + \sigma_n^{-\varpi})).$$

Putting these estimates together, we have

$$\begin{aligned}\int_{M_n} \mathcal{L}_{n-1,0,\lambda}\rho &= \int_{M_n} \mathcal{L}_{n-1,L_n,\lambda}\left(h_{L_n,0,\lambda} + \mathcal{O}(\alpha_{L_n,0,\lambda}(\lambda\sigma_n^{-1}L_n + \sigma_n^{-\varpi}))\right)\\ &= \alpha_{n,L_n+1,\lambda}\left(\ell_{n,L_n,\lambda}(h_{L_n,0,\lambda}) + \mathcal{O}(\alpha_{L_n,0,\lambda}(\lambda\sigma_n^{-1}L_n + \sigma_n^{-\varpi}) + e^{-c|n-L_n|})\right)\end{aligned}$$

Next, applying first (O-2) and Lemma 4.4(a) and then Lemma 4.2 yields,

$$\begin{aligned}\left|\int_{M_n} \mathcal{L}_{n-1,0,\lambda}\rho\right| &\geq K^{-1}\left|\alpha_{n,L_n+1,\lambda}\alpha_{L_n,0,\lambda}\left(\ell_{n,L_n,\lambda}(h_{L_n,0,\lambda}) + \mathcal{O}\big(\lambda\sigma_n^{-1}L_n + \sigma_n^{-\varpi}\big)\right)\right|\\ &= K^{-1}|\alpha_{n,0,\lambda}|\frac{\ell_{n,L_n,\lambda}(h_{L_n,0,\lambda}) + \mathcal{O}\big(\lambda\sigma_n^{-1}L_n + \sigma_n^{-\varpi}\big)}{\ell_{n,L_n,\lambda}(h_{L_n,0,\lambda}) + \mathcal{O}\big(e^{-c\min\{L_n,n-L_n\}}\big)}\\ &\geq c_{\#}|\alpha_{n,0,\lambda}|\,,\end{aligned}$$

for ϖ large enough and λ sufficiently small, where in the last line we have used the lower bound on $|\ell_{n,L_n,\lambda}(h_{L_n,0,\lambda})|$ from (O-3). □

5. Cones

In this section, we describe the machinery of complex cones necessary to verify the contraction required by (O-3), and which can also imply (O-2). For the reader's convenience, we provide a self contained theory of complex cones more than sufficient for our needs. Since we strive for simplicity, our results are not optimal; see [38, 17] for a more complete, general, but, we believe, less readable exposition. We will present the theory in a general context, with some tools to aid in the application of this method to the examples: expanding maps in Section 6 and billiards in Section 7. We start by recalling few needed facts about real cones.

5.1. **Real Cones.**

Let V be a topological real vector space and $\mathcal{S} \in V'$ such that $\ell(x) = 0$ for all $\ell \in \mathcal{S}$ implies $x = 0$. Define

$$C_{\mathbb{R}} = \{h \in V \setminus \{0\} \ : \ \ell(h) \geq 0, \forall \ell \in \mathcal{S}\}. \tag{5.1}$$

Note that

$$C_{\mathbb{R}} \cap -C_{\mathbb{R}} = \emptyset, \tag{5.2}$$

since if $h \in C_{\mathbb{R}} \cap -C_{\mathbb{R}}$, then $\ell(h) = 0$ for all $\ell \in \mathcal{S}$; hence $h = 0$, contrary to definition (5.1).

Moreover, assume that there exists $\mathbb{e} \in C_{\mathbb{R}}$ such that

$$\text{for all } h \in V \text{ there exists } \lambda \in \mathbb{R}_+ \text{ such that } \lambda\mathbb{e} - h \in C_{\mathbb{R}}. \tag{5.3}$$

Then one can easily check that

$$\|h\| := \inf\{\lambda \in \mathbb{R}_+ \ : \ \ell(\lambda\mathbb{e} \pm h) \geq 0, \ \forall \ell \in \mathcal{S}\} \tag{5.4}$$

is a norm.[9] Note that $\|\mathbb{e}\| = 1$. Let $\mathcal{B}_{\mathbb{R}}$ be the completion of V in such a norm.[10] Since, for all $\ell \in \mathcal{S}$, $\ell(\|h\|\mathbb{e} \pm h) \geq 0$, it follows that $\mathcal{S} \subset \mathcal{B}'_{\mathbb{R}}$.

Finally, let $\mathcal{S}_*$ be the weak-$*$ closure of the convex hull of $\{\lambda\ell \ : \ \lambda \in \mathbb{R}_+, \ell \in \mathcal{S}\}$. To simplify matters further, we restrict to the special case in which there exists $\mathbb{m} \in \mathcal{S}_*$ and $\kappa \in (0,1)$ such that

$$\mathbb{m}(\mathbb{e}) = 1 \tag{5.5}$$

and, for all $h \in \mathcal{C}_{\mathbb{R}}$,

$$\mathbb{m}(h) \geq \kappa\|h\|. \tag{5.6}$$

[9] Alternatively, it is equivalent to define this norm by $\|h\| = \inf\{\lambda \in \mathbb{R} \ : \ -\lambda\mathbb{e} \preceq h \preceq \lambda\mathbb{e}\}$, where $h \preceq g$ iff $g - h \in C_{\mathbb{R}}$.

[10] Note that the expression (5.4) defines the norm also on $\mathcal{B}_{\mathbb{R}}$.

Having described the abstract setting, we can now proceed to explore its properties. We start defining a larger cone $\mathcal{C}_{\mathbb{R}} \supset C_{\mathbb{R}}$ by

$$\mathcal{C}_{\mathbb{R}} = \{h \in \mathcal{B}_{\mathbb{R}} \setminus \{0\} \ : \ \ell(h) \geq 0, \forall \ell \in \mathcal{S}\}. \tag{5.7}$$

Next, we define the dual cone as

$$\mathcal{C}'_{\mathbb{R}} = \{\ell \in \mathcal{B}'_{\mathbb{R}} \ : \ \ell(h) \geq 0 \quad \forall h \in \mathcal{C}_{\mathbb{R}}\} \supset \mathcal{S} \tag{5.8}$$

Note that we have

$$\mathcal{C}_{\mathbb{R}} = \{h \in \mathcal{B}_{\mathbb{R}} \setminus \{0\} \ : \ \ell(h) \geq 0, \forall \ell \in \mathcal{C}'_{\mathbb{R}}\}. \tag{5.9}$$

Hence, again, $\mathcal{C}_{\mathbb{R}} \cap -\mathcal{C}_{\mathbb{R}} = \emptyset$.[11] The cone $\mathcal{C}_{\mathbb{R}}$ is associated with the Hilbert metric (see [8, Equation (D.1.2)]): for all $g, h \in \mathcal{C}_{\mathbb{R}}$,

$$\begin{aligned} \alpha(h,\, g) &= \sup\{\lambda \in \mathbb{R}^+ \mid g - \lambda h \in \mathcal{C}_{\mathbb{R}}\} \\ \beta(h,\, g) &= \inf\{\mu \in \mathbb{R}^+ \mid \mu h - g \in \mathcal{C}_{\mathbb{R}}\} \\ d_{H,\mathcal{C}_{\mathbb{R}}}(h,\, g) &= \ln\left[\frac{\beta(h,\, g)}{\alpha(h,\, g)}\right]. \end{aligned}$$

From now on we will write simply d_H, if the cone is clear from the context.
Since by (5.7) $g - \lambda h \in \mathcal{C}_{\mathbb{R}}$ iff $\ell(g - \lambda h) \geq 0$ for all $\ell \in \mathcal{S}$, taking the limit $\lambda \uparrow \alpha$ yields $\alpha = \inf_{\ell \in \mathcal{S}} \frac{\ell(g)}{\ell(h)}$. Alternatively, using (5.9) in the same way, we have $\alpha = \inf_{\ell \in \mathcal{C}'_{\mathbb{R}}} \frac{\ell(g)}{\ell(h)} = \inf_{\ell \in \mathcal{S}} \frac{\ell(g)}{\ell(h)}$. Arguing similarly for β yields

$$d_H(h, g) = \sup_{\ell, p \in \mathcal{S}} \ln \frac{\ell(h)p(g)}{\ell(g)p(h)} = \sup_{\ell, p \in \mathcal{C}'_{\mathbb{R}}} \ln \frac{\ell(h)p(g)}{\ell(g)p(h)}. \tag{5.10}$$

Let us recall the relevance of real cones for our context.

Theorem 5.1 ([31, Theorem 1.1])**.** *Let $\mathcal{B}_1, \mathcal{B}_2$ be real Banach spaces with real cones $\mathcal{C}_1, \mathcal{C}_2$ as above. Let $L \in L(\mathcal{B}_1, \mathcal{B}_2)$, such that $L\mathcal{C}_1 \subset \mathcal{C}_2$ and*

$$\Delta = \sup_{x, y \in \mathcal{C}_1} d_{H,\mathcal{C}_2}(Lx, Ly).$$

Then, for all $h, g \in \mathcal{C}_1$ we have

$$d_{H,\mathcal{C}_2}(Lh, Lg) \leq \tanh(\Delta/4) d_{H,\mathcal{C}_1}(h, g).$$

Lemma 5.2 ([31, Lemma 1.3])**.** *Given $h, g \in \mathcal{C}$ for which $\|f\| = \|g\|$,*

$$\|h - g\| \leq \left(e^{d_H(h,g)} - 1\right) \|f\|.$$

In the next section, we describe how the above results can be extended to the case of operators acting on complex Banach spaces, see, in particular, Theorem 5.15 and Lemma 5.11.

5.2. **Complex Cones.**

Let $\mathcal{B}_{\mathbb{C}}$ be the complexification of $\mathcal{B}_{\mathbb{R}}$ as detailed in Lemma 2.5.

Definition 5.3 (Complex cone and its dual)**.** *The associated complex cone is defined as $\mathcal{C}_{\mathbb{C}} := \mathbb{C}_* \cdot (\mathcal{C}_{\mathbb{R}} + i\mathcal{C}_{\mathbb{R}})$, where $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$. We may also write $\mathcal{C}_{\mathbb{C}} = \mathbb{C}_1 \cdot (\mathcal{C}_{\mathbb{R}} + i\mathcal{C}_{\mathbb{R}})$, where $\mathbb{C}_1 = \{z \in \mathbb{C} : |z| = 1\}$. We define the dual cone by*

$$\mathcal{C}'_{\mathbb{C}} = \{\ell \in \mathcal{B}'_{\mathbb{C}} \ : \ \ell(h) \neq 0 \quad \forall h \in \mathcal{C}_{\mathbb{C}}\}.$$

For any element $\ell \in \mathcal{B}'_{\mathbb{R}}$, its action on $h + ig \in \mathcal{B}_{\mathbb{C}}$, $h, g \in \mathcal{B}_{\mathbb{R}}$, is naturally defined by $\ell(h + ig) = \ell(h) + i\ell(g)$. Then, for all $h \in \mathcal{C}_{\mathbb{C}}$, letting $z \in \mathbb{C}$ such that $|z| = 1$ and $zh = x + iy$ with $x, y \in \mathcal{C}_{\mathbb{R}}$,

$$\begin{aligned} |\mathrm{m}(h)| = |\mathrm{m}(x + iy)| &= \sqrt{\mathrm{m}(x)^2 + \mathrm{m}(y)^2} \geq \kappa \sqrt{\|x\|^2 + \|y\|^2} \\ &\geq \frac{\kappa}{\sqrt{2}} \|x + iy\| = \frac{\kappa}{\sqrt{2}} \|h\|, \end{aligned} \tag{5.11}$$

[11] Indeed, if $h \in \mathcal{C}_{\mathbb{R}} \cap -\mathcal{C}_{\mathbb{R}}$, then $\ell(h) = 0$ for all $\ell \in \mathcal{S}$, and $\|h\| = 0$, by (5.4).

where in the second line we have used (A.2). In addition, if $\|u\| \le 1$, then

$$\mathbb{e} + \frac{1}{2}u \succeq \mathbb{e} - \frac{1}{2}\|u\|\mathbb{e} \succeq \frac{1}{2}\mathbb{e} \succeq 0$$

that is the interior of $\mathcal{C}_{\mathbb{R}}$, and hence of $\mathcal{C}_{\mathbb{C}}$, is not empty. That is $\mathcal{C}_{\mathbb{C}}$ is a *regular cone* in the language of [38, Defintion 3.2-(3)]. Moreover, for each $\ell \in \mathcal{B}'_{\mathbb{R}}$ such that $\|\ell\| < \kappa$ we have $\mathbb{m} + \ell \in \mathcal{C}'_{\mathbb{R}}$, so also $\mathcal{C}'_{\mathbb{R}}$ has non empty interior.

The first statement of the following Lemma is contained in [38, Proposition 5.2], while the second is the first part of the proof of [17, Lemma 4.1].[12]

Lemma 5.4. *We have the following characterizations.*

a) $\mathcal{C}_{\mathbb{C}} = \{h \in \mathcal{B}_{\mathbb{C}} \setminus \{0\} \ : \ \forall \ell, m \in \mathcal{C}'_{\mathbb{R}},\ \Re\left(\ell(h)\overline{m(h)}\right) \ge 0\}$.

b) $\mathcal{C}'_{\mathbb{C}} = \{\ell \in \mathcal{B}'_{\mathbb{C}} \ : \ \forall x, y \in \mathcal{C}_{\mathbb{R}},\ \Re(\ell(x)\overline{\ell(y)}) > 0\}$.

c) $\mathcal{C}'_{\mathbb{C}} \supset \{\pm\ell \pm ip \ : \ \ell, p \in \mathring{\mathcal{C}}'_{\mathbb{R}}\} =: \hat{\mathcal{C}}'_{\mathbb{C}}$, *where* $\mathring{\mathcal{C}}'_{\mathbb{R}} = \{\ell \in \mathcal{C}'_{\mathbb{R}} \ : \ \ell(x) > 0\, \forall x \in \mathcal{C}_{\mathbb{R}}\}$.

Proof. (a) To prove that $\mathcal{C}_{\mathbb{C}} \subseteq \{h \in \mathcal{B}_{\mathbb{C}} \setminus \{0\} \ : \ \forall \ell, m \in \mathcal{C}'_{\mathbb{R}},\ \Re\left(\ell(h)\overline{m(h)}\right) \ge 0\}$, check that $\Re\left(\ell(zh)\overline{m(zh)}\right) = |z|^2\Re\left(\ell(h)\overline{m(h)}\right)$, for all $z \in \mathbb{C}_*$. Also, by defintion, for each $h \in \mathcal{C}_{\mathbb{C}}$ there exists $z \in \mathbb{C}$ such that $zh = x + iy$ with $x, y \in \mathcal{C}_{\mathbb{R}}$. Finally, for $x, y \in \mathcal{C}_{\mathbb{R}}$, $\Re\left(\ell(x+iy)\overline{m(x+iy)}\right) = \ell(x)m(x) + \ell(y)m(y) \ge 0$.

To prove the opposite inclusion, let $h \in \{h \in \mathcal{B}_{\mathbb{C}} \ : \ \forall \ell, m \in \mathcal{C}'_{\mathbb{R}},\ \Re\left(\ell(h)\overline{m(h)}\right) \ge 0\}$. We claim that there exists $z \in \mathbb{C}$ such that $\tilde{h} := zh = x + iy$ with $x \in \mathcal{B}_{\mathbb{R}}$ and $y \in \mathcal{C}_{\mathbb{R}}$. Indeed, let $h = u + iv$, $u, v \in \mathcal{B}_{\mathbb{R}}$ and assume that $u, v \notin \mathcal{C}_{\mathbb{R}} \cup -\mathcal{C}_{\mathbb{R}}$. If $u = \lambda v$, $\lambda \in \mathbb{R}$, then we can choose $\ell, m \in \mathcal{C}'_{\mathbb{R}}$ such that $\ell(u) > 0$ and $\lambda m(u) < 0$, so $\Re\left(\ell(h)\overline{m(h)}\right) = \lambda\ell(u)m(u) < 0$ contrary to assumptions. Hence u, v must be linearly independent in $\mathcal{B}_{\mathbb{R}}$. Note that there must exist $\ell \in \mathcal{C}'_{\mathbb{R}}$ such that $\ell(u) = 0$ and $\ell(v) \ne 0$.[13] Thus $\Re\left(\ell(h)\overline{m(h)}\right) = \ell(v)m(v)$, and choosing $m \in \mathcal{C}'_{\mathbb{R}}$ such that $\ell(v)m(v) < 0$ leads to a contradiction. Hence, if $v \in \mathcal{C}_{\mathbb{R}}$ we are done, if $v \in -\mathcal{C}_{\mathbb{R}}$ then we choose $z = -1$, if $u \in \mathcal{C}_{\mathbb{R}}$ we choose $z = i$ and finally if $u \in -\mathcal{C}_{\mathbb{R}}$ we choose $z = -i$.

Then for each $\ell \in \mathcal{C}'_{\mathbb{R}}$ such that $\ell(\tilde{h}) \ne 0$, we can write $\ell(\tilde{h})|\ell(\tilde{h})|^{-1} = e^{i\theta_\ell}$, $\theta_\ell \in [0, \pi]$. Suppose that there exists $\ell_1, \ell_2 \in \mathcal{C}'_{\mathbb{R}}$ such that $\pi \ge |\theta_{\ell_2} - \theta_{\ell_1}| > \pi/2$, then

$$\Re(\ell_2(h)\overline{\ell_1(h)}) = |z|^{-2}|\ell_2(\tilde{h})\ell_1(\tilde{h})|\cos(\theta_{\ell_2} - \theta_{\ell_1}) < 0$$

contrary to the assumption. Thus there exists $\varphi \in [0, \pi/2]$ such that for each $\ell \in \mathcal{C}'_{\mathbb{R}}$, $\ell(\tilde{h}) \ne 0$, $\theta_\ell \in [\varphi - \pi/4, \varphi + \pi/4]$. Let $e^{-i\varphi}\tilde{h} = u + iv$, then $\ell(u + iv) = |\ell(u + iv)|e^{i(\theta_\ell - \varphi)}$ which implies $\ell(u) \ge |\ell(v)|$, that is $\ell(u \pm v) \ge 0$. This means that $u \pm v \in \mathcal{C}_{\mathbb{R}}$. Accordingly, setting $x + iy = (1 + i)e^{-i\varphi}\tilde{h} = (1 + i)(u + iv) = u - v + i(u + v)$ we have that $x, y \in \mathcal{C}_{\mathbb{R}}$, hence $h \in \mathcal{C}_{\mathbb{C}}$.

Next, we prove statement (b). Let $\ell \in \{\ell \in \mathcal{B}'_{\mathbb{C}} \ : \ \forall x, y \in \mathcal{C}_{\mathbb{R}},\ \Re(\ell(x)\overline{\ell(y)}) > 0\}$. If there exist $x, y \in \mathcal{C}_{\mathbb{R}}$ such that $\ell(x + iy) = 0$, then $\ell(x) = -i\ell(y)$ and $\Re(\ell(x)\overline{\ell(y)}) = \Re(-i|\ell(y)|^2) = 0$ contrary to the assumption. Hence $\{\ell \in \mathcal{B}'_{\mathbb{C}} \ : \ \forall x, y \in \mathcal{C}_{\mathbb{R}},\ \Re(\ell(x)\overline{\ell(y)}) > 0\} \subset \mathcal{C}'_{\mathbb{C}}$.

To prove the opposite inclusion, let $\ell \in \mathcal{C}'_{\mathbb{C}}$. Suppose that there exist $x, y \in \mathcal{C}_{\mathbb{R}}$ such that $\Re(\ell(x)\overline{\ell(y)}) \le 0$ and write $\ell(x) = re^{i\theta} \ne 0$, $\ell(y) = se^{i\varphi} \ne 0$, with $\theta, \varphi \in [-\pi, \pi)$. Then $0 \ge \Re(\ell(x)\overline{\ell(y)}) = rs\cos(\theta - \varphi)$, so $\frac{3}{2}\pi \ge |\theta - \varphi| \ge \pi/2$. Define

$$z = -se^{-i(\theta-\varphi)}x + ry = -s\cos(\theta - \varphi)x + ry + is\sin(\theta - \varphi)x =: u + iv$$

then $\ell(z) = 0$. Note that $z \ne 0$ since $-s\cos(\theta - \varphi) \ge 0$ hence $0 \ne u \in \mathcal{C}_{\mathbb{R}}$. If $\sin(\theta - \varphi) \ge 0$, then also $v \in \mathcal{C}_{\mathbb{R}}$. Otherwise $iz = u_1 + iv_1$ with $u_1, v_1 \in \mathcal{C}_{\mathbb{R}}$. Thus there exists $w \in \mathcal{C}_{\mathbb{C}}$ such that $\ell(w) = 0$, contradicting the hypothesis.

Finally, we prove statement (c). Let $\pm\ell, \pm p \in \mathring{\mathcal{C}}'_{\mathbb{R}}$, then for each $x, y \in \mathcal{C}_{\mathbb{R}}$ we have

$$\Re((\ell + ip)(x) \cdot (\ell - ip)(y)) = \ell(x)\ell(y) + p(x)p(y) > 0,$$

[12] Note however that in [17, Lemma 4.1] it is assumed that x, y are linearly independent, which is not necessary.

[13] Indeed, by definition $\mathcal{S} \subset \mathcal{C}'_{\mathbb{R}}$ and if $\pm u \notin \mathcal{C}_{\mathbb{R}}$, then there must exist $\ell_1, p \in \mathcal{S}$ such that $\ell_1(u), p(-u) < 0$. Hence, $\ell = \ell_1 - \frac{\ell_1(u)}{p(u)}p \in \mathcal{C}'_{\mathbb{R}}$ and $\ell(u) = 0$. Next, if for all $\ell \in \mathcal{C}'_{\mathbb{R}}$ such that $\ell(u) = 0$ we have also $\ell(v) = 0$, then the same must happen for all $\ell \in \mathcal{B}'_{\mathbb{R}}$ (since $\mathcal{C}'_{\mathbb{R}}$ has a non empty interior) and this is impossible by the Hahn-Banch theorem.

which implies $\ell + ip \in \mathcal{C}'_{\mathbb{C}}$ by part (b) of the lemma. □

Lemma 5.5. *For each $\ell \in \mathcal{C}'_{\mathbb{C}}$ we have*

$$\|\ell\|' \leq \sqrt{2}|\ell(\mathbb{e})|.$$

Proof. If $\ell \in \mathcal{C}'_{\mathbb{C}}$, then, when acting on $\mathcal{B}_{\mathbb{R}}$ it can be written as $\ell(x) = \ell_1(x) + i\ell_2(x)$. By Lemma 5.4(b) we have, for each $x \in \mathcal{B}_{\mathbb{R}}$,

$$0 \leq \Re(\ell(\|x\|\mathbb{e} - x)\overline{\ell(\|x\|\mathbb{e} + x)}) = \ell_1(\|x\|\mathbb{e} - x)\ell_1(\|x\|\mathbb{e} + x) + \ell_2(\|x\|\mathbb{e} - x)\ell_2(\|x\|\mathbb{e} + x)$$

which implies

$$|\ell(x)| \leq \|x\||\ell(\mathbb{e})|.$$

If $x + iy \in \mathcal{B}_{\mathbb{C}}$, then

$$|\ell(x + iy)| \leq |\ell(\mathbb{e})|(\|x\| + \|y\|) \leq \sqrt{2}|\ell(\mathbb{e})|\|x + iy\|.$$

□

Definition 5.6. *We call the complex cone* linearly convex *if for each $g \in \mathcal{B}_{\mathbb{C}} \setminus \mathcal{C}_{\mathbb{C}}$ we can find $\ell \in \hat{\mathcal{C}}'_{\mathbb{C}}$ such that $\ell(g) = 0$.*[14]

Remark 5.7. *Note that if $\mathcal{C}_{\mathbb{C}}$ is linearly convex, then*

$$\mathcal{C}_* := \{h \in \mathcal{B}_{\mathbb{C}} \ : \ \ell(h) \neq 0 \ \forall \ \ell \in \hat{\mathcal{C}}'_{\mathbb{C}}\} = \mathcal{C}_{\mathbb{C}}.$$

Indeed, if $h \in \mathcal{C}_*$, then $h \in \mathcal{C}_{\mathbb{C}}$, otherwise, by linear convexity, there exists $\ell \in \hat{\mathcal{C}}'_{\mathbb{C}}$ such that $\ell(h) = 0$, contrary to the assumption. On the other hand, if $h \in \mathcal{C}_{\mathbb{C}}$, then for each $\ell \in \hat{\mathcal{C}}'_{\mathbb{C}}$, by Lemma 5.4(c) and Definition 5.3, we have that $\ell(h) \neq 0$; hence, $h \in \mathcal{C}_*$.

The next Lemma corresponds to [18, Lemma 4.1].

Lemma 5.8. *The cone $\mathcal{C}_{\mathbb{C}}$ is linearly convex.*

Proof. Let $x + iy \notin \mathcal{C}_{\mathbb{C}}$, $x, y \in \mathcal{B}_{\mathbb{R}}$, then by the first statement of Lemma 5.4, there exists $\tilde{\ell}, \tilde{p} \in \mathcal{C}'_{\mathbb{R}}$ such that

$$\tilde{\ell}(x)\tilde{p}(x) + \tilde{\ell}(y)\tilde{p}(y) = \Re(\tilde{\ell}(x + iy)\tilde{p}(x - iy)) < 0.$$

Note that, setting $p = \tilde{p} + \varepsilon\mathbb{m}$ and $\ell = \tilde{\ell} + \varepsilon\mathbb{m}$, with $\varepsilon > 0$, then $\ell, p \in \mathring{\mathcal{C}}'_{\mathbb{R}}$ and for ε sufficiently small we still have

$$\ell(x)p(x) + \ell(y)p(y) < 0. \tag{5.12}$$

Define

$$q = [\alpha_p\ell(x) + p(x)]\ell - i\ell(y)[\alpha_p\ell + p]$$
$$\alpha_p = -\frac{p(x)\ell(x) + \ell(y)p(y)}{\ell(x)^2 + \ell(y)^2} > 0.$$

One can check that $q(x + iy) = 0$ and $q \in \hat{\mathcal{C}}'_{\mathbb{C}}$. □

In order to construct a metric on the cone, for each $h, g \in \mathcal{C}_{\mathbb{C}}$ define

$$E_{\mathcal{C}}(h, g) = \left\{ \frac{\ell(h)}{\ell(g)} \ : \ \ell \in \mathcal{C}'_{\mathbb{C}} \right\}. \tag{5.13}$$

For future use, it is convenient to record the following facts.

Lemma 5.9. *For each $h, g \in \mathcal{C}_{\mathbb{C}}$ we have*

$$E_{\mathcal{C}}(h, g) = \left\{ \frac{\ell(h)}{\ell(g)} \ : \ \ell \in \hat{\mathcal{C}}'_{\mathbb{C}} \right\} =: \widehat{E}_{\mathcal{C}}(h, g).$$

[14] Note that this definition is a bit different from the one in [18, Section 3]. But it has the advantage of simplifying the proof of Lemma 5.8.

Proof. Clelarly $\widehat{E}_{\mathcal{C}}(h,g) \subset E_{\mathcal{C}}(h,g)$. On the other hand, note that if $\zeta \in E_{\mathcal{C}}(h,g)$, then there exists $\ell \in \mathcal{C}'_{\mathbb{C}}$ such that $\ell(h-\zeta g)=0$. But this implies $h-\zeta g \notin \mathcal{C}_{\mathbb{C}}$, otherwise, by definition, there would exist z such that $z(h-\zeta g) = x+iy$ with $x,y \in \mathcal{C}_{\mathbb{R}}$ and this would imply $\Re(\ell(x)\overline{\ell(y)})=0$ contrary to Lemma 5.4(b). Accordingly, by Lemma 5.8 there exists $\ell \in \hat{\mathcal{C}}'_{\mathbb{C}}$ such that $\ell(h-\zeta g)=0$, hence $\zeta \in \widehat{E}_{\mathcal{C}}(h,g)$. That is $E_{\mathcal{C}}(h,g) = \widehat{E}_{\mathcal{C}}(h,g)$ and the lemma follows. □

Lemma 5.10. *For each $h,g \in \mathcal{C}_{\mathbb{C}}$, $z \notin E_{\mathcal{C}}(h,g)$ iff $zg-h \in \mathcal{C}_{\mathbb{C}}$.*

Proof. If $z \notin E_{\mathcal{C}}(h,g)$, then, by Lemma 5.9, we have that for all $\ell \in \hat{\mathcal{C}}'_{\mathbb{C}}$ $z \neq \frac{\ell(h)}{\ell(g)}$, that is $\ell(h-zg) \neq 0$. Hence, by Remark 5.7, $zg-h \in \mathcal{C}_{\mathbb{C}}$. On the other hand, if $zg-h \in \mathcal{C}_{\mathbb{C}}$, then again by by Remark 5.7, we have that, for each $\ell \in \hat{\mathcal{C}}'_{\mathbb{C}}$, $\ell(zg-h) \neq 0$. Hence, $z \neq \frac{\ell(h)}{\ell(g)}$, that is $z \notin E_{\mathcal{C}}(h,g)$. □

Next, we define the key object in the theory,

$$\delta_{\mathcal{C}}(h,g) = \ln \frac{\sup_{z\in E_{\mathcal{C}}(h,g)}|z|}{\inf_{z\in E_{\mathcal{C}}(h,g)}|z|} = \sup_{z,w\in E_{\mathcal{C}}(h,g)} \ln\left|\frac{z}{w}\right|. \tag{5.14}$$

As a first result, we have a version of [17, Lemma 2.6] showing the relevance of $\delta_{\mathcal{C}}$.

Lemma 5.11. *Let $h,g \in \mathcal{C}_{\mathbb{C}}$, such that $\mathbb{m}(h)=\mathbb{m}(g)$, $|\mathbb{m}(g)|=1$. Then*

$$\|h-g\| \le \frac{\sqrt{2}}{\kappa}\delta_{\mathcal{C}}(h,g).$$

Proof. There exists $z\in\mathbb{C}$, $|z|=1$ such that $\mathbb{m}(zh)=\mathbb{m}(zg)=1$. Let $\tilde{h}=zh$ and $\tilde{g}=zg$. Also, by definition $\delta_{\mathcal{C}}(h,g)=\delta_{\mathcal{C}}(\tilde{h},\tilde{g})$. Then, since $\mathbb{m}\in\mathcal{S}_*$,

$$a := \inf_{z\in E_{\mathcal{C}}(\tilde{h},\tilde{g})}|z| \le \left|\frac{\mathbb{m}(\tilde{h})}{\mathbb{m}(\tilde{g})}\right| = 1 \le \sup_{z\in E_{\mathcal{C}}(\tilde{h},\tilde{g})}|z| =: b$$

Note that if $a=0$ or $b=\infty$, then the Lemma is trivially satisfied. For each $0<\alpha<a\le 1\le b<\beta$ we have $\alpha\tilde{g}-\tilde{h}, \beta\tilde{g}-\tilde{h}\in\mathcal{C}_{\mathbb{C}}$. Indeed, if $\alpha\tilde{g}-\tilde{h}\notin\mathcal{C}_{\mathbb{C}}$, then, since the cone is linearly convex (see Definition 5.6), there would exist $\ell\in\hat{\mathcal{C}}'_{\mathbb{C}}$ such that $\ell(\alpha\tilde{g}-\tilde{h})=0$, but then $\alpha=\frac{\ell(\tilde{h})}{\ell(\tilde{g})}\ge a$ contrary to the assumption; the argument for $\beta\tilde{g}-\tilde{h}\in\mathcal{C}_{\mathbb{C}}$ is the same.

We can then write $(\beta-\alpha)(\tilde{g}-\tilde{h}) = (\beta-1)(\alpha\tilde{g}-\tilde{h}) + (1-\alpha)(\beta\tilde{g}-\tilde{h})$, and, recalling (5.11), we have

$$\begin{aligned}
\|\tilde{g}-\tilde{h}\| &\le \frac{\beta-1}{\beta-\alpha}\|\alpha\tilde{g}-\tilde{h}\| + \frac{1-\alpha}{\beta-\alpha}\|\beta\tilde{g}-\tilde{h}\| \\
&\le \frac{\sqrt{2}}{\kappa}\left[\frac{\beta-1}{\beta-\alpha}|\mathbb{m}(\alpha\tilde{g}-\tilde{h})| + \frac{1-\alpha}{\beta-\alpha}|\mathbb{m}(\beta\tilde{g}-\tilde{h})|\right] \\
&\le \frac{2\sqrt{2}}{\kappa}\frac{(\beta-1)(1-\alpha)}{\beta-\alpha} = \frac{2\sqrt{2}}{\kappa}\frac{(\sqrt{\beta}-\sqrt{\alpha})^2-(1-\sqrt{\alpha\beta})^2}{\beta-\alpha} \\
&\le \frac{2\sqrt{2}}{\kappa}\frac{\sqrt{\beta}-\sqrt{\alpha}}{\sqrt{\beta}+\sqrt{\alpha}} \le \frac{2\sqrt{2}}{\kappa}\ln\sqrt{\frac{\beta}{\alpha}} = \frac{\sqrt{2}}{\kappa}\ln\frac{\beta}{\alpha}.
\end{aligned}$$

where, in the last line, we have used that, for $x\ge 1$, $\ln x \ge \frac{x-1}{x+1}$. Taking the inf on β and the sup on α yields

$$\|g-h\| = \|\tilde{g}-\tilde{h}\| \le \frac{\sqrt{2}}{\kappa}\delta_{\mathcal{C}}(\tilde{h},\tilde{g}) = \frac{\sqrt{2}}{\kappa}\delta_{\mathcal{C}}(h,g).$$

□

For $z\in\mathbb{C}$ and all $h\in\mathcal{C}_{\mathbb{C}}$ we have $\delta_{\mathcal{C}}(h,zh)=0$. It is then natural to define the equivalence relation $h\sim g$ iff $h=zg$ for some $z\in\mathbb{C}$. Let $\widetilde{\mathcal{C}}_{\mathbb{C}}$ be the quotient of $\{h\in\mathcal{C}_{\mathbb{C}} \,:\, \delta_{\mathcal{C}}(h,\mathbb{e})<\infty\}$. The following is similar to [18, Theorem 3.1].[15]

Theorem 5.12. *The space $(\widetilde{\mathcal{C}}_{\mathbb{C}},\delta_{\mathcal{C}})$ is a complete metric space.*

[15] This result is not used in the present paper; we include it for completeness.

Proof. Clearly, $\delta_{\mathcal{C}}(g,h)=\delta_{\mathcal{C}}(h,g)$. For each $h,g,f\in\widetilde{\mathcal{C}}_{\mathbb{C}}$ and $\ell\in\hat{\mathcal{C}}'_{\mathbb{R}}$ we have

$$\inf_{z\in E_{\mathcal{C}}(h,g)}|z|\inf_{z\in E_{\mathcal{C}}(g,f)}|z|\le\left|\frac{\ell(h)}{\ell(g)}\right|\left|\frac{\ell(g)}{\ell(f)}\right|=\left|\frac{\ell(h)}{\ell(f)}\right|\le\sup_{z\in E_{\mathcal{C}}(h,g)}|z|\sup_{z\in E_{\mathcal{C}}(g,f)}|z|,$$

hence, by (5.14) and taking the sup and inf on ℓ, the triangle inequality follows.

Next, if $\delta_{\mathcal{C}}(h,g)=0$, then by Lemma 5.11 we have $\frac{h}{\mathbb{m}(h)}=\frac{g}{\mathbb{m}(g)}$, that is $h\sim g$.

As for the completeness, it follows from Lemma 5.11 that if $\{h_n\}\subset\mathcal{C}_{\mathbb{C}}$ is a Cauchy sequence with respect to $\delta_{\mathcal{C}}$, then

$$\left\|\frac{1}{\mathbb{m}(h_n)}h_n-\frac{1}{\mathbb{m}(h_m)}h_m\right\|\le\frac{\sqrt{2}}{\kappa}\delta_{\mathcal{C}}(h_n,h_m).$$

So $\frac{1}{\mathbb{m}(h_n)}h_n$ is a Cauchy sequence in $\mathcal{B}_{\mathbb{C}}$, hence it has a limit h_*, and by Lemma 5.4(a), $h_*\in\mathcal{C}_{\mathbb{C}}$. Note that, for each $g\in\mathcal{C}_{\mathbb{C}}$ and $q\in\hat{\mathcal{C}}'_{\mathbb{C}}$ with $|q(\mathbb{e})|=1$, recalling equation (5.11),[16]

$$e^{\delta_{\mathcal{C}}(\mathbb{e},g)}\ge\left|\frac{\mathbb{m}(g)q(\mathbb{e})}{\mathbb{m}(\mathbb{e})q(g)}\right|\ge c_{\#}\frac{\|g\|}{|q(g)|}\,.$$

Hence, $|q(h_*)|\ge c_{\#}e^{-\delta_{\mathcal{C}}(\mathbb{e},h_*)}\|h_*\|\ge c_{\#}$, since $\delta_{\mathcal{C}}(\mathbb{e},h_*)\le\delta_{\mathcal{C}}(\mathbb{e},h_m)+1$, for m large enough. For all $\ell,p\in\hat{\mathcal{C}}'_{\mathbb{C}}$ let $\tilde{\ell}=\ell(\mathbb{e})^{-1}\ell$ and $\tilde{p}=p(\mathbb{e})^{-1}p$, then, setting $\tilde{h}_n=h_n/\mathbb{m}(h_n)$ and recalling Lemma 5.5,

$$\left|\frac{\ell(h_*)p(h_n)}{\ell(h_n)p(h_*)}\right|=\left|\frac{\tilde{\ell}(h_*)\tilde{p}(\tilde{h}_n)}{\tilde{\ell}(\tilde{h}_n)\tilde{p}(h_*)}\right|\le\frac{1+\frac{|\tilde{p}(h_*-\tilde{h}_n)|}{\tilde{p}(h_*)}}{1-\frac{|\tilde{\ell}(h_*-\tilde{h}_n)|}{\tilde{\ell}(h_*)}}\le\frac{1+C_{\#}\|h_*-\tilde{h}_n\|}{1-C_{\#}\|h_*-\tilde{h}_n\|}$$

which, taking the sup on ℓ,p implies $\lim_{n\to\infty}\delta_{\mathcal{C}}(h_n,h_*)=0$. □

The previous result is nontrivial provided $\{h\in\mathcal{C}_{\mathbb{C}}:\delta_{\mathcal{C}}(h,\mathbb{e})<\infty\}\neq\emptyset$. This follows from the fact that $\mathbb{e}$ is in the interior of $\mathcal{C}_{\mathbb{C}}$, which is verified by the following lemma.

Lemma 5.13. *For any $h\in\mathcal{B}_{\mathbb{C}}$ and for all $z\in\mathbb{C}$ with $|z|<(\sqrt{2}\|h\|)^{-1}$ we have $\mathbb{e}+zh\in\mathcal{C}_{\mathbb{C}}$ and $\delta_{\mathcal{C}}(\mathbb{e},\mathbb{e}+zh)\le\ln\left(\frac{1+|z|\sqrt{2}\|h\|}{1-|z|\sqrt{2}\|h\|}\right)<\infty$.*

Proof. We begin by claiming that if $h\in\mathcal{B}_{\mathbb{C}}$, then $h+z\mathbb{e}\in\mathcal{C}_{\mathbb{C}}$ for all $z\in\mathbb{C}$ with $|z|>\sqrt{2}\|h\|$. Indeed, by Remark 5.7 and Lemma 5.8, if $h+z\mathbb{e}\notin\mathcal{C}_{\mathbb{C}}$, then there exists $\ell\in\hat{\mathcal{C}}'_{\mathbb{C}}$ such that $\ell(h+z\mathbb{e})=0$. But then applying Lemma 5.5, one has

$$|z\ell(\mathbb{e})|=|\ell(h)|\le\sqrt{2}\|h\||\ell(\mathbb{e})|,$$

which is a contradiction.

With the claim proved, it follows that $\mathbb{e}+zh\in\mathcal{C}_{\mathbb{C}}$ for all $|z|<(\sqrt{2}\|h\|)^{-1}$. Moreover, for any $\ell\in\hat{\mathcal{C}}'_{\mathbb{C}}$,

$$\frac{\ell(\mathbb{e}+zh)}{\ell(\mathbb{e})}=1+z\frac{\ell(h)}{\ell(\mathbb{e})}\,,$$

so that according to the definition (5.14) and using again Lemma 5.5,

$$\delta_{\mathcal{C}}(\mathbb{e}+zh,\mathbb{e})\le\ln\left(\frac{1+|z|\sqrt{2}\|h\|}{1-|z|\sqrt{2}\|h\|}\right)\,,$$

as required. □

The metric $\delta_{\mathcal{C}}$ also provides a partial converse to Lemma 5.5, as follows.

Lemma 5.14. *For any $h\in\mathcal{C}_{\mathbb{C}}$ and all $\ell\in\mathcal{C}'_{\mathbb{C}}$,*

$$|\ell(h)|\ge e^{-\delta_{\mathcal{C}}(h,\mathbb{e})}\frac{\kappa}{\sqrt{2}}\|h\||\ell(\mathbb{e})|\,.$$

Proof. Let $h\in\mathcal{C}_{\mathbb{C}}$ and suppose $D=\delta_{\mathcal{C}}(h,\mathbb{e})$. Then by definition (5.14), for any $\ell\in\mathcal{C}'_{\mathbb{C}}$,

$$e^{D}\ge\frac{|\ell(\mathbb{e})\mathbb{m}(h)|}{|\ell(h)\mathbb{m}(\mathbb{e})|}\ge\frac{|\ell(\mathbb{e})|}{|\ell(h)|}\frac{\kappa}{\sqrt{2}}\|h\|\,,$$

where we have used (5.11). The lemma follows. □

[16]Note that, since $\mathbb{m}\in\mathcal{S}_*$, $\left|\frac{\mathbb{m}(h)}{\mathbb{m}(g)}\right|\in\overline{E_{\mathcal{C}}(h,g)}$.

The next is a basic result in [17]. We provide a detailed proof, although it is essentially as in [17], because our definitions differ slightly from the ones in [17].

Theorem 5.15 ([18, Theorem 3.1 (iii)]). *Let $\mathcal{B}_1, \mathcal{B}_2$ be complex Banach spaces with complex cones $\mathcal{C}_1, \mathcal{C}_2$, satisfying hyotheses (5.1), (5.3) and (5.6). Let $L \in L(\mathcal{B}_1, \mathcal{B}_2)$, such that $L\mathcal{C}_1 \subset \mathcal{C}_2$ and $\Delta = \sup_{x,y\in\mathcal{C}_1} \delta_{\mathcal{C}_2}(Lx, Ly)$. Then, for all $h, g \in \mathcal{C}_1$ we have*

$$\delta_{\mathcal{C}_2}(Lh, Lg) \le \tanh(\Delta/4)\delta_{\mathcal{C}_1}(h, g).$$

Proof. Let $g, h \in \mathcal{C}_1$. If $g = zh$, $z \in \mathbb{C}$, then we have seen that $\delta_{\mathcal{C}_2}(Lh, Lg) = \delta_{\mathcal{C}_1}(h, g) = 0$, hence the theorem is trivially true. We can then assume g, h are linearly independent. Let $\mu, \lambda \in \mathbb{C}$ be such that

$$\begin{aligned} |\mu| &= M := \sup\{|z| \ : \ z \in E_{\mathcal{C}_1}(h, g)\} \\ |\lambda| &= m := \inf\{|z| \ : \ z \in E_{\mathcal{C}_1}(h, g)\}. \end{aligned} \tag{5.15}$$

Since if $M = \infty$ or $m = 0$, the statement is trivial, we can assume $\infty > M > m > 0$. Indeed it must be that $M > m$, otherwise $M = m$ and $E_{\mathcal{C}_1}(h, g) \subset \{z \in \mathbb{C} \ : \ |z| = m\}$. But Lemma 5.4(a) implies that $\mathcal{C}_{\mathbb{C}} \cup \{0\}$ is closed, hence its complement is open, and so is $E_{\mathcal{C}_1}(h, g)$ by Lemma 5.10, contradicting the hypothesis that it is contained in a circle. The openness of $E_{\mathcal{C}_1}(h, g)$ implies that $\lambda, \mu \notin E_{\mathcal{C}_1}(h, g)$. Hence, by Lemma 5.10,

$$\mu g - h \in \mathcal{C}_1 \,, \quad h - \lambda g \in \mathcal{C}_1.$$

Accordingly, by assumption,

$$L(\mu g - h) \in \mathcal{C}_2 \,, \quad L(h - \lambda g) \in \mathcal{C}_2$$

and,

$$\delta_{\mathcal{C}_2}(L(\mu g - h), L(h - \lambda g)) \le \Delta. \tag{5.16}$$

Let $z \notin E_{\mathcal{C}_2}(L(h - \lambda g), L(\mu g - h))$. By Lemma 5.10 this is equivalent to

$$zL(\mu g - h) - L(h - \lambda g) \in \mathcal{C}_2$$

or

$$\frac{z\mu + \lambda}{z + 1} Lg - Lh \in \mathcal{C}_2.$$

Accordingly, if we define the Möbious transformation $\psi(z) = \frac{z\mu+\lambda}{z+1}$, then

$$\psi(E_{\mathcal{C}_2}(L(h - \lambda g), L(\mu g - h))) = E_{\mathcal{C}_2}(Lh, Lg).$$

Recalling (5.16) and (5.14)

$$e^{\Delta} \ge \sup_{z,w\in\psi^{-1}(E_{\mathcal{C}_2}(Lh,Lg))} \left|\frac{z}{w}\right| = \sup_{\alpha,\beta\in E_{\mathcal{C}_2}(Lh,Lg)} \left|\frac{\mu - \alpha}{\mu - \beta}\right| \left|\frac{\lambda - \beta}{\lambda - \alpha}\right| . \tag{5.17}$$

Note that, by Lemma 5.10 and the cone invariance, if $z \notin E_{\mathcal{C}_1}(h, g)$, then $zLg - Lh \in \mathcal{C}_2$, hence $z \notin E_{\mathcal{C}_2}(Lh, Lg)$. It follows that $E_{\mathcal{C}_2}(Lh, Lg) \subset E_{\mathcal{C}_1}(h, g)$, hence in equation (5.17) we have $m < |\alpha|, |\beta| < M$.

Next, we choose λ, μ to maximize the right-hand side of the above expression. Note that $\phi(z) = \frac{z-\beta}{z-\alpha}$ is a Möbius transformation, hence it maps the circle $|z| = r$ into the circle of center $c(r) = \frac{\beta\overline{\alpha}-r^2}{|\alpha|^2-r^2}$ and radius $R(r) = \frac{r|\beta-\alpha|}{||\alpha|^2-r^2|}$.[17] As observed above, the entire circles of radius m and M lie in the complement of $E_{\mathcal{C}_1}(h, g)$. Thus,

$$\begin{aligned} \sup_{|\lambda|=m} \left|\frac{\lambda - \beta}{\lambda - \alpha}\right| = |c(m)| + R(m) &= \frac{|\beta\overline{\alpha} - m^2| + m|\alpha - \beta|}{|\alpha|^2 - m^2} \\ &\ge \frac{|\beta||\alpha| - m^2 + m|\beta| - m|\alpha|}{|\alpha|^2 - m^2} = \frac{|\beta| - m}{|\alpha| - m}. \end{aligned}$$

[17] Indeed, the reader can easily check that $|\phi^{-1}(x + iy)| = r$ implies

$$x^2 + y^2 - 2\frac{\Re(\beta\overline{\alpha} - r^2)x + \Im(\beta\overline{\alpha} - r^2)y}{|\alpha|^2 - r^2} + \frac{|\beta|^2 - r^2}{|\alpha|^2 - r^2} = 0.$$

Reversing the roles of α and β yields the analogous estimate on $C(M)$,

$$\sup_{|\mu|=M}\left|\frac{\mu-\alpha}{\mu-\beta}\right| = \frac{|M^2-\alpha\overline{\beta}|+M|\beta-\alpha|}{M^2-|\beta|^2} \geq \frac{M-|\alpha|}{M-|\beta|}.$$

Substituting in (5.17) yields

$$e^{\Delta} \geq \sup_{\alpha,\beta\in E(Lh,Lg)} \frac{(M-|\alpha|)(|\frac{\beta}{\alpha}||\alpha|-m)}{(M-|\frac{\beta}{\alpha}||\alpha|)(|\alpha|-m)}.$$

Remark that the above supremum occurs when $|\alpha|<|\beta|$, which we will assume going forward. With this condition, the ratio $\frac{(M-|\alpha|)(|\beta|-m)}{(M-|\beta|)(|\alpha|-m)} \geq 1$. We can then study the function $\phi(t) = \frac{(M-t)(|\frac{\beta}{\alpha}|t-m)}{(M-|\frac{\beta}{\alpha}|t)(t-m)}$ for $m<t<M\frac{|\alpha|}{|\beta|}$. This function is positive, has vertical asymptotes at the endpoints of its domain[18] and has a minimum at $t_0=\sqrt{\frac{mM|\alpha|}{|\beta|}} \in (m, M\frac{|\alpha|}{|\beta|})$. Hence,

$$e^{\Delta} \geq \sup_{\alpha,\beta\in E_{\mathcal{C}_2}(Lh,Lg)} \phi(t_0) = \sup_{\alpha,\beta\in E_{\mathcal{C}_2}(Lh,Lg)} \frac{\left(\sqrt{\frac{M|\beta|}{m|\alpha|}}-1\right)^2}{\left(\sqrt{\frac{M}{m}}-\sqrt{\frac{|\beta|}{|\alpha|}}\right)^2}.$$

Since the above right-hand side is increasing with respect to $\frac{|\beta|}{|\alpha|}$ for $0<\frac{|\beta|}{|\alpha|}<\frac{M}{m}$, and, by definition, $\sup_{\alpha,\beta\in E_{\mathcal{C}_2}(Lh,Lg)}\frac{|\beta|}{|\alpha|} = e^{\delta_{\mathcal{C}_2(Lg,Lh)}} =: e^{\theta_2}$, we have, setting $e^{\theta_1} := \frac{M}{m} = e^{\delta_{\mathcal{C}_1}(h,g)}$, and recalling that by construction, $\theta_1>\theta_2$,

$$\begin{aligned} e^{\frac{\Delta}{2}} &\geq \frac{e^{\frac{\theta_2+\theta_1}{2}}-1}{e^{\frac{\theta_1}{2}}-e^{\frac{\theta_2}{2}}} = \frac{e^{\frac{\theta_2+\theta_1}{4}}-e^{-\frac{\theta_2+\theta_1}{4}}}{e^{\frac{\theta_1-\theta_2}{4}}-e^{-\frac{\theta_1-\theta_2}{4}}} = \frac{\sinh\left(\frac{\theta_2+\theta_1}{4}\right)}{\sinh\left(\frac{\theta_1-\theta_2}{4}\right)} \\ &= \frac{\sinh\left(\frac{\theta_1}{4}\right)\cosh\left(\frac{\theta_2}{4}\right)+\cosh\left(\frac{\theta_1}{4}\right)\sinh\left(\frac{\theta_2}{4}\right)}{\sinh\left(\frac{\theta_1}{4}\right)\cosh\left(\frac{\theta_2}{4}\right)-\cosh\left(\frac{\theta_1}{4}\right)\sinh\left(\frac{\theta_2}{4}\right)} = \frac{\tanh\left(\frac{\theta_1}{4}\right)+\tanh\left(\frac{\theta_2}{4}\right)}{\tanh\left(\frac{\theta_1}{4}\right)-\tanh\left(\frac{\theta_2}{4}\right)}. \end{aligned}$$

The above implies

$$\tanh\left(\frac{\theta_2}{4}\right) \leq \frac{e^{\frac{\Delta}{2}}-1}{1+e^{\frac{\Delta}{2}}}\tanh\left(\frac{\theta_1}{4}\right) = \tanh\left(\frac{\Delta}{4}\right)\tanh\left(\frac{\theta_1}{4}\right) \leq \tanh\left(\frac{\theta_1\tanh\left(\frac{\Delta}{4}\right)}{4}\right)$$

since, for $\gamma\in(0,1)$ and $x\geq 0$, $\tanh(\gamma x)\geq\gamma\tanh(x)$. Because the hyperbolic tangent is increasing, we finally have $\theta_2\leq\tanh\left(\frac{\Delta}{4}\right)\theta_1$, and the theorem follows. □

Next, we provide a simplified but less optimal version of [18, Theorem 4.5], which suffices for our needs.

For $i\in\{1,2\}$, let $\mathcal{B}_{i,\mathbb{R}}$, $\mathcal{B}_{i,\mathbb{C}}$, $\mathcal{S}_i$, $\mathbb{e}_i$ and $\mathbb{m}_i$ be as in the introduction of the appendix. We assume (5.4) and (5.6) hold for both $i=1$ and $i=2$. We start by relating the real diameter with the complex one.

Lemma 5.16. *Let $A\in L(\mathcal{B}_{1,\mathbb{R}},\mathcal{B}_{2,\mathbb{R}})$ such that $A(\mathcal{C}_{1,\mathbb{R}})\subset\mathcal{C}_{2,\mathbb{R}}$ with*

$$\operatorname{diam}_{d_H}(A(\mathcal{C}_{1,\mathbb{R}})) = \Delta_{\mathbb{R}} < \infty$$

then[19]

$$\Delta_{\mathbb{C}} := \operatorname{diam}_{\delta_{\mathcal{C}_{2,\mathbb{C}}}}(A(\mathcal{C}_{1,\mathbb{C}})) \leq 8\Delta_{\mathbb{R}} + 2\ln[3\sqrt{2}\kappa^{-2}].$$

Proof. By (5.10) we have, for all $h\in\mathcal{C}_{1,\mathbb{R}}$ and $\ell\in\mathcal{C}'_{2,\mathbb{R}}$

$$e^{\Delta_{\mathbb{R}}} \geq \frac{\ell(Ah)\mathbb{m}_2(\mathbb{e}_2)}{\mathbb{m}_2(Ah)\ell(\mathbb{e}_2)} \geq e^{-\Delta_{\mathbb{R}}}$$

which, recalling (5.6) and the normalization $\mathbb{m}_i(\mathbb{e}_i)=1$, implies

$$\ell(\mathbb{e}_2)\|Ah\|e^{\Delta_{\mathbb{R}}} \geq \ell(Ah) \geq \ell(\mathbb{e}_2)\kappa\|Ah\|e^{-\Delta_{\mathbb{R}}}. \tag{5.18}$$

[18] Note that $t=|\alpha|$ is indeed in this domain since, recalling also (5.15), $m<|\alpha|$ by definition and $M|\alpha|/|\beta|>|\alpha|$ since $M>|\beta|$.

[19] A more sophisticated argument would yield a better constant, independent on κ, see [18], but this estimate is more than sufficient for our needs and in the spirit to present the argument in its simplest form.

Accordingly, for all $h = z(x+iy)$, $z \in \mathbb{C} \setminus \{0\}$, $x, y \in \mathcal{C}_{1,\mathbb{R}}$, and $\ell, p \in \mathcal{C}'_{2,\mathbb{R}}$

$$\Re(\ell(Ah)\overline{p(Ah)}) = |z|^2 \left[\ell(Ax)p(Ax) + \ell(Ay)p(Ay)\right] \geq 0.$$

It follows from Lemma 5.4(a) that $A(\mathcal{C}_{1,\mathbb{C}}) \subset \mathcal{C}_{2,\mathbb{C}}$, and, for all $\tilde{\ell}, \tilde{p} \in \hat{\mathcal{C}}'_{2,\mathbb{C}}$ and all $z \in \mathbb{C}$ and $x, y \in \mathcal{C}_{1,\mathbb{R}}$, setting $\ell = |\tilde{\ell}(\mathbb{e}_2)|^{-1}\tilde{\ell} = \ell_1 + i\ell_2$ and $p = |\tilde{p}(\mathbb{e}_2)|^{-1}\tilde{p} = p_1 + ip_2$

$$\left|\frac{z\tilde{\ell}(A(x+iy))\tilde{p}(\mathbb{e}_2)}{\tilde{\ell}(\mathbb{e}_2)z\tilde{p}(A(x+iy))}\right|^2 \leq \frac{\left[\ell_1(Ax) - \ell_2(Ay)\right]^2 + \left[\ell_2(Ax) + \ell_1(Ay)\right]^2}{\left[p_1(Ax) - p_2(Ay)\right]^2 + \left[p_2(Ax) + p_1(Ay)\right]^2}. \tag{5.19}$$

We use (5.18) to bound the numerator,

$$\begin{aligned}\left[\ell_1(Ax) - \ell_2(Ay)\right]^2 + \left[\ell_2(Ax) + \ell_1(Ay)\right]^2 &\leq 2\big(\ell_1(Ax)^2 + \ell_2(Ay)^2\big) + 2\big(\ell_2(Ax)^2 + \ell_1(Ay)^2\big) \\ &\leq 2e^{2\Delta_{\mathbb{R}}}\left(\|Ax\|^2 + \|Ay\|^2\right),\end{aligned}$$

where we have used the normalization $\ell_1(\mathbb{e}_2)^2 + \ell_2(\mathbb{e}_2)^2 = 1$.

To derive a lower bound on the denominator in (5.19), we argue by cases. Fix $b \geq 2$ and $c > 0$, $c < \frac{e^{-2\Delta_{\mathbb{R}}}\kappa}{\sqrt{e^{-4\Delta_{\mathbb{R}}}\kappa^2 + b^2}} < 1/\sqrt{5}$.

Case 1: $p_1(\mathbb{e}_2)^2, p_2(\mathbb{e}_2)^2 \geq c^2$. Note that, by definition, there exists $\sigma_i \in \{-1, +1\}$ such that $\sigma_i p_i \in \mathcal{C}'_{2,\mathbb{R}}$. If $\sigma_1\sigma_2 = 1$, then, recalling (5.18),

$$\begin{aligned}&\left[p_1(Ax) - p_2(Ay)\right]^2 + \left[p_2(Ax) + p_1(Ay)\right]^2 \geq \left[p_2(Ax) + p_1(Ay)\right]^2 \\ &\geq p_2(Ax)^2 + p_1(Ay)^2 \geq \kappa^2 c^2(\|Ax\|^2 + \|Ay\|^2)e^{-2\Delta_{\mathbb{R}}}.\end{aligned}$$

while, If $\sigma_1\sigma_2 = -1$, then similarly,

$$\begin{aligned}&\left[p_1(Ax) - p_2(Ay)\right]^2 + \left[p_2(Ax) + p_1(Ay)\right]^2 \geq \left[p_1(Ax) - p_2(Ay)\right]^2 \\ &\geq p_1(Ax)^2 + p_2(Ay)^2 \geq \kappa^2 c^2(\|Ax\|^2 + \|Ay\|^2)e^{-2\Delta_{\mathbb{R}}}.\end{aligned}$$

Case 2: Either $p_1(\mathbb{e}_2)^2 < c^2$ *or* $p_2(\mathbb{e}_2)^2 < c^2$. We shall assume $p_1(\mathbb{e}_2)^2 < c^2$, the argument for the other alternative being analogous. In this case then, $p_2(\mathbb{e}_2)^2 > 1 - c^2$ due to the normalization of p.

We further refine into subcases.

Subcase A: $\|Ay\|b^{-1} \leq \|Ax\| \leq b\|Ay\|$. Then by choice of c and using (5.18),

$$\begin{aligned}\left[p_1(Ax) - p_2(Ay)\right]^2 &\geq \left[\sqrt{1-c^2}\kappa e^{-\Delta_{\mathbb{R}}}\|Ay\| - ce^{\Delta_{\mathbb{R}}}b\|Ay\|\right]^2 \\ &= \|Ay\|^2\left(\sqrt{1-c^2}\kappa e^{-\Delta_{\mathbb{R}}} - bce^{\Delta_{\mathbb{R}}}\right)^2.\end{aligned}$$

Adding the analogous estimate for the second term yields the lower bound,

$$\left[p_1(Ax) - p_2(Ay)\right]^2 + \left[p_2(Ax) + p_1(Ay)\right]^2 \geq \left(\|Ax\|^2 + \|Ay\|^2\right)\left(\sqrt{1-c^2}\kappa e^{-\Delta_{\mathbb{R}}} - bce^{\Delta_{\mathbb{R}}}\right)^2.$$

Subcase B: Either $\|Ay\| < b^{-1}\|A_x\|$ *or* $\|Ax\| < b^{-1}\|Ay\|$. Again, it is sufficient to argue only one of the alternatives. Suppose that $\|Ay\| < b^{-1}\|Ax\|$. Then,

$$\begin{aligned}&\left[p_1(Ax) - p_2(Ay)\right]^2 + \left[p_2(Ax) + p_1(Ay)\right]^2 \geq \left[p_2(Ax) + p_1(Ay)\right]^2 \\ &\geq \|Ax\|^2\left(\sqrt{1-c^2}\kappa e^{-\Delta_{\mathbb{R}}} - b^{-1}ce^{\Delta_{\mathbb{R}}}\right)^2 \\ &\geq \frac{b^2}{b^2+1}\left(\|Ax\|^2 + \|Ay\|^2\right)\left(\sqrt{1-c^2}\kappa e^{-\Delta_{\mathbb{R}}} - b^{-1}ce^{\Delta_{\mathbb{R}}}\right)^2.\end{aligned}$$

We proceed to optimize our choice of b and c to obtain a common lower bound from Cases 1 and 2. Let us choose $b = \sqrt{2}$. Then set

$$c = \frac{\kappa e^{-2\Delta_{\mathbb{R}}}}{\sqrt{\kappa^2 e^{-4\Delta_{\mathbb{R}}} + 4b^2}} \implies \sqrt{c^{-2} - 1} = 2b\kappa^{-1}e^{2\Delta_{\mathbb{R}}}.$$

This implies that the lower bound in Subcase A is at least,

$$\left(\|Ax\|^2 + \|Ay\|^2\right)\frac{\kappa^2 c^2}{4}\left(c^{-2} - 1\right)e^{-2\Delta_{\mathbb{R}}}. \tag{5.20}$$

Moreover, the lower bound in Subcase B is at least as large since with $b = \sqrt{2}$,

$$\frac{b}{\sqrt{b^2+1}}\left(\sqrt{c^{-2}-1} - \kappa^{-1}b^{-1}e^{2\Delta_{\mathbb{R}}}\right) \geq \sqrt{c^{-2}-1}\left(1 - \frac{1}{2b^2}\right)\frac{b}{\sqrt{b^2+1}} > \frac{1}{2}\sqrt{c^{-2}-1}.$$

It remains to compare the lower bounds from Subcase A and Case 1. Due to (5.20), the lower bound from Case 1 is smaller if

$$c^2 < c^2\frac{1}{4}(c^{-2}-1) \implies c < \frac{1}{\sqrt{5}},$$

which is true given our choice of c whenever $b \geq 1$.

Finally, putting together this common lower bound with our upper bound for the numerator in (5.19), yields,

$$\left|\frac{z\tilde{\ell}(A(x+iy))\tilde{p}(\mathbb{e}_2)}{\tilde{\ell}(\mathbb{e}_2)z\tilde{p}(A(x+iy))}\right| \leq \sqrt{2}\kappa^{-1}c^{-1}e^{2\Delta_{\mathbb{R}}} \leq 3\sqrt{2}\kappa^{-2}e^{4\Delta_{\mathbb{R}}}. \tag{5.21}$$

Since

$$\sup_{h\in\mathcal{C}_{1,\mathbb{C}}} \delta_{\mathcal{C}_{2,\mathbb{C}}}(A(h),\mathbb{e}_2) \leq \operatorname{diam}_{\delta_{\mathcal{C}_{2,\mathbb{C}}}}(A(\mathcal{C}_{1,\mathbb{C}})) \leq 2\sup_{h\in\mathcal{C}_{1,\mathbb{C}}} \delta_{\mathcal{C}_{2,\mathbb{C}}}(Ah,\mathbb{e}_2), \tag{5.22}$$

we have the lemma. □

We finally provide the key result that is needed in the applications to verify the hypothesis on the finiteness of the complex diameter in Theorem 5.15.

Theorem 5.17. *Let, for $i \in \{1,2\}$, $\mathcal{B}_{i,\mathbb{R}}$, $\mathcal{B}_{i,\mathbb{C}}$, $\mathcal{S}_i$, $\mathbb{e}_i$ and $\mathbb{m}_i$ be as in the introduction to this section (in particular, the $\mathbb{e}_i$ and $\mathbb{m}_i$ satisfy equations (5.3) and (5.6), respectively). Let $\mathcal{L} \in L(\mathcal{B}_{1,\mathbb{R}},\mathcal{B}_{2,\mathbb{R}})$ and $\mathcal{L}_{\mathbb{C}} \in L(\mathcal{B}_{1,\mathbb{C}},\mathcal{B}_{2,\mathbb{C}})$. Assume that $\mathcal{L}(\mathcal{C}_{1,\mathbb{R}}) \subset \mathcal{C}_{2,\mathbb{R}}$ and $\operatorname{diam}_H(\mathcal{L}(\mathcal{C}_{1,\mathbb{R}})) := \Delta_{\mathbb{R}} < \infty$. If there exists $\varepsilon \in (0, \frac{\kappa^2}{12\sqrt{2}}e^{-2\Delta_{\mathbb{R}}})$, such that for all $\ell \in \mathcal{S}_2$ and all $h \in \mathcal{C}_{1,\mathbb{R}}$,*

$$|\ell(\mathcal{L}_{\mathbb{C}}h) - \ell(\mathcal{L}h)| \leq \varepsilon\ell(\mathcal{L}h). \tag{5.23}$$

Then $\mathcal{L}_{\mathbb{C}}(\mathcal{C}_{1,\mathbb{C}}) \subset \mathcal{C}_{2,\mathbb{C}}$, and we have

$$\operatorname{diam}_{\delta_{\mathcal{C}_{2,\mathbb{C}}}}(\mathcal{L}_{\mathbb{C}}(\mathcal{C}_{1,\mathbb{C}})) \leq 8\Delta_{\mathbb{R}} + 2\ln[3\sqrt{2}\kappa^{-2}] + \tfrac{\sqrt{2}}{3}\kappa^2 e^{-2\Delta_{\mathbb{R}}}.$$

Proof. For all $x, y \in \mathcal{C}_{1,\mathbb{R}}$ and $\ell \in \mathcal{S}_2$ we have using (5.23),

$$\begin{aligned}|\ell(\mathcal{L}_{\mathbb{C}}(x+iy)) - \ell(\mathcal{L}(x+iy))| &\leq |\ell(\mathcal{L}_{\mathbb{C}}x) - \ell(\mathcal{L}x)| + |\ell(\mathcal{L}_{\mathbb{C}}y) - \ell(\mathcal{L}y)| \\ &\leq \varepsilon[\ell(\mathcal{L}x) + \ell(\mathcal{L}y)] \leq \varepsilon\sqrt{2}|\ell(\mathcal{L}(x+iy))|.\end{aligned} \tag{5.24}$$

It follows that for all $h \in \mathcal{C}_{1,\mathbb{C}}$ and $\ell, p \in \mathring{\mathcal{C}}'_{2,\mathbb{R}}$, provided $\varepsilon \leq \frac{1}{\sqrt{2}}$,

$$\begin{aligned}\Re(\ell(\mathcal{L}_{\mathbb{C}}h)\overline{p(\mathcal{L}_{\mathbb{C}}h)}) &= \Re\Big(\ell(\mathcal{L}h)\overline{p(\mathcal{L}h)} + \ell((\mathcal{L}_{\mathbb{C}}-\mathcal{L})h)\overline{p((\mathcal{L}_{\mathbb{C}}-\mathcal{L})h)} \\ &\quad + \ell((\mathcal{L}_{\mathbb{C}}-\mathcal{L})h)\overline{p(\mathcal{L}h)} + \ell(\mathcal{L}h)\overline{p((\mathcal{L}_{\mathbb{C}}-\mathcal{L})h)}\Big) \\ &\geq \Re\left(\ell(\mathcal{L}h)\overline{p(\mathcal{L}h)}\right) - 3\sqrt{2}\varepsilon|\ell(\mathcal{L}h)||p(\mathcal{L}h)| \\ &\geq \Re\left(\ell(\mathcal{L}h)\overline{p(\mathcal{L}h)}\right) - 6\sqrt{2}\varepsilon\ell(\mathbb{e}_2)p(\mathbb{e}_2)\|\mathcal{L}h\|\|\mathcal{L}h\|\end{aligned}$$

where, in the first inequality we have used (5.24) and in the second inequality we have used Lemma 5.4(c) and Lemma 5.5. Since we can assume $h = x + iy$ with $x, y \in \mathcal{C}_{1,\mathbb{R}}$, recalling (5.18) we have

$$\begin{aligned}\Re\left(\ell(\mathcal{L}h)\overline{p(\mathcal{L}h)}\right) &= \ell(\mathcal{L}(x))p(\mathcal{L}x) + \ell(\mathcal{L}y)p(\mathcal{L}y) \\ &\geq \ell(\mathbb{e}_2)p(\mathbb{e}_2)\kappa^2 e^{-2\Delta_{\mathbb{R}}}\left(\|\mathcal{L}x\|^2 + \|\mathcal{L}y\|^2\right) \\ &\geq 2^{-1}\ell(\mathbb{e}_2)p(\mathbb{e}_2)\kappa^2 e^{-2\Delta_{\mathbb{R}}}\|\mathcal{L}h\|^2,\end{aligned}$$

where we have used (A.2). Hence, $\Re(\ell(\mathcal{L}_{\mathbb{C}}(h))\overline{p(\mathcal{L}_{\mathbb{C}}(h))}) > 0$, provided $\varepsilon < \frac{\kappa^2}{12\sqrt{2}}e^{-2\Delta_{\mathbb{R}}}$.

Note that if $\ell \in \mathcal{C}'_{2,\mathbb{R}}$, then $\ell + \alpha \mathrm{m} \in \mathring{\mathcal{C}}'_{2,\mathbb{R}}$ for each $\alpha > 0$. Thus the above implies $\Re(\ell(\mathcal{L}_{\mathbb{C}}(h))\overline{p(\mathcal{L}_{\mathbb{C}}(h))}) \geq 0$ for all $\ell, p \in \mathcal{C}'_{2,\mathbb{R}}$. Applying Lemma 5.4(a), we have $\mathcal{L}_{\mathbb{C}}(\mathcal{C}_{1,\mathbb{C}}) \subset \mathcal{C}_{2,\mathbb{C}}$.

Finally, for all $\ell, p \in \mathcal{C}'_{2,\mathbb{R}}$ and $h \in \mathcal{C}_{1,\mathbb{C}}$, it follows by (5.24) that,

$$\begin{aligned}\left|\frac{\ell(\mathcal{L}_{\mathbb{C}}h)p(\mathcal{L}h)}{\ell(\mathcal{L}h)p(\mathcal{L}_{\mathbb{C}}h)}\right| &\leq \frac{|\ell(\mathcal{L}h)p(\mathcal{L}h)| + |\ell([\mathcal{L}_{\mathbb{C}} - \mathcal{L}]h)p(\mathcal{L}h)|}{|\ell(\mathcal{L}h)p(\mathcal{L}h)| - |\ell(\mathcal{L}h)p([\mathcal{L}_{\mathbb{C}} - \mathcal{L}]h)|} \\ &\leq \frac{1+\sqrt{2}\varepsilon}{1-\sqrt{2}\varepsilon} \leq 1 + 4\varepsilon,\end{aligned}$$

since we assumed $\varepsilon < \frac{1}{12\sqrt{2}}$.

The above implies

$$\begin{aligned}\delta_{\mathcal{C}_{2,\mathbb{C}}}(\mathcal{L}_{\mathbb{C}}h, \mathbb{e}_2) &= \sup_{\ell,p} \ln \frac{|\ell(\mathcal{L}_{\mathbb{C}}h)p(\mathbb{e}_2)|}{|\ell(\mathbb{e}_2)p(\mathcal{L}_{\mathbb{C}}h)|} \\ &\leq \sup_{\ell,p} \ln \frac{|\ell(\mathcal{L}h)p(\mathbb{e}_2)|}{|\ell(\mathbb{e}_2)p(\mathcal{L}h)|} + \sup_{\ell,p} \ln \frac{|\ell(\mathcal{L}_{\mathbb{C}}h)p(\mathcal{L}h)|}{|\ell(\mathcal{L}h)p(\mathcal{L}_{\mathbb{C}}h)|} \\ &\leq \delta_{\mathcal{C}_{2,\mathbb{C}}}(\mathcal{L}h, \mathbb{e}_2) + 4\varepsilon \leq 4\Delta_{\mathbb{R}} + \ln[3\sqrt{2}\kappa^{-2}] + 4\varepsilon \\ &\leq 4\Delta_{\mathbb{R}} + \ln[3\sqrt{2}\kappa^{-2}] + \tfrac{1}{3\sqrt{2}}\kappa^2 e^{-2\Delta_{\mathbb{R}}},\end{aligned} \tag{5.25}$$

where, in the next to last inequality, we have applied (5.21). We conclude using (5.22). □

We conclude with a comment on (5.23) that may facilitate checking it.

For each $h \in \mathcal{C}_{1,\mathbb{R}}$ let $a(h) = \Re(\mathcal{L}_{\mathbb{C}}h), b(h) = \Im(\mathcal{L}_{\mathbb{C}}h)$. By defintion a, b are real linear operators on $\mathcal{B}_{2,\mathbb{R}}$ and $\mathcal{L}_{\mathbb{C}}h = a(h) + ib(h)$.

Lemma 5.18. *If, for each $h \in \mathcal{C}_{1,\mathbb{R}}$,*

$$\begin{aligned}&\varepsilon\mathcal{L}h \pm 2\left[\mathcal{L}h - a(h)\right] \in \mathcal{C}_{2,\mathbb{R}} \\ &\varepsilon\mathcal{L}h \pm 2b(h) \in \mathcal{C}_{2,\mathbb{R}}\end{aligned}$$

then condition (5.23) is satisfied.

Proof. By hypothesis, for each $\ell \in \mathcal{C}'_{2,\mathbb{R}}$ we have

$$\begin{aligned}&\varepsilon\ell(\mathcal{L}h) \geq 2\left|\ell(\mathcal{L}h - a(h))\right| \\ &\varepsilon\ell(\mathcal{L}h) \geq 2\left|\ell(b(h))\right|.\end{aligned}$$

Accordingly,

$$\varepsilon\ell(\mathcal{L}h) \geq (|\ell(\mathcal{L}h - a(h))| + |\ell(-ib(h))|) \geq |\ell(\mathcal{L}h - \mathcal{L}_{\mathbb{C}}(h))|.$$

□

5.3. **Cone Contraction Implies Loss of Memory.**

In this section, we show that cone contraction implies loss of memory. This provides a tool to easily verify, in some cases, condition (O-3).

Consider complex Banach spaces $\mathbb{B}_k$, cones $\mathcal{C}_k = \mathcal{C}_{\mathbb{C},k} \subset \mathbb{B}_k$, vectors $\mathbb{e}_k \in \mathbb{B}_k$ as described in Sections 2 and 5.2. Suppose there exist functionals $\mathrm{m}_k \in \mathbb{B}'_k$ satisfying (5.6) with

a) $\inf_{k\geq 0} \kappa_k = \bar{\kappa} > 0$.

In addition, there are operators $\mathbb{L}_k : \mathbb{B}_k \to \mathbb{B}_{k+1}$ satisfying the following:

b) there exists $K > 0$ such that, for all $k, j \in \mathbb{N}$, we have

$$|\mathrm{m}_k\left(\mathbb{L}_k \cdots \mathbb{L}_j \mathbb{e}_j\right)| \leq K;$$

c) the operators $\mathbb{L}_k$ satisfy $\mathbb{L}_k(\mathcal{C}_k) \subset \mathcal{C}_{k+1}$, and the diameter in the complex projective metric of $\mathbb{L}_k(\mathcal{C}_k)$ in $\mathcal{C}_{k+1}$ is uniformly bounded by $\Delta < \infty$.

Define $\mathbb{L}_{k,j} := \mathbb{L}_k \cdots \mathbb{L}_j$ and

$$\begin{aligned} \boldsymbol{\alpha}_{k,j} &= \mathbb{m}_k(\mathbb{L}_{k-1,j}\mathbb{e}_j) \\ \boldsymbol{h}_{k,j} &= \frac{\mathbb{L}_{k-1,j}\mathbb{e}_j}{\mathbb{m}_k(\mathbb{L}_{k-1,j}\mathbb{e}_j)} \\ \boldsymbol{\ell}_{k,j}(h) &= \frac{\mathbb{m}_k\left(\mathbb{L}_{k-1,j}h\right)}{\mathbb{m}_k(\mathbb{L}_{k-1,j}\mathbb{e}_j)}. \end{aligned} \tag{5.26}$$

The next lemma will be helpful to check condition (O-2).

Lemma 5.19 (Uniform Boundedness)**.** *For each $h \in \mathbb{B}_j$, we have*

$$\begin{aligned} &|\boldsymbol{\alpha}_{k,j}| \le K, \quad \|\boldsymbol{h}_{k,j}\|_k \le \frac{\sqrt{2}}{\bar{\kappa}}, \quad \|\boldsymbol{\ell}_{k,j}\|'_j \le \sqrt{2} \\ &\|\mathbb{L}_{k,j}h\|_k \le \frac{2K}{\bar{\kappa}}\|h\|_j. \end{aligned}$$

Proof. By assumption $|\boldsymbol{\alpha}_{k,j}| \le K$. Since, by assumption, $\mathbb{m}_k$ satisfies (5.11), by Definition 5.3, $\mathbb{m} \in \mathcal{C}'_{\mathbb{C}}$ so that, for all $h \in \mathcal{C}_{\mathbb{C}}$,

$$|\mathbb{m}_k(\mathbb{L}_{k-1,j}h)| \ge \frac{\bar{\kappa}}{\sqrt{2}}\|\mathbb{L}_{k-1,j}h\|_k > 0. \tag{5.27}$$

Hence, $\|\boldsymbol{h}_{k,j}\|_k \le \frac{\sqrt{2}}{\bar{\kappa}}$. In addition, $\boldsymbol{\ell}_{k,j}(h) \ne 0$ for all $h \in \mathcal{C}_{\mathbb{C}}$. Accordingly, $\boldsymbol{\ell}_{k,j} \in \mathcal{C}'_{\mathbb{C}}$ and so by Lemma 5.5 $\|\boldsymbol{\ell}_{k,j}\|' \le \sqrt{2}$. Hence, recalling (5.11)

$$\|\mathbb{L}_{k-1,j}h\|_k \le \frac{\sqrt{2}}{\bar{\kappa}}\left|\mathbb{m}_k\left(\mathbb{L}_{k-1,j}h\right)\right| = \frac{\sqrt{2}}{\bar{\kappa}}\left|\boldsymbol{\ell}_{k,j}\left(h\right)\mathbb{m}_k\left(\mathbb{L}_{k-1,j}\mathbb{e}_j\right)\right| \le \frac{2K}{\bar{\kappa}}\|h\|_j.$$

□

The following lemma will be helpful to check condition (O-3).

Lemma 5.20 (Loss of Memory)**.** *Let $\theta = \tanh(\Delta/4)$, then for each $k, j \in \mathbb{N}$, $h \in \mathbb{B}_j$,*

$$\|\mathbb{L}_{k-1,j}h - \boldsymbol{\alpha}_{k,j}\boldsymbol{\ell}_{k,j}(h)\boldsymbol{h}_{k,j}\|_j \le \frac{2(1+\sqrt{2})}{\bar{\kappa}\theta^2}\Delta\theta^{k-j}|\boldsymbol{\alpha}_{k,j}|\,\|h\|_j.$$

Proof. Applying Lemma 5.11 and Theorem 5.15, we have for all $h, g \in \mathcal{C}_j$,

$$\begin{aligned} &\left\|\frac{\mathbb{L}_{k,j}h}{\mathbb{m}_{k+1}(\mathbb{L}_{k,j}h)} - \frac{\mathbb{L}_{k,j}g}{\mathbb{m}_{k+1}(\mathbb{L}_{k,j}g)}\right\|_{k+1} \le \frac{\sqrt{2}}{\bar{\kappa}}\delta_{\mathbb{C},k}(\mathbb{L}_{k,j}h, \mathbb{L}_{k,j}g) \\ &\le \frac{\sqrt{2}}{\bar{\kappa}}(\tanh(\Delta/4))^{k-j-1}\delta_{\mathbb{C},j+1}(\mathbb{L}_j h, \mathbb{L}_j g) \\ &= \frac{\sqrt{2}}{\bar{\kappa}}\theta^{k-j-1}\delta_{\mathbb{C},j+1}(\mathbb{L}_j h, \mathbb{L}_j g) \le \frac{\sqrt{2}}{\bar{\kappa}}\theta^{k-j-1}\Delta. \end{aligned}$$

By definition $\boldsymbol{\ell}_{k,j}(\mathbb{e}_j) = 1$ and $\mathbb{m}_k(\boldsymbol{h}_{k,j}) = 1$. Moreover, by Lemma 5.19 and Lemma 5.5, $|\boldsymbol{\alpha}_{k,j}| \le K$, $\|\boldsymbol{h}_{k,j}\|_k \le \sqrt{2}/\bar{\kappa}$ and $\|\boldsymbol{\ell}_{k,j}\|'_j \le \sqrt{2}$. Hence, recalling (5.26), for all $h \in \mathcal{C}_j$ we have

$$\begin{aligned} \|\mathbb{L}_{k-1,j}h - \boldsymbol{\alpha}_{k,j}\boldsymbol{\ell}_{k,j}(h)\boldsymbol{h}_{k,j}\|_k &= \|\mathbb{L}_{k-1,j}h - \mathbb{m}_k(\mathbb{L}_{k-1,j}h)\boldsymbol{h}_{k,j}\|_k \\ &\le \frac{\sqrt{2}}{\theta^2\bar{\kappa}}\theta^{k-j}\Delta\left|\mathbb{m}_k(\mathbb{L}_{k-1,j}h)\right| \\ &\le \frac{2}{\theta^2\bar{\kappa}}\theta^{k-j}\Delta|\boldsymbol{\alpha}_{k,j}|\|h\|_j. \end{aligned} \tag{5.28}$$

For $h \in \mathbb{B}_j$, define $\bar{h} := h + \sqrt{2}\|h\|_j\mathbb{e}_j$. As in Lemma 5.13, $\bar{h}, \|h\|_j\mathbb{e}_j \in \mathcal{C}_{\mathbb{C},j} \cup \{0\}$. Recalling $\boldsymbol{\ell}_{k,j}(\|h\|_j\mathbb{e}_j) = \|h\|_j$, as well as $\mathbb{m}_k(\mathbb{L}_{k-1,j}\bar{h}) = \mathbb{m}_k(\boldsymbol{\alpha}_{k,j}\boldsymbol{\ell}_{k,j}(\bar{h})\boldsymbol{h}_{k,j})$, we apply (5.28) to obtain,

$$\begin{aligned} \|\mathbb{L}_{k-1,j}h - \boldsymbol{\alpha}_{k,j}\boldsymbol{\ell}_{k,j}(h)\boldsymbol{h}_{k,j}\|_k &= \|\mathbb{L}_{k-1,j}\bar{h} - \boldsymbol{\alpha}_{k,j}\boldsymbol{\ell}_{k,j}(\bar{h})\boldsymbol{h}_{k,j}\|_k \\ &\le \|\bar{h}\|_j\frac{2}{\bar{\kappa}\theta^2}\Delta\theta^{k-j}|\boldsymbol{\alpha}|_{k,j} \le \|h\|_j\frac{2(1+\sqrt{2})}{\bar{\kappa}\theta^2}\Delta\theta^{k-j}|\boldsymbol{\alpha}_{k,j}|. \end{aligned}$$

□

5.4. **A Useful Lower Bound.** In this section, we show under the assumptions of Theorem 5.17, that the lower bound $|\boldsymbol{\ell}_{k,j}(\boldsymbol{h}_{j,l})| \geq K^{-1}$ holds uniformly in j,k,l, with $\boldsymbol{\ell}_{k,j}, \boldsymbol{h}_{j,l}$ defined as in (5.26). In addition to properties (a)-(c) of Section 5.3, we assume that there exist real operators $\mathcal{L}_k : \mathcal{B}_k \to \mathcal{B}_{k+1}$ satisfying

d) $\mathcal{L}_k(\mathcal{C}_{k,\mathbb{R}}) \subset \mathcal{C}_{k+1,\mathbb{R}}$ and $\operatorname{diam}_H(\mathcal{L}_k(\mathcal{C}_{k,\mathbb{R}})) \leq \Delta_{\mathbb{R}} < \infty$ for all $k \geq 1$;

e) for all $\ell \in \mathcal{S}_{k+1}$ and all $h \in \mathcal{C}_{k,\mathbb{R}}$, $|\ell(\mathbb{L}_k h) - \ell(\mathcal{L}_k h)| \leq \varepsilon \ell(\mathcal{L}_k h)$ for some $\varepsilon \in \big(0, \frac{\bar{\kappa}^2}{12\sqrt{2}} e^{-2\Delta_{\mathbb{R}}}\big)$.

Thus we are in the setting of Theorem 5.17.

In this setting, we prove the lemma,

Lemma 5.21. *Under the assumption (a)–(e), there exists $K > 0$ such that $\boldsymbol{\ell}_{k,j}(\boldsymbol{h}_{j,l}) \geq K^{-1}$ for all $0 \leq l \leq j \leq k$.*

Proof. By definition,

$$\boldsymbol{\ell}_{k,j}(h) = \frac{\mathbb{m}(\mathbb{L}_{k-1,j} h)}{\boldsymbol{\alpha}_{\ell,j}} .$$

Since $\mathbb{L}_{k,j}\mathcal{C}_{j,\mathbb{C}} \subset \mathcal{C}_{k,\mathbb{C}}$ and invoking (5.11) and property (a) of Section 5.3, we have $\boldsymbol{\ell}_{k,j}(h) \neq 0$ for all $h \in \mathcal{C}_{j,\mathbb{C}}$. Thus $\boldsymbol{\ell}_{k,j} \in \mathcal{C}'_{j,\mathbb{C}}$ and since $\boldsymbol{h}_{j,l} \in \mathcal{C}_{j,\mathbb{C}}$, we may apply Lemma 5.14 to obtain,

$$|\boldsymbol{\ell}_{k,j}(\boldsymbol{h}_{j,l})| \geq e^{-\delta_{\mathcal{C}_{j,\mathbb{C}}}(\boldsymbol{h}_{j,l}, \mathbb{e}_j)} \tfrac{\bar{\kappa}}{\sqrt{2}} \|\boldsymbol{h}_{j,l}\|_j ,$$

where we have used $\boldsymbol{\ell}_{k,j}(\mathbb{e}_j) = 1$. Recalling (5.26) and using Lemma 5.5, and equation (5.5), we have

$$1 = \mathbb{m}_j(\boldsymbol{h}_{j,l}) \leq \|\mathbb{m}_j\|'_j \|\boldsymbol{h}_{j,l}\|_j \leq \sqrt{2}\|\boldsymbol{h}_{j,l}\|_j.$$

It remains to show that $\delta_{\mathcal{C}_{j,\mathbb{C}}}(\boldsymbol{h}_{j,l}, \mathbb{e}_j) \leq C$, where $C < \infty$ is independent of j and l. This follows from the cone contraction provided for each $\mathbb{L}_i$. In particular, remark that $\mathbb{L}_{j-1,l}\mathcal{C}_{l,\mathbb{C}} \subset \mathbb{L}_{j-1}\mathcal{C}_{j-1,\mathbb{C}}$, so that $\boldsymbol{h}_{j,l} \in \mathbb{L}_{j-1}\mathcal{C}_{j-1,\mathbb{C}}$. Thus it suffices to show that $\delta_{j,\mathcal{C}}(h, \mathbb{e}_j) \leq C$ for each $h \in \mathbb{L}_{j-1}\mathcal{C}_{j-1,\mathbb{C}}$ for some C independent of j.

To see this, apply Theorem 5.17, and in particular (5.25), with $\mathcal{L}_{\mathbb{C}} = \mathbb{L}_{j-1}$ and $\mathcal{L} = \mathcal{L}_{j-1}$ to obtain for each $h \in \mathcal{C}_{j-1,\mathbb{C}}$,

$$\delta_{\mathcal{C}_{j,\mathbb{C}}}(\mathbb{L}_{j-1}h, \mathbb{e}_j) \leq \delta_{\mathcal{C}_{j,\mathbb{C}}}(\mathcal{L}_{j-1}h, \mathbb{e}_j) + 4\varepsilon ,$$

with $\varepsilon = \frac{\bar{\kappa}^2}{12\sqrt{2}} e^{-2\Delta_{\mathbb{R}}}$, applying properties (d) and (e). Finally, applying Lemma 5.16 and in particular (5.21) with $A = \mathcal{L}_{j-1}$ yields,

$$\delta_{\mathcal{C}_{j,\mathbb{C}}}(\mathbb{L}_{j-1}h, \mathbb{e}_j) \leq 4\Delta_{\mathbb{R}} + \ln\big(3\sqrt{2}\bar{\kappa}^{-2}\big) + 4\varepsilon ,$$

completing the proof of the claim. □

6. Application: Smooth Expanding Maps

6.1. Sequential Expanding Maps.

We consider the setting illustrated in Section 2.1 and prove Theorem 2.1. To this end, we first check the conditions of Theorem 2.7. Theorem 2.1 will then follow trivially as explained in Section 6.5.

6.1.1. *Verifying Conditions (C-1)-(C-4).*

Let M be a smooth, connected d-dimensional Riemannian manifold. Without loss of generality, we can rescale the distance d so that the diameter of M is one and the Riemannian volume so that the volume of M is also one. We set $M_k = M$ and $\mu_k =$ Lebesgue. First of all, for $a > 0$, we consider the cones $C_k = \mathcal{C}_a$ for all $k \in \mathbb{N}$,

$$\mathcal{C}_a = \big\{h \in C^1(M, \mathbb{R}) \ : \ \|\nabla h(x)\| \leq a h(x) \ \forall x \in M\big\} \setminus \{0\}.$$

Note that (C-1) is trivially satisfied. Then setting $\mathbb{e}(x) = 1 \in \mathcal{C}_a$, if $h \in C^1$, then $(a^{-1}\|\nabla h\|_\infty + \|h\|_\infty)\mathbb{e} + h \in \mathcal{C}_a$; that is, (C-3) holds true. Indeed, it follows immediately from (2.5) that for $h \in C^1(M, \mathbb{C})$,

$$\|g\| := \|g\|_k = \sup_{x \in M} \{a^{-1}\|\nabla g(x)\| + |g(x)|\},$$

thus $\mathcal{B}_k = C^1(M, \mathbb{R})$ and $\mathbb{B}_k = C^1(M, \mathbb{C})$.

Next, suppose $g \in \mathcal{C}_a$ and $x_0 \in M$ is such that[20] $g(x_0) = \int_M g$.
If $\gamma \in C^1([0,1], M)$ is a geodesic (parametrized by arclength) that connects x and x_0, then

$$\left|\frac{d}{dt} g(\gamma(t))\right| \le \|\nabla g(\gamma(t))\| \le a g(\gamma(t)),$$

hence

$$e^a \int_M g \ge g(x) \ge e^{-a} \int_M g\,, \quad \text{for all } x \in M. \tag{6.1}$$

Defining the transfer operators $\mathcal{L}_k$ as in (2.3) with respect to the volume measure yields,

$$\mathcal{L}_k h(x) = \sum_{y \in f_k^{-1}(x)} \frac{h(y)}{|\det(D_y f_k)|}.$$

Hence, for some $\bar{D} \le C_\# \sup_{k\in\mathbb{N}} \|D^2 f_k\|_\infty = C_\# A$,

$$\|\nabla \mathcal{L}_k h\| \le \vartheta^{-1} \mathcal{L}_k \|\nabla h\| + \bar{D} \mathcal{L}_k h \le (\vartheta^{-1} a + \bar{D}) \mathcal{L}_k h \tag{6.2}$$

which implies $\mathcal{L}_k \mathcal{C}_a \subset \mathcal{C}_{\nu a}$, $\nu \in (\vartheta^{-1}, 1)$, for all $k \in \mathbb{N}$, provided $a \ge \bar{D}(\nu - \vartheta^{-1})^{-1}$.

From this point forward, we fix $\nu \in (\vartheta^{-1}, 1)$ and $a > 1$ such that $a > \bar{D}(\nu - \vartheta)^{-1}$.

Now for each $h \in C^1$, and using (6.1) since both $\|h\|$ and $h + \|h\|$ are in $\mathcal{C}_a$,

$$\begin{aligned} |\mathcal{L}_k \cdots \mathcal{L}_j h| &\le |\mathcal{L}_k \cdots \mathcal{L}_j (h + \|h\|)| + \|h\| |\mathcal{L}_k \cdots \mathcal{L}_j 1| \\ &\le e^a \int \mathcal{L}_k \cdots \mathcal{L}_j (h + 2\|h\|) = 3e^a \|h\|, \end{aligned} \tag{6.3}$$

which, iterating (6.2), proves (C-4).

One can easily compute the Hilber metric d_H and prove that for all $g \in \mathcal{C}_{\nu a}$, $\nu \in (0,1)$,

$$d_H(h, 1) \le 2a + \ln \frac{1+\nu}{1-\nu} =: \Delta/2. \tag{6.4}$$

So (C-2) holds true.

6.2. Conditions (O-1), (O-2) (Applicability of Theorem 2.6).

Assume that we have a sequence of observables g_k satisfying $\sup_k(\|g_k\|_\infty + \|\nabla g_k\|_\infty) = K < \infty$. This implies in particular that $\sup_k \|g_k\| \le (1 + a^{-1})K$, hence (O-1) holds with $j_0 = \infty$.

Let $\rho \in C^1$, $\rho \ge 0$ with $\int \rho \, d\mu_0 = 1$. Define $\hat{g}_k$ as in (2.8). For $n \in \mathbb{N}$, define $\hat{S}_n$ as in (2.10) and the variance σ_n^2 as in (2.11). Remark that $\sigma_n \le C_\# \sqrt{n}$ by Lemma 2.4.

With n fixed and $k \le n$, define the weighted operators for $\lambda \in \mathbb{R}$ as in (2.14),

$$\mathcal{L}_{k,\lambda} h := \mathcal{L}_{k,\lambda,n} h = \mathcal{L}_k (e^{i\sigma_n^{-1}\lambda g_k} h)\,.$$

Taking the derivative as in (6.2) we have

$$\|\nabla \mathcal{L}_{k,\lambda} h\| \le \vartheta^{-1} \mathcal{L}_{k,0} \|\nabla h\| + (\bar{D} + |\lambda| \sigma_n^{-1} \|g_k\|_\infty) \mathcal{L}_{k,0} |h|.$$

Which can be iterated, yielding

$$\begin{aligned} \|\nabla \mathcal{L}_{k,\lambda} \cdots \mathcal{L}_{j,\lambda} h\| &\le \vartheta^{-k+j+1} \mathcal{L}_k \cdots \mathcal{L}_j \|\nabla h\| + \sum_{l=0}^{k-j} \vartheta^{-l} (\bar{D} + |\lambda| \sigma_n^{-1} \|g_{k-l}\|_\infty) \mathcal{L}_{k-l} \cdots \mathcal{L}_j |h| \\ &\le \|h\| \frac{3e^a (\bar{D} + |\lambda| \sigma_n^{-1} \sup_j \|g_j\|_\infty)}{1 - \vartheta^{-1}}. \end{aligned}$$

where we have used (6.3) in the last step. Applying (6.3) again proves (O-2).

Accordingly, Theorem 2.6 applies. Note that Theorem 2.6 provides nontrivial information only if $\sigma_n \ge C_\# n^\alpha$ for some $\alpha > \frac{1}{3}$.

[20] Here and throughout this section, the integral is taken with respect to the volume measure on M, which has been normalized to be a probability measure.

6.3. **Complex Cones for Expanding Maps.**
To obtain the sharper results of Theorem 2.7, we have to verify condition (O-3). In order to do that, we use the complex cone theory introduced by Rugh and further developed by Dubois [38, 17, 18], as summarized in Section 5.

First we need to verify that we are in the setting of Section 5. As noted earlier, with $\mathbb{e} = 1$, the norm associated the cone is $\|h\| = \sup_x\{|h(x)| + a^{-1}\|\nabla h(x)\|\}$ which is equivalent to the C^1 norm, so $\mathcal{B}_{\mathbb{R}} = C^1(M, \mathbb{R})$.

For each $x \in M$ and $v \in \mathbb{R}^d$, $\|v\| \le 1$, we define

$$\ell_{x,v}(h) = ah(x) - \langle v, \nabla h(x)\rangle.$$

Clearly $\ell_{x,v} \in \mathcal{B}'_{\mathbb{R}}$. In addition, it is easy to check that

$$\mathcal{C}_a = \{h \in \mathcal{B}_{\mathbb{R}} \ : \ \ell(h) \ge 0 \ \forall \ell \in \mathcal{S}\}$$
$$\mathcal{S} = \{\ell_{x,v} \ : \ x \in M, \|v\| \le 1\}$$

This shows that (5.1) and (5.7) are satisfied. The complex cone $\mathcal{C}_{\mathbb{C}}$ is defined precisely as in Definition 5.3.

Next, set $\mathbb{m}(h) = \int h$, then, recalling (6.1), we have, for all $h \in \mathcal{C}_a$,

$$\mathbb{m}(h) \ge e^{-a}\|h\|_\infty \ge \frac{1}{2}e^{-a}\|h\| := \kappa\|h\|\,, \tag{6.5}$$

which verifies (5.6), and by extension (5.11).

To conclude checking the hypotheses of Theorem 5.15 we must to prove that the diameter of the image of the complex cone is finite. Since (6.4) states that the diameter of the real cone is finite, we can apply Theorem 5.17, provided we check (5.23). To check (5.23) let $\ell_{x,v} \in \mathcal{S}$, $h \in \mathcal{C}_a$,

$$\begin{aligned} |\ell_{x,v}\left((\mathcal{L}_{k,\lambda} - \mathcal{L}_k)h\right)| \le & \left|(\mathcal{L}_{k,\lambda} - \mathcal{L}_k)\langle (Df)^{-1}v, \nabla h\rangle\right| + |(\mathcal{L}_{k,\lambda} - \mathcal{L}_k)\mathcal{D}_v h| \\ & + \left|\mathcal{L}_{k,\lambda}(i\lambda\sigma_n^{-1}\langle (Df)^{-1}v, \nabla g_k\rangle h)\right| + a\,|(\mathcal{L}_{k,\lambda} - \mathcal{L}_k)h|\,, \end{aligned} \tag{6.6}$$

where $\mathcal{D}_v = -\operatorname{sign}(\det Df_k)\frac{\langle Df_k^{-1}v, \nabla \det Df_k\rangle}{(\det Df_k)^2}$.

Since $\mathcal{L}_k 1 \in \mathcal{C}_a$, (6.1) implies the following bound on the first term above,

$$\begin{aligned} \vartheta^{-1}\,|(\mathcal{L}_{k,\lambda} - \mathcal{L}_k)\|\nabla h\|| &\le \vartheta^{-1}|\lambda|\sigma_n^{-1}|g_k|_\infty|\mathcal{L}_k 1|_\infty\|\nabla h\|_\infty \le \vartheta^{-1}ae^a|\lambda|\sigma_n^{-1}|g_k|_\infty|h|_\infty \\ &\le \vartheta^{-1}ae^{2a}|\lambda|\sigma_n^{-1}|g_k|_\infty\int h \le \vartheta^{-1}ae^{2a}|\lambda|\sigma_n^{-1}|g_k|_\infty\int \mathcal{L}_k h \\ &\le \vartheta^{-1}ae^{3a}|\lambda|\sigma_n^{-1}|g_k|_\infty\mathcal{L}_k h(x)\,, \end{aligned}$$

where we have used $h, \mathcal{L}_k h \in \mathcal{C}_a$. Similarly, we bound the second, third and fourth terms of (6.6) using the positivity of h,

$$|\lambda|\sigma_n^{-1}(\bar{D} + a)|g_k|_\infty\mathcal{L}_k h(x) + \vartheta^{-1}|\lambda|\sigma_n^{-1}\|\nabla g_k\|_\infty\mathcal{L}_k h(x)\,.$$

Putting these estimates together in (6.6) implies,

$$|\ell_{x,v}\left((\mathcal{L}_{k,\lambda} - \mathcal{L}_k)h\right)| \le |\lambda|\sigma_n^{-1}\|g_k\|_{\mathcal{C}^1}\left[\vartheta^{-1}ae^{3a} + \bar{D} + a\right]\mathcal{L}_k h(x).$$

On the other hand, $\mathcal{L}_k h \in \mathcal{C}_{\nu a}$, hence

$$\ell_{x,v}(\mathcal{L}_k h) = a\mathcal{L}_k h - \langle v, \nabla\mathcal{L}_k h\rangle \ge a(1-\nu)\mathcal{L}_k h(x).$$

Accordingly,

$$|\ell_{x,v}\left((\mathcal{L}_{k,\lambda} - \mathcal{L}_k)h\right)| \le |\lambda|\sigma_n^{-1}a^{-1}(1-\nu)^{-1}\|g_k\|_{\mathcal{C}^1}\left[\vartheta^{-1}ae^{3a} + \bar{D} + a\right]\ell_{x,v}(\mathcal{L}_k h).$$

which satisfies (5.23) provided λ is small enough. Hence Theorem 5.17 proves the finite diameter of the complex cone, and Theorem 5.15 applies.

6.4. **Verification of Condition (O-3).**

We can finally check condition (O-3). By the above discussion it follows that, for λ_0 small enough, and $|\lambda| \leq \lambda_0 \sigma_n$, we have that Theorem 5.15 applies to each $\mathcal{L}_{k,\lambda,n}$ with complex diameter $\Delta_{\mathbb{C}} < \infty$ uniform in λ, k and n. Moreover, since $\mathcal{C}_{\mathbb{R},k} = \mathcal{C}_a$ and $\mathrm{m}_k = \mathrm{m} = \mu$, the Riemannian volume, for each k, we are in the setting of Section 5.3 with $\bar{\kappa} = \kappa = \frac{1}{2}e^{-a}$, K the constant from (O-2) and $\mathbb{L}_k = \mathcal{L}_{k,\lambda,n}$. We can then apply Lemma 5.20 to obtain (O-3) with $\alpha_{k,j,\lambda} = \boldsymbol{\alpha}_{k,j}$, $h_{k,j,\lambda} = \boldsymbol{h}_{k,j}$ and $\ell_{k,j,\lambda} = \boldsymbol{\ell}_{k,j}$, as defined in Section 5.3.

By definition $\ell_{k,j,\lambda}(1) = 1$ and $\mathrm{m}(h_{k,j,\lambda}) = 1$. Note that this normalization is compatible with that required in (O-3) since $\mathrm{m}(h_{k,j,\lambda}) = \int h_{k,j,\lambda}$. Moreover, $|\alpha_{k,j,\lambda}| \leq 1$ and $\|h_{k,j,\lambda}\| \leq \sqrt{2}/\kappa$, $\|\ell_{k,j,\lambda}\|' \leq \sqrt{2}$ by Lemma 5.19. Note that all these bounds are independent of n.

Then since condition (b) of Section 5.3 holds with $K = 1$, Lemma 5.20 implies that the property (2.15) of (O-3) holds with $K = \frac{2(1+\sqrt{2})}{\kappa\theta^2}\Delta_{\mathbb{C}}$, $\theta = \tanh(\Delta_{\mathbb{C}}/4)$ and $c = -\log\theta$.

Since we have already verified the hypotheses of Theorem 5.17, conditions (d) and (e) of Section 5.4 are satisfied. Thus the final claim that there exists $K > 0$ such that $|\ell_{k,j,\lambda}(h_{j,l,\lambda})| \geq K^{-1}$ for all $0 \leq l \leq j \leq k \leq n$, follows immediately from Lemma 5.21.

6.5. **Central Limit Theorem and Variance for Sequential Expanding Maps.**

We have verified hypotheses (C-1)-(C-4) and (O-1)-(O-3). Thus Theorem 2.7 and Corollary 2.9 apply to the setting of sequential expanding maps. Hence, Theorem 2.1 follows.

As mentioned in Section 2.1, Theorem 2.1 is conditional on the growth of σ_n. To find conditions implying that the variance grows sufficiently fast, in the abstract sequential case setting, is nonobvious. Here, we provide a partial answer for the case at hand.

To simplify matters, we assume that the maps and observables appearing in the sequence all belong to a finite set[21]

$$\mathcal{F} := \{\bar{f}_j\}_{j=1}^N \qquad \lambda = \inf_j \inf_x \|(D_x \bar{f}_j)^{-1}\|^{-1} > 1$$
$$\mathcal{G} = \{\bar{g}_j\}_{j=1}^N \subset \mathcal{C}^1(M, \mathbb{R}).$$

For any given sequences $\omega^f, \omega^g \in \{1, \dots, N\}^{\mathbb{N}\cup\{0\}}$, we set $f_k = \bar{f}_{\omega_k^f}$, $g_k = \bar{g}_{\omega_k^g}$.
The maps $\mathcal{F}$ satisfy specification: for each $\varepsilon, L > 0$, there exists $D_{\mathcal{F}}(\varepsilon, L)$ such that for each set of intervals $J_k = \{a_k, \dots, a_k + L\}$, such that $a_{k+1} - a_k - L \geq D_{\mathcal{F}}(\varepsilon, L)$, and points $x_k \in M$, there exists $z \in M$ such that

$$|f_{a_k+i} \circ \cdots \circ f_0(z) - f_{a_k+i} \circ \cdots \circ f_{a_k}(x_k)| \leq \varepsilon,$$

for all $k \in \mathbb{N}$ and $i \in \{0, \dots, L-1\}$.

The main result in this section is the following criterion. See Lemma 6.6 for a quantitative version.

Proposition 6.1. *Let $\ln\rho \in \mathcal{C}^1(M)$. If there exists $a \in (0,1)$ and $L \in \mathbb{N}$ such that for each sequence $\overline{\omega} = \{(\omega_i^1, \omega_i^2)\}_{i=0}^L$ there exists a point $x_{\overline{\omega}}$ such that* [22]

$$\sum_{i=1}^L \hat{g}_{\omega_i^2}(\bar{f}_{\omega_{i-1}^1} \circ \cdots \circ \bar{f}_{\omega_0^1}(x_{\overline{\omega}})) \geq aL$$
$$\text{where} \quad \varepsilon \leq \frac{a}{4\sup_j \|D\bar{g}_j\|_{\mathcal{C}^0}} \quad \text{and} \quad L \geq \frac{8}{a} D_{\mathcal{F}}(\varepsilon, L) \max_j \|\hat{g}_j\|_\infty.$$

then there exists $B > 0$ such that $\sigma_n^2 = \mathbb{E}(S_n^2) \geq Bn$.

The rest of the section is devoted to the proof of the above Proposition. But first, it is appropriate to discuss its relevance and generality.

Remark 6.2. *For simplicity, in Proposition 6.1, we assume that $\ln\rho \in \mathcal{C}^1$. This is done so that ρ belongs to a real cone $\mathcal{C}_a$, as defined in Section 6.1. The interested reader can generalize to the case $\rho \in C^1$, $\rho \geq 0$, since then there exists a $k_0 \in \mathbb{N}$ such that $\mathcal{L}_k \cdots \mathcal{L}_0 \rho \in \mathcal{C}_a$ for all $k \geq k_0$.*

[21] The finiteness allows us to obtain a condition that can be checked on a finite number of finite time trajectories; otherwise some type of compactness seems to be needed.

[22] For the definition of $\hat{g}_{\omega_i}$ see equation (2.1).

Remark 6.3. *Note that in the case of a single dynamical system and a single observable (that is $\bar{f}_k = f$, and $g_k = g$, a zero average observable), the usual condition for the linear growth of the variance is that there exists a periodic orbit $\{p, f(p), \dots, f^{q-1}(p)\}$, $f^q(p) = p$, such that*[23] $\sum_{k=0}^{q-1} g(f^k(p)) \geq b$, *for some $b > 0$.*[24] *This implies the condition in Proposition 6.1. To see it choose μ_k to be the invariant measure of f, so $\hat{g} = g$. Then, setting $L = Kq$, $K \in \mathbb{N}$,*

$$\sum_{i=1}^{L} g(f^{i-1}(p)) = K \sum_{i=1}^{q} g(f^{i-1}(p)) \geq Kb = L\frac{b}{q} =: aL.$$

Accordingly, the conditions of Proposition 6.1 are satisfied with the choice $x_\omega = p$. This shows that Proposition 6.1 is a natural generalization to the sequential case of the usual condition. It remains to see if the condition is explicitly checkable; this is verified in Corollary 6.4, although it is unclear if our condition can be reduced to checking that the observables are not coboundaries w.r.t. some dynamics.

Note that, by equation (6.11), $\|\hat{g}_k\|_\infty \leq 2\|g_k\|_\infty$. Moreover, the $\mathcal{L}_k$ have a uniform spectral gap. Hence, there exists a computable $A > 0$, $\nu \in (0,1)$ such that, for all $S \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\begin{aligned} &\left| \int_M g_k \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho - \gamma_{S,k} \right| \leq A\nu^S \\ &\gamma_{S,k} := \frac{1}{\int_M 1} \int_M g_k \mathcal{L}_{k-1} \cdots \mathcal{L}_{\max\{k-S,0\}} 1. \end{aligned} \tag{6.7}$$

Corollary 6.4. *There exist computable constants $\kappa, b, L_0 > 0$ such that, if for some $L \geq L_0$ and each sequence $\overline{\omega} = \{\omega_i^1, \omega_i^2\}_{i=0}^L$, there exists a point $x_{\overline{\omega}}$ such that*

$$\sum_{i=1}^{L} \tilde{g}_{\omega_i^2}(\bar{f}_{\omega_{i-1}^1} \circ \cdots \circ \bar{f}_{\omega_0^1}(x_{\overline{\omega}})) \geq 2\kappa \ln L, \tag{6.8}$$

where $\tilde{g}_k = g_k - \gamma_{S,k}$ and $S = b \ln L$, then the hypothesis of Proposition 6.1 are verified. Thus, there exists $B > 0$ such that $\sigma_n^2 \geq Bn$.

Proof. Let $a = \kappa \frac{\ln L}{L}$. We use (6.7): if $b > -1/\ln \nu$, then $A\nu^S \leq a$ for all L larger than some computable L_0, and

$$\sum_{i=1}^{L} \hat{g}_{\omega_i^2}(\bar{f}_{\omega_{i-1}^1} \circ \cdots \circ \bar{f}_{\omega_0^1}(x_{\overline{\omega}})) \geq aL.$$

Set $\Gamma = \sup_j \max\{\|Dg_j\|_\infty, 2\|g_j\|_\infty\}$ and choose $\varepsilon = \frac{a}{4\Gamma}$. Then Proposition 6.1 applies if

$$L \geq \frac{D_{\mathcal{F}}(\varepsilon, L)\Gamma}{a}. \tag{6.9}$$

By Lemma 6.6, there exists a computable $c_\star > 0$ such that we can choose $D_{\mathcal{F}}(\varepsilon, L) = c_\star \ln a^{-1}$. Consequently, equation (6.9) is satisfied if, for $L \geq L_0$,

$$1 \geq \frac{c_\star \ln a^{-1}\Gamma}{aL} = c_\star \Gamma \frac{1}{\kappa \ln L} \ln\left[\frac{L}{\kappa \ln L}\right] = \frac{c_\star \Gamma}{\kappa} \frac{\ln L - \ln\ln L - \ln \kappa}{\ln L}$$

which is satisfied if $\kappa = c_\star \Gamma$. □

Note that condition (6.8) is checkable by trial and error. This is similar to the problem of finding a periodic orbit with non-zero average in the nonsequential case (see Remark 6.3).

To prove Proposition 6.1, we will use a martingale decomposition, similar to [15], but we must first introduce some notation. Let

$$h_k = \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho \tag{6.10}$$

[23] Possibly substituting $-g$ for g.

[24] Which implies that g is not a coboundary.

and $\widehat{\mathcal{L}}_k(g) = h_{k+1}^{-1}\mathcal{L}_k(h_k g)$. Note that $\widehat{\mathcal{L}}_k \cdots \widehat{\mathcal{L}}_i(g) = h_{k+1}^{-1}\mathcal{L}_k \cdots \mathcal{L}_i(h_i g)$, moreover $\widehat{\mathcal{L}}_k 1 = 1$. Also, recalling (2.8),

$$\hat{g}_k = g_k - \int_M g_k \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho \tag{6.11}$$

We would like to find $\phi_k, Y_k \in \mathcal{C}^1$ such that

$$\begin{aligned} &\hat{g}_k = \phi_{k+1} \circ f_k - \phi_k + Y_k \\ &\phi_0 = 0 \\ &\widehat{\mathcal{L}}_k Y_k = 0. \end{aligned} \tag{6.12}$$

This decomposition would imply,

$$\widehat{\mathcal{L}}_k \hat{g}_k = \phi_{k+1} - \widehat{\mathcal{L}}_k \phi_k \,.$$

With this motivation, let us define $\phi_0 = 0$ and for each $k \geq 1$,

$$\phi_k = \sum_{j=0}^{k-1} \widehat{\mathcal{L}}_{k-1} \cdots \widehat{\mathcal{L}}_j \hat{g}_j, \tag{6.13}$$

and set $Y_k = \hat{g}_k - \phi_{k+1} \circ f_k + \phi_k$. Then indeed,

$$-\widehat{\mathcal{L}}_k Y_k = \widehat{\mathcal{L}}_k \left[\phi_{k+1} \circ f_k - \phi_k - \hat{g}_k\right] = \sum_{j=0}^{k} \left[\widehat{\mathcal{L}}_k \cdots \widehat{\mathcal{L}}_j \hat{g}_j\right] - \sum_{j=0}^{k-1} \widehat{\mathcal{L}}_k \cdots \widehat{\mathcal{L}}_j \hat{g}_j - \widehat{\mathcal{L}}_k \hat{g}_k = 0.$$

Note that, by usual arguments, there exists $C_\star > 0$ such that

$$\begin{aligned} &\|\phi_k\|_{\mathcal{C}^1} \leq C_\star \,, \quad \|Y_k\|_{\mathcal{C}^1} \leq C_\star \\ &C_\star^{-1} \leq h_k \leq C_\star. \end{aligned} \tag{6.14}$$

Let $S_n = \sum_{k=0}^{n-1} \hat{g}_k \circ f_{k-1} \circ \cdots \circ f_0$. We want to compute[25]

$$\mathbb{E}(S_n^2) = \mathbb{E}\left(\left[\phi_{n+1} \circ f_n \circ \cdots \circ f_0 + \sum_{k=0}^{n-1} Y_k \circ f_{k-1} \circ \cdots \circ f_0\right]^2\right)$$

Note that, recalling (6.10) and also (6.12), for all $k < n$,

$$\begin{aligned} &\mathbb{E}\left(\phi_{n+1} \circ f_n \circ \cdots \circ f_0 \cdot Y_k \circ f_{k-1} \circ \cdots \circ f_0\right) = \int \phi_{n+1} \circ f_n \circ \cdots \circ f_k \cdot Y_k \cdot \mathcal{L}_{k-1} \cdots \mathcal{L}_0 \rho \, dx \\ &= \int \phi_{n+1} \circ f_n \circ \cdots \circ f_k \cdot Y_k h_k \, dx \\ &= \int \phi_{n+1} \circ f_n \circ \cdots \circ f_{k+1} \cdot h_{k+1} \cdot \widehat{\mathcal{L}}_k(Y_k) \, dx = 0 \end{aligned}$$

[25] As usual, by $\mathbb{E}(\varphi)$ we mean $\int_M \varphi(x)\, \rho \, dx$.

Accordingly, recalling (6.14) and (6.12),

$$
\begin{aligned}
\mathbb{E}(S_n^2) &= \mathbb{E}\left(\left[\phi_{n+1}\circ f_n\circ\cdots\circ f_0 + \sum_{k=0}^{n-1} Y_k\circ f_{k-1}\circ\cdots\circ f_0\right]^2\right)\\
&= \mathbb{E}\left(\left[\sum_{k=0}^{n-1} Y_k\circ f_{k-1}\circ\cdots\circ f_0\right]^2\right) + \mathcal{O}(1)\\
&= \sum_{k=0}^{n-1}\mathbb{E}\left(h_k Y_k^2\right) + 2\sum_{k>i}\mathbb{E}\left(h_i Y_k\circ f_{k-1}\circ\cdots\circ f_i Y_i\right) + \mathcal{O}(1)\\
&= \sum_{k=0}^{n-1}\mathbb{E}\left(h_k Y_k^2\right) + 2\sum_{k>i}\mathbb{E}\left(h_k Y_k \widehat{\mathcal{L}}_{k-1}\cdots\widehat{\mathcal{L}}_i Y_i\right) + \mathcal{O}(1)\\
&= \sum_{k=0}^{n-1}\mathbb{E}\left(h_k Y_k^2\right) + \mathcal{O}(1).
\end{aligned}
\tag{6.15}
$$

Next, we want to connect the growth of S_n along a single orbit with the growth of its L^2 norm.

Lemma 6.5. *There exists $B>0$ such that if, for some $p\in M$, $|S_n(p)|\geq 3An^\alpha$, $\alpha>\frac{d+1}{d+2}$, then we have $\mathbb{E}(S_n^2)\geq A^{d+2}Bn^{(d+2)\alpha-d-1}$.*

Proof. Equation (6.12) allows us to write

$$
\begin{aligned}
|S_n(p)| &= \left|\phi_{n+1}\circ f_n\circ\cdots\circ f_0(p) + \sum_{k=0}^{n-1} Y_k\circ f_{k-1}\circ\cdots\circ f_0(p)\right|\\
&\leq \sum_{k=0}^{n-1}|Y_k|\circ f_{k-1}\circ\cdots\circ f_0(p) + \mathcal{O}(1).
\end{aligned}
$$

Note that, for each $q\in M$, $|Y_k(x)|\geq |Y_k(q)| - C_\star|x-q|$. If follows that, for all $x\in\{y\in M\ :\ |y-q|\leq |Y_k(q)|/(2C_\star)\}$, one has $|Y_k(x)|\geq\frac{1}{2}|Y_k(q)|$. Thus, recalling (6.14),

$$
\mathbb{E}(h_k Y_k^2) = C_\star^{-1}\mathbb{E}(Y_k^2)\geq c_d|Y_k|^{d+2}\circ f_{k-1}\circ\cdots\circ f_0(p),
$$

for some constant c_d. Let $J_n=\{k\in\{0,\dots,n-1\}\ :\ |Y_k|\circ f_{k-1}\circ\cdots\circ f_0(p)\geq An^{-1+\alpha}\}$, then using (6.15),

$$
\begin{aligned}
3An^\alpha \leq |S_n(p)| &\leq \frac{n^{(d+1)(1-\alpha)}}{A^{d+1}}\sum_{k\in J_n}|Y_k|^{d+2}\circ f_{k-1}\circ\cdots\circ f_0(p) + An^\alpha + \mathcal{O}(1)\\
&\leq \frac{n^{(d+1)(1-\alpha)}}{A^{d+1}c_d}\sum_{k=0}^{n-1}\mathbb{E}(h_k Y_k^2) + An^\alpha + \mathcal{O}(1) \leq \frac{n^{(d+1)(1-\alpha)}}{A^{d+1}c_d}\mathbb{E}(S_n^2) + An^\alpha + \mathcal{O}(n^{(d+1)(1-\alpha)}+1).
\end{aligned}
$$

For n large enough we have $An^\alpha + \mathcal{O}(n^{(d+1)(1-\alpha)}+1)\leq 2An^\alpha$, consequently

$$
\mathbb{E}(S_n^2)\geq\frac{A^{d+1}c_d}{n^{(d+1)(1-\alpha)}}An^\alpha,
$$

from which the lemma follows. □

Proof of Proposition 6.1. Let $D := D_{\mathcal{F}}(\varepsilon, L)$. For any $\omega^f,\omega^g\in\{1,\dots N\}^{\mathbb{N}\cap\{0\}}$, let $f_k=\bar{f}_{\omega_k^f}$ and $g_k=\bar{g}_{\omega_k^g}$ and $r_k=k(L+D)$. We can assume, without loss of generality, that $n\geq\frac{L}{D}+1$. By hypothesis

there exists $p \in M$ such that

$$\begin{aligned} S_n(p) &\geq \sum_{k=0}^{\lfloor \frac{n}{L+D} \rfloor} \sum_{i=0}^{L-1} \hat{g}_{r_k+i+1}(f_{r_k+i} \circ \cdots \circ f_0(p)) - \left[\frac{nD}{L+D} + 1\right] \sup_j \|\bar{g}_j\|_{\mathcal{C}^0} \\ &\geq \sum_{k=0}^{\lfloor \frac{n}{L+D} \rfloor} \sum_{i=0}^{L-1} \hat{g}_{r_k+i+1}(f_{r_k+i} \circ \cdots \circ f_{r_k}(x_k)) - \frac{\varepsilon L n}{L+D} \sup_j \|D\bar{g}_j\|_{\mathcal{C}^0} - \frac{2nD}{L+D} \sup_j \|\bar{g}_j\|_{\mathcal{C}^0} \\ &\geq \frac{aLn}{L+D} - \frac{aLn}{4(L+D)} - \frac{anL}{4(L+D)} \geq \frac{a}{2} n. \end{aligned}$$

We can then apply Lemma 6.5 with $\alpha = 1$, which proves the proposition for B small enough. □

It remains to discuss the Lemma used to obtain Corollary 6.4.

Lemma 6.6. *There exists $\varepsilon_0 \in (0,1)$ such that, for all $\varepsilon \leq \varepsilon_0$, we can choose $D_{\mathcal{F}}(\varepsilon, L) = \inf\{n \in \mathbb{N} : \varepsilon\lambda^n \geq \operatorname{diam}(M)\}$.*

Proof. Set ε_0 so that all the maps $\bar{f}_j$ are locally invertible on balls of size ε_0. For $\varepsilon \leq \varepsilon_0$, consider the set

$$A_k = \{y \in M \ : \ |f_{a_k+i} \circ \cdots \circ f_{a_k}(y) - f_{a_k+i} \circ \cdots \circ f_{a_k}(x_k)| \leq \varepsilon, \, 0 \leq i \leq L-1\}.$$

Then $f_{a_k+L-1} \circ \cdots \circ f_{a_k}(A_k)$ is a ball of radius ε. Accordingly,

$$f_{a_k+D_{\mathcal{F}}(\varepsilon,L)} \circ \cdots \circ f_{a_k}(A_k) = M.$$

Thus $f_{a_k+D_{\mathcal{F}}(\varepsilon,L)} \circ \cdots \circ f_{a_k}(A_k) \supset A_{k+1}$. Taking the intersection of the preimages of the A_k, we have a nested sequence of closed sets and we can choose as z any point in the intersection. □

7. Application: Sequential Billiard Maps

In this section, we show that Theorems 2.6 and 2.7 apply to sequential billiards as described in [9]. Our strategy will be to define real and complex cones on which the relevant operators act as strict contractions, and so apply the theorems of Section 5 to verify conditions (O-1)-(O-3). We begin by recalling the setting of [9].

Fixing constants $K \in \mathbb{N}$ and $\ell_i > 0$, $i = 1, \ldots K$, we consider billiard configurations in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ formed by placing K scatterers B_i having C^3 boundaries and with arclengths given by ℓ_i and curvatures $\mathcal{K} > 0$. We denote such a billiard table by $Q = \mathbb{T}^2 \setminus \cup_i B_i$. The billiard map T is defined by the motion of a point particle undergoing elastic collisions at the boundaries and traveling with unit speed in straight lines between collisions. We adopt the standard collision coordinates, $x = (r, \varphi)$, where r is the arclength coordinate on ∂Q and φ is the angle made by the post-collision velocity with the normal to the boundary. Thus the phase space for T is given by $\mathcal{M} = \cup_{i=1}^K I_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, where for each i, $I_i = [0, \ell_i]/\sim$ is an interval of length ℓ_i with endpoints identified. Since we have fixed K and ℓ_i, $\mathcal{M}$ is the same for each such table Q.

Let $\mathcal{K}_{\min}(Q) \leq \mathcal{K}_{\max}(Q)$ denote the minimum and maximum curvatures of scatterers in the table Q, and let $\tau_{\min}(Q) \leq \tau_{\max}(Q)$ denote the minimum and maximum distances between consecutive collisions.

These scatterer configurations are subject to the following three constraints: Fix $\tau_*, \mathcal{K}_*, E_* > 0$. Then for each configuration Q, we assume: $\tau_* \leq \tau_{\min}(Q) \leq \tau_{\max}(Q) \leq \tau_*^{-1}$, $\mathcal{K}_* \leq \mathcal{K}_{\min}(Q) \leq \mathcal{K}_{\max}(Q) \leq \mathcal{K}_*^{-1}$, the C^3 norm of ∂Q is at most E_*. Let $\mathcal{Q}(\tau_*, \mathcal{K}_*, E_*)$ denote the set of billiard tables with K scatterers satisfying these conditions and let $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ denote the corresponding set of billiard maps. As noted above, each $T \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ acts on the same phase space $\mathcal{M}$.

Fixing these constants ensures that maps in $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ enjoy the uniform properties listed as **(H1)**-**(H5)** in [9]: a common set of invariant cones in the tangent space, uniform distortion bounds, uniform growth lemma, etc.

We choose as our reference measure on $\mathcal{M}$, $\mu_{\text{SRB}} := \frac{1}{2|\partial Q|} \cos\varphi \, dr d\varphi$, which is preserved by each $T \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$. The associated transfer operator $\mathcal{L}_T$ acting on measurable functions is defined by,

$$\mathcal{L}_T h = h \circ T^{-1}.$$

In order to generate sufficient hyperbolicity to strictly contract the cones $\mathcal{C}_{c,A,L}(\delta)$ defined in Section 7.1 below, we require a notion of distance in $\mathcal{Q}(\tau_*, \mathcal{K}_*, E_*)$. For a table $Q \in \mathcal{Q}(\tau_*, \mathcal{K}_*, E_*)$, the boundary ∂B_i of each scatterer can be parametrized according to arclength by a function $u_{i,\theta}$, where $\theta \in [0, \ell_i)$ represents

the tagged point in ∂B_i where the parametrization begins. Given two tables $Q, \tilde{Q} \in \mathcal{Q}(\tau_*, \mathcal{K}_*, E_*)$ with respective parametrizations $u_{i,\theta}$, $\tilde{u}_{i,\theta}$, let Π_K denote the set of permutations π on $\{1, \dots, K\}$ satisfying $\tilde{\ell}_{\pi(i)} = \ell_i$. Define the distance,

$$\mathbb{d}(Q, \tilde{Q}) = \min_{\pi \in \Pi_K} \min_{\theta \in [0, \ell_i)} \sum_{i=1}^{K} |u_{i,\theta} - \tilde{u}_{\pi(i),\theta}|_{C^2(I_i, \mathbb{R}^2)}.$$

For $Q_0 \in \mathcal{Q}(\tau_*, \mathcal{K}_*, E_*)$ and $\varepsilon \leq \frac{1}{2} \min\{\tau_*, \mathcal{K}_*\}$, define

$$\mathcal{Q}(Q_0, E_*; \varepsilon) = \{Q \in \mathcal{Q}(\tau_*/2, \mathcal{K}_*/2, E_*) : \mathbb{d}(Q, Q_0) < \varepsilon\}.$$

Let $\mathcal{F}(Q_0, E_*; \varepsilon)$ denote the corresponding set of billiard maps.

Definition 7.1. *Fix $Q_k \in \mathcal{Q}(\tau_*, \mathcal{K}_*, E_*)$, $\varepsilon > 0$ from* [9, Lemma 6.6(b)] *and $N_{\mathcal{F}} \in \mathbb{N}$ from* [9, Theorem 2.3]. *We define each of our maps f_k (in the notation of Section 2) by $f_k = T_{N_{\mathcal{F}}} \circ \cdots \circ T_1$, where $T_j \in \mathcal{F}(Q_k, E_*; \varepsilon)$.*

*With this definition of f_k, the sequence $f_{k,0} = f_{k-1} \circ \cdots \circ f_0$ is necessarily an $N_{\mathcal{F}}$-*admissible sequence *in the terminology of* [9, Definition 2.4].

Note that each f_k is comprised of a collection of $N_{\mathcal{F}}$ billiard maps all of which are close with respect to the distance $\mathbb{d}$, but the maps comprising f_{k+1} are not necessarily close to the maps comprising f_k.[26]

In the notation of Section 2 then, we have $M_k = \mathcal{M}$, $\mu_k = \mu_{\text{SRB}}$, while $f_k = T_{N_{\mathcal{F}}} \circ \cdots \circ T_1$, and $\mathcal{L}_k = \mathcal{L}_{T_{N_{\mathcal{F}}}} \cdots \mathcal{L}_{T_1}$, where $T_j \in \mathcal{F}(Q_k, E_*; \varepsilon)$ as in Definition 7.1.

7.1. Definition and Contraction of the Real Cone $C_{\mathbb{R}} = \mathcal{C}_{c,A,L}(\delta)$.

Next, we must define precisely the cone of functions constructed in [9] which is strictly contracted by $\mathcal{L}_k$.

According to [9, Section 3.1], there exist stable and unstable cones C^s and C^u in the tangent space of $\mathcal{M}$ (not to be confused with the projective cones of functions defined below), that are strictly contracted by DT^{-1} and DT, respectively, for all $T \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$. For fixed $k_0 \in \mathbb{N}$, define the usual homogeneity strips,

$$\mathbb{H}_{\pm k} = \{(r, \varphi) \in \mathcal{M} : (k+1)^{-2} \leq |\pm \tfrac{\pi}{2} - \varphi| \leq k^{-2}\}, \text{ for all } k \geq k_0.$$

For convenience, label $\mathbb{H}_0 = \mathcal{M} \setminus \cup_{|k| \geq k_0} \mathbb{H}_k$. We say a C^1 curve $W \subset \mathcal{M}$ is stable or cone-stable if its tangent vector at each point lies in C^u. We call it homogeneous if it lies in a single homogeneity strip.

Let $\mathcal{W}^s$ denote the set of homogeneous cone-stable curves whose curvature is bounded by $B_0 > 0$. We choose B_0 so that $T^{-1}\mathcal{W}^s \subset \mathcal{W}^s$ for all $T \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$.

Since we will subdivide curves when they cross $\partial \mathbb{H}_k$, we will refer to the extended singularity sets for T (respectively T^{-1}) as $\mathcal{S}_1^{\mathbb{H}} = \mathcal{S}_0 \cup T^{-1}(\mathcal{S}_0 \cup_{|k| \geq k_0} \partial \mathbb{H}_k)$ (respectively $\mathcal{S}_{-1}^{\mathbb{H}} = \mathcal{S}_0 \cup T(\mathcal{S}_0 \cup_{|k| \geq k_0} \partial \mathbb{H}_k)$), where $\mathcal{S}_0 = \{(r, \varphi) \in \mathcal{M} : \varphi = \pm \pi/2\}$.

For $\alpha \in (0, 1]$, $a \geq 1$ and $W \in \mathcal{W}^s$, we define the following cone of test functions, following [9, Section 4.1]. Let $d(\cdot, \cdot)$ define distance on W induced by arclength.

$$\mathcal{D}_{a,\alpha}(W) := \left\{ \psi \in C^0(W) : \psi > 0, \frac{\psi(x)}{\psi(y)} \leq e^{ad(x,y)^\alpha} \right\}$$

To define the cone for $\mathcal{L}_k$, we shall need a notion of distance between stable curves. To this end, we view each $W \in \mathcal{W}^s$ as the graph of a C^2 function of the r coordinate,

$$W = \{G_W(r) = (r, \varphi_W(r)) : r \in I_W\}. \tag{7.1}$$

For $W^1, W^2 \in \mathcal{W}^s$, if W^1 and W^2 lie in the same homogeneity strip with $|I_{W^1} \cap I_{W^2}| > 0$, define the distance between them by

$$d_{\mathcal{W}^s}(W^1, W^2) = |\varphi_{W^1} - \varphi_{W^2}|_{C^1(I_{W^1} \cap I_{W^2})} + |I_{W^1} \triangle I_{W^2}|.$$

Otherwise, define $d_{\mathcal{W}^s}(W^1, W^2) = \infty$. Although $d_{\mathcal{W}^s}$ is not a metric, it is sufficient for our needs.

When $d_{\mathcal{W}^s}(W^1, W^2) < \infty$, we will also need to measure a distance between test functions. For $\beta < \alpha$ as defined below and $\psi_i \in \mathcal{D}_{a,\beta}(W^i)$, define,

$$d_*(\psi_1, \psi_2) = \left| \psi_1 \circ G_{W^1} \|G'_{W^1}\| - \psi_2 \circ G_{W^2} \|G'_{W^2}\| \right|_{C^\beta(I_{W^1} \cap I_{W^2})},$$

[26]The notion of $N_{\mathcal{F}}$-admissible sequence can also be simplified by choosing $f_k = T_k^{N_{\mathcal{F}}}$, where $T_k \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$, i.e. simply requiring $N_{\mathcal{F}}$ iterates of the same map before changing to a different (but not necessarily close) map.

where $\|G'_W\| = \sqrt{1+(d\varphi_W/dr)^2}$.

For a given length scale δ, define

$$\mathcal{W}^s_-(\delta) = \{W \in \mathcal{W}^s : |W| \le 2\delta\} \quad \text{and} \quad \mathcal{W}^s(\delta) = \{W \in \mathcal{W}^s : |W| \in [\delta, 2\delta]\}.$$

Next, let $\mathcal{A}_*$ denote the set of functions on $\mathcal{M}$ whose restriction to each $W \in \mathcal{W}^s$ is integrable with respect to the arclength measure dm_W. Define,

$$|||h|||^{\sim}_+ = \sup_{\substack{W\in\mathcal{W}^s_-(\delta)\\ \psi\in\mathcal{D}_{a,\beta}(W)}} \frac{|\int_W h\psi\, dm_W|}{\int_W \psi\, dm_W}, \quad \text{for } h \in \mathcal{A}_*.$$

Set $\mathcal{A}_0 = \{h \in \mathcal{A}_* : |||h|||^{\sim}_+ < \infty\}$, and note that $|||\cdot|||^{\sim}_+$ defines a seminorm on $\mathcal{A}_0$. Thus identifying two functions g and h in $\mathcal{A}_0$ if $|||g-h|||^{\sim}_+ = 0$, we define $\mathcal{A}$ to be the resulting normed vector space of equivalence classes. Note that if $g \sim h$, then $g = h$ almost everywhere with respect to both the Lebesgue measure and μ_{SRB}.

Next, for $h \in \mathcal{A}$, define the following two quantities,

$$|||h|||_+ = \sup_{\substack{W\in\mathcal{W}^s(\delta)\\ \psi\in\mathcal{D}_{a,\beta}(W)}} \frac{\left|\int_W h\psi\, dm_W\right|}{\int_W \psi\, dm_W}, \qquad |||h|||_- = \inf_{\substack{W\in\mathcal{W}^s(\delta)\\ \psi\in\mathcal{D}_{a,\beta}(W)}} \frac{\int_W h\psi\, dm_W}{\int_W \psi\, dm_W}. \tag{7.2}$$

Denote the average value of ψ on W by $⨍_W \psi\, dm_W = \frac{1}{|W|}\int_W \psi\, dm_W$. Since all of our integrals on $W \in \mathcal{W}^s$ will be taken with respect to the arc-length dm_W, to keep our notation concise, we will drop the measure from our integral notation in the following.

For exponents, $\alpha, \beta, \gamma, q \in (0,1)$ and constants $a, c, A, L > 1$, $\delta > 0$, we define the cone

$$\mathcal{C}_{c,A,L}(\delta) = \Bigg\{ h \in \mathcal{A}\setminus\{0\} :$$

$$|||h|||_+ \le L|||h|||_-; \tag{7.3}$$

$$\sup_{W\in\mathcal{W}^s_-(\delta)} \sup_{\psi\in\mathcal{D}_{a,\beta}(W)} |W|^{-q} \frac{|\int_W h\psi|}{⨍_W \psi} \le A\delta^{1-q}|||h|||_-; \tag{7.4}$$

$$\forall\, W^1, W^2 \in \mathcal{W}^s_-(\delta) : d_{\mathcal{W}^s}(W^1, W^2) \le \delta, \forall \psi_i \in \mathcal{D}_{a,\alpha}(W^i) : d_*(\psi_1, \psi_2) = 0,$$

$$\left| \frac{\int_{W^1} h\psi_1}{⨍_{W^1} \psi_1} - \frac{\int_{W^2} h\psi_2}{⨍_{W^2} \psi_2} \right| \le d_{\mathcal{W}^s}(W^1, W^2)^\gamma\, \delta^{1-\gamma} cA|||h|||_- \Bigg\}. \tag{7.5}$$

Let δ_0 be small enough so that [9, eq. (3.8)] holds true. Next, we assume the parameters of the cone satisfy the constraints detailed in [9, Section 5.3]. In particular

$$\begin{aligned} &q \in (0, 1/2), \quad 0 < \beta < \alpha \le 1/3, \quad \gamma \le \min\{\alpha - \beta, q\} \\ &3\delta < \delta_0, \quad e^{a2(\delta_0)^\beta} \le 2, \quad 4A\bar{C}_0\delta \le \delta_0/4, \quad c > 16C_s^q \end{aligned} \tag{7.6}$$

where $\bar{C}_0 > 1$ is defined in [9, Lemm 3.3] and $C_s := \sqrt{1+(\mathcal{K}_*^{-1}+\tau_*^{-1})^2}$ is the maximum slope of cone-stable curves in $\mathcal{W}^s$. The fundamental property of the above cone is the following.

Theorem 7.2. [9, Theorem 6.12] *Fix constants c, L and A satisfying the conditions of* [9, Section 5.3]. *Then there exists $\delta_* > 0$ such that for all $\delta \in (0, \delta_*)$ there exists $N_{\mathcal{F}} = N_{\mathcal{F}}(\delta)$ and $\varepsilon > 0$ such that $\mathcal{L}_{T_{N_{\mathcal{F}}}} \cdots \mathcal{L}_{T_1}(\mathcal{C}_{c,A,L}(\delta)) \subset \mathcal{C}_{c,A,L}(\delta)$ with uniformly finite diameter in $\mathcal{C}_{c,A,L}(\delta)$ for all choices of $Q_k \in \mathcal{Q}(\tau_*, \mathcal{K}_*, E_*)$ with $T_j \in \mathcal{F}(Q_k, E_*; \varepsilon)$, $j = 1, \dots, N_{\mathcal{F}}$.*

Since all the parameters are fixed from here forward, to simplify notation and to match notation with Section 5, we will refer to $\mathcal{C}_{c,A,L}(\delta)$ simply as $C_{\mathbb{R}}$ in what follows.

7.2. Conditions (C-1)-(C-4).

Having defined the cone we can now check the conditions on the dynamics.

Proposition 7.3. *Fix $\tau_*, \mathcal{K}_*, E_* > 0$, and let $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ be as defined above. Then conditions (C-1)-(C-4) of Section 2 hold with $M_k = \mathcal{M}$, $\mu_k = \mu_{SRB}$, $C_k = \mathcal{C}_{c,A,L}(\delta)$ and $f_k = T_{N_{\mathcal{F}}} \circ \cdots \circ T_1$, $T_j \in \mathcal{F}(Q_k, E_*; \varepsilon)$.*

Proof. Remark that since $C_k = \mathcal{C}_{c,A,L}(\delta)$ is the same for all k, then $\|\cdot\|_k = \|\cdot\|$ is independent of k.

Condition (C-1) follows from [9, Remark 7.2], while Condition (C-2) follows from [9, Theorems 6.12 and 6.13].

Condition (C-3) instead follows from the beginning of the proof of [9, Lemma 8.15] where $\mathbb{e}$ is chosen to be the constant function 1.

To prove condition (C-4) note that by choice of our reference measure, we have $\mathcal{L}_k 1 = 1$. Hence,

$$-\|h\|\mathbb{e} = -\|h\|\mathcal{L}_k\mathbb{e} \preceq \mathcal{L}_k h \preceq \|h\|\mathcal{L}_k\mathbb{e} \preceq \|h\|\mathbb{e}$$

which implies $\|\mathcal{L}_k h\| \le \|h\|$, which proves (C-4) with $C_* = 1$. □

Before introducing the observables for which we will prove the CLT and verifying conditions (O-1)-(O-3), we need to present a description of the cone adapted to the theory developed in Section 5.

7.3. Alternative Description of the Cone $C_{\mathbb{R}}$.

So that our notation for the cone $C_{\mathbb{R}}$ coincides with the formulation in Section 5, we present the following equivalent description of $C_{\mathbb{R}}$ in terms of the positivity of a set of linear functionals. With the parameters $a, \alpha, \beta, \gamma, q, c, A, L, \delta$ fixed as above, define a corresponding set of linear functionals $\mathcal{S}$ on $\mathcal{A}$ as follows:

$$\ell_{W^1,W^2,\psi_1,\psi_2,L}(h) = L\frac{\int_{W^1} h\psi_1}{\int_{W^1}\psi_1} - \frac{\int_{W^2} h\psi_2}{\int_{W^2}\psi_2}, \quad W^i \in \mathcal{W}^s(\delta), \psi_i \in \mathcal{D}_{a,\beta}(W^i). \tag{7.7}$$

$$\begin{aligned}\ell_{W^1,W^2,\psi_1,\psi_2,q,\pm}(h) = A\delta^{1-q}\frac{\int_{W^0} h\psi_0}{\int_{W^0}\psi_0} \pm |W^1|^{1-q}\frac{\int_{W^1} h\psi_1}{\int_{W^1}\psi_1},\\ W^0 \in \mathcal{W}^s(\delta), W^1 \in \mathcal{W}^s_-(\delta), \psi_i \in \mathcal{D}_{a,\beta}(W^i).\end{aligned} \tag{7.8}$$

$$\begin{aligned}\ell_{W^0,W^1,W^2,\psi_0,\psi_1,\psi_2,\gamma,\pm}(h) &= d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}cA\delta^{1-q}\frac{\int_{W^0} h\psi_0}{\int_{W^0}\psi_0}\\ &\quad \pm\left[\frac{|W^1|\int_{W^1} h\psi_2}{\int_{W^1}\psi_1} - \frac{|W^2|\int_{W^2} h\psi_3}{\int_{W^1}\psi_2}\right],\\ &W^0 \in \mathcal{W}^s(\delta), \psi_0 \in \mathcal{D}_{a,\beta}(W^0), W^1, W^2 \in \mathcal{W}^s_-(\delta), \psi_i \in \mathcal{D}_{a,\alpha}(W^i),\\ &d_{\mathcal{W}^s}(W^1,W^2) \le \delta, d_*(\psi_1,\psi_2) = 0.\end{aligned} \tag{7.9}$$

Let $\mathcal{S}$ denote the collection of all such linear functionals as W^i and ψ_i range over the stated sets of stable curves and test functions.

Lemma 7.4. *Let $\mathcal{S}$ be as defined above. Then,*

$$C_{\mathbb{R}} = \{h \in \mathcal{A}\setminus\{0\} \ : \ \ell(h) \ge 0, \forall \ell \in \mathcal{S}\}.$$

Proof. The equivalences between the three classes of linear functionals defined in (7.7)-(7.9) and the three cone conditions (7.3)-(7.5) is immediate.

Consider, for example, (7.3). If $|||h|||_+ \le L|||h|||_-$, then by definition of $|||\cdot|||_+$ and $|||\cdot|||_-$ it follows that $\ell_{W^1,W^2,\psi_1,\psi_2,L}(h) \ge 0$ for all $W^1, W^2 \in \mathcal{W}^s(\delta)$ and $\psi_i \in \mathcal{D}_{a,\beta}(W^i)$. On the other hand if $\ell_{W^1,W^2,\psi_1,\psi_2,L}(h) \ge 0$ for all such W^i and ψ_i, then it follows that

$$L \inf_{\substack{W^1\in\mathcal{W}^s(\delta)\\ \psi_1\in\mathcal{D}_{a,\beta}(W^1)}} \frac{\int_{W^1} h\psi_1}{\int_{W^1}\psi_1} \ge \sup_{\substack{W^2\in\mathcal{W}^s(\delta)\\ \psi_2\in\mathcal{D}_{a,\beta}(W^2)}} \frac{\int_{W^2} h\psi_2}{\int_{W^2}\psi_2},$$

which is precisely (7.3).

The other conditions are similarly parallel, with (7.4) corresponding to the set of functionals in (7.8) and (7.5) corresponding to the set of functionals in (7.9). □

We now define the norm $\|h\|$ precisely as in (5.4), letting $\mathbb{e} = 1$. Then $\mathcal{B}_{\mathbb{R}}$ is the completion of $\mathcal{A}$ in this norm, It follows that $\mathcal{S} \subset \mathcal{B}'_{\mathbb{R}}$.

Finally, following (5.7) we define the larger cone $\mathcal{C}_{\mathbb{R}} \supset C_{\mathbb{R}}$ by

$$\mathcal{C}_{\mathbb{R}} = \{h \in \mathcal{B}_{\mathbb{R}}\setminus\{0\} : \ell(h) \ge 0, \forall \ell \in \mathcal{S}\}.$$

7.4. **Introduction of Observables and Some Basic Properties.**

Although we work with sequential maps of the form $f_k = T_{N_{\mathcal{F}}} \circ \cdots \circ T_1$, $T_j \in \mathcal{F}(Q_k, E_*; \varepsilon)$ according to Proposition 7.3, we would like to prove the Central Limit Theorem for observables sampled at each billiard collision rather than at each $N_{\mathcal{F}}$ collisions. To this end, we will work with observables of the form

$$g_k = \sum_{j=0}^{N_{\mathcal{F}}-1} \tilde{g}_{k,j} \circ T_j \circ \cdots \circ T_1, \tag{7.10}$$

where each $\tilde{g}_{k,j} \in C^\alpha(M)$,[27] with $\alpha \in (0, 1/3)$ from the definition of $C_{\mathbb{R}}$, and with $|\tilde{g}_{k,j}|_{C^\alpha} \le \tilde{K}$ for some $\tilde{K} > 0$ and all $k \in \mathbb{N}$, $j \le N_{\mathcal{F}} - 1$.

For this class of observables, we will prove (O-1) in several steps. The first step is the following Lemma, whose proof is postponed to Section 7.10.

Lemma 7.5. *There exists $C_0 > 0$ such that for all $g \in C^\alpha(M)$ and $h \in \mathcal{C}_{\mathbb{R}}$, $\|gh\| \le C_0 |g|_{C^\alpha} |||h|||_-$. In particular, setting $C_\star = c + 8C_s^q + 4(c + 2^q \delta^\gamma 10a) + 8\delta^\gamma + 12C_s(2/c)^{\frac{1}{q}-1}$, if*

$$\begin{aligned} &\lambda > \Xi_* |g|_{C^\alpha} |||h|||_- \,, \quad \textit{with} \\ &\Xi_* = \max\left\{ \frac{(3+2L)L(1+2a)}{L-1}, \frac{(5+2L)A(1+2a)}{A-2^{1-q}}, \frac{C_\star + 2cA(1+2a)L}{cA - 2C_s} \right\}, \end{aligned} \tag{7.11}$$

then $\lambda \pm gh \in \mathcal{C}_{\mathbb{R}}$ and so $\|gh\| \le \lambda$.

Remark 7.6. *Note that if $\lambda > 0$, $h \in \mathcal{C}_{\mathbb{R}}$ and $\lambda - h \in \mathcal{C}_{\mathbb{R}}$, then necessarily,*

$$\frac{\int_W (\lambda - h)\psi}{\int_W \psi} > 0 \quad \Longrightarrow \quad \lambda > \frac{\int_W h\psi}{\int_W \psi},$$

for each $W \in \mathcal{W}^s(\delta)$ and $\psi \in \mathcal{D}_{a,\beta}(W)$. Taking the supremum over W and ψ, we conclude that $\|h\| \ge |||h|||_+$.

On the other hand, according to Lemma 7.5 with $g \equiv 1$, $\|h\| \le C_0 |||h|||_-$. Thus for $h \in \mathcal{C}_{\mathbb{R}}$, the two quantities are equivalent, i.e.

$$|||h|||_- \le |||h|||_+ \le \|h\| \le C_0 |||h|||_- \,.$$

Corollary 7.7. *If $h \in \mathcal{B}_{\mathbb{R}}$ and $g \in C^\alpha(M)$, then $gh \in \mathcal{B}_{\mathbb{R}}$ and $\|gh\| \le 3C_0 |g|_{C^\alpha} \|h\|$, where $C_0 > 0$ from Lemma 7.5 is independent of g and h.*

Proof. For $h \in \mathcal{B}_{\mathbb{R}}$, $h \ne 0$, by (5.4), $h + \|h\|, \|h\| \in \mathcal{C}_{\mathbb{R}} \cup \{0\}$ (recalling that in the present setting $\mathbb{e} = 1$). Then writing $gh = g(h + \|h\|) - g\|h\|$ implies

$$\|gh\| \le \|g(h + \|h\|)\| + \|g\|h\|\,\| \le C_0 |g|_{C^\alpha} \|h + \|h\|\,\| + C_0 |g|_{C^\alpha} \|h\|$$

where we have used the fact that $\|1\| = 1$ and applied Lemma 7.5 twice. □

Remark 7.8. *Theorem 7.2 provides a bound on the norm only for sequences of operators of length $N_{\mathcal{F}}$. To obtain finer information (for example, a CLT for observables sampled at each billiard collision rather than sampled once every $N_{\mathcal{F}}$ collisions) requires us to estimate uniformly the norms of the operators $\mathcal{L}_T$ for each $T \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$. Note that this does not follow from [9]. In fact, the estimates in [9] cannot even be used to prove that $\mathcal{L}_T \in L(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$.*

The issue pointed out in Remark 7.8 is solved by the next Proposition proven in Section 7.11.

Proposition 7.9. *There exists $C_L > 0$ such that for all $T \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ and all $h \in \mathcal{B}_{\mathbb{R}}$, $\|\mathcal{L}_T h\| \le C_L \|h\|$.*

7.5. **Verification of Hypothesis (O-1).**

To verify (O-1), we begin with $j = 1$. We must bound the norm of $\mathcal{L}_k(g_k h)$ for $h \in \mathcal{B}_{\mathbb{R}}$. Since $\mathcal{L}_k = \mathcal{L}_{T_{N_{\mathcal{F}}}} \cdots \mathcal{L}_{T_1}$, we have

$$\mathcal{L}_k(g_k h) = \sum_{j=0}^{N_{\mathcal{F}}-1} \mathcal{L}_k(\tilde{g}_{k,j} \circ T_j \circ \cdots \circ T_1 \cdot h) = \sum_{j=0}^{N_{\mathcal{F}}-1} \mathcal{L}_{T_{N_{\mathcal{F}}}} \cdots \mathcal{L}_{T_{j+1}} (\tilde{g}_{k,j} \mathcal{L}_{T_j} \cdots \mathcal{L}_{T_1} h).$$

[27] We use the norm $|g|_{C^\alpha} = \|g\|_\infty + \sup_{x,y} \frac{|g(x)-g(y)|}{d(x,y)^\alpha}$, so that C^α is a Banach algebra.

Using the triangle inequality and applying Proposition 7.9 $N_{\mathcal{F}} - j$ times to the jth term yields,

$$\begin{aligned} \|\mathcal{L}_k(g_k h)\| &\leq \sum_{j=0}^{N_{\mathcal{F}}-1} C_L^{N_{\mathcal{F}}-j} \|\tilde{g}_{k,j}\mathcal{L}_{T_j}\cdots\mathcal{L}_{T_1}h\| \\ &\leq \sum_{j=0}^{N_{\mathcal{F}}-1} C_L^{N_{\mathcal{F}}} 3C_0|\tilde{g}_{k,j}|_{C^\alpha}\|h\| \leq 3N_{\mathcal{F}}C_L^{N_{\mathcal{F}}}C_0\tilde{K}\|h\|\,, \end{aligned} \tag{7.12}$$

where we have applied Corollary 7.7 followed by Proposition 7.9 again in the second line. This proves (O-1) for $j = 1$ with $K = 3N_{\mathcal{F}}C_L^{N_{\mathcal{F}}}C_0\tilde{K}$.

For $j = 2$, we proceed similarly. Now,

$$\begin{aligned} g_k^2 &= \sum_{i,j=0}^{N_{\mathcal{F}}-1} \tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1\cdot\tilde{g}_{k,i}\circ T_i\circ\cdots T_1 \\ &= \sum_{j=0}^{N_{\mathcal{F}}-1} \tilde{g}_{k,j}^2\circ T_j\circ\cdots\circ T_1 + 2\sum_{j=1}^{N_{\mathcal{F}}-1}\sum_{i=0}^{j-1}\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1\cdot\tilde{g}_{k,i}\circ T_i\circ\cdots T_1\,. \end{aligned}$$

Multiplying by h and applying $\mathcal{L}_k = \mathcal{L}_{T_{N_{\mathcal{F}}}}\cdots\mathcal{L}_{T_1}$ yields,

$$\begin{aligned} \mathcal{L}_k(g_k h) = &\sum_{j=0}^{N_{\mathcal{F}}-1} \mathcal{L}_{T_{N_{\mathcal{F}}}}\cdots\mathcal{L}_{T_{j+1}}(\tilde{g}_{k,j}^2\mathcal{L}_{T_j}\cdots\mathcal{L}_{T_1}h) \\ &+ 2\sum_{j=1}^{N_{\mathcal{F}}-1}\sum_{i=0}^{j-1}\mathcal{L}_{T_{N_{\mathcal{F}}}}\cdots\mathcal{L}_{T_{j+1}}(\tilde{g}_{k,j}\mathcal{L}_{T_j}\cdots\mathcal{L}_{T_{i+1}}(\tilde{g}_{k,i}\mathcal{L}_{T_i}\cdots\mathcal{L}_{T_1}h))\,. \end{aligned}$$

And applying Lemma 7.5 and Proposition 7.9 and Corollary 7.7 as in (7.12) yields,

$$\|\mathcal{L}_k(g_k^2 h)\| \leq N_{\mathcal{F}}C_L^{N_{\mathcal{F}}}3C_0\tilde{K}^2\|h\| + N_{\mathcal{F}}(N_{\mathcal{F}}-1)C_L^{N_{\mathcal{F}}}9C_0^2\tilde{K}^2\|h\| \leq 9C_0^2N_{\mathcal{F}}^2\tilde{K}^2C_L^{N_{\mathcal{F}}}\|h\|,$$

where we have used the fact that $|\tilde{g}_{k,j}^2|_{C^\alpha} \leq |\tilde{g}_{k,j}|_{C^\alpha}^2$. This proves (O-1) for $j = 2$ since the constant is $\leq K^2$.

For $j = 3$ the process is similar, using the fact that

$$g_k^3 = \sum_{i,j,\ell=0}^{N_{\mathcal{F}}-1} \tilde{g}_{k,\ell}\circ T_\ell\circ\cdots\circ T_1\cdot\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1\cdot\tilde{g}_{k,i}\circ T_i\circ\cdots T_1\,.$$

Then estimating precisely as in the case $j = 2$, we obtain,

$$\|\mathcal{L}_k(g_k^3 h)\| \leq N_{\mathcal{F}}^3 27C_0^3\tilde{K}^3C_L^{N_{\mathcal{F}}}\|h\|\,,$$

which is the required estimate for $j = 3$. The same strategy works for any $j \in \mathbb{N}$, completing the verification of (O-1) with $j_0 = \infty$.

7.6. Contraction of the Complex Cone $\mathcal{C}_{\mathbb{C}}$.

With $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{C}_{\mathbb{R}}$ defined as in Section 7.3, we proceed to define their complexifications $\mathcal{B}_{\mathbb{C}}$ and $\mathcal{C}_{\mathbb{C}}$ precisely as in Section 5, specifically Definition 5.3.

In light of Remark 7.6, we define

$$\mathbb{m}(h) := \int_M h\,d\mu_{\mathrm{SRB}} \qquad \text{for } h \in C_{\mathbb{R}}. \tag{7.13}$$

By [9, Remark 7.2], there exists $\bar{C} \geq 1$ such that

$$\bar{C}^{-1}|||h|||_- \leq \mathbb{m}(h) \leq \bar{C}|||h|||_+, \tag{7.14}$$

and since $|||\cdot|||_-$ is equivalent to $\|\cdot\|$ by Remark 7.6, $\mathbb{m}$ extends to a linear functional on $\mathcal{C}_{\mathbb{R}}$ and thus to $\mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{\mathbb{C}}$. Indeed, disintegrating μ_{SRB} as in the proof of [9, Lemma 7.1] it follows that $\mathbb{m} \in \mathcal{S}_*$, where, as in Section 5 (see definition (5.6)), $\mathcal{S}_*$ denotes the weak-* closure of the convex hull of $\{\lambda\ell : \lambda \in \mathbb{R}_+, \ell \in \mathcal{S}\}$. Now $\mathbb{m}(1) = 1$ and combining (7.14) with Remark 7.6 implies that $\mathbb{m}$ satisfies (5.6) and therefore (5.11) with $\kappa = C_0^{-1}\bar{C}^{-1}$.

For fixed $n \in \mathbb{N}$, and $j \le k \le n$, $\lambda > 0$, define the complex operators $\mathcal{L}_{k,\lambda}$ and their concatenations as in (3.7),[28]

$$\mathcal{L}_{k,\lambda} = \mathcal{L}_k(e^{i\lambda\sigma_n^{-1}g_k}h), \quad \mathcal{L}_{k,j,\lambda} = \mathcal{L}_k \cdots \mathcal{L}_j(e^{i\lambda\sigma_n^{-1}\sum_{m=j}^k g_m \circ f_{m,j}}h)\,.$$

The main result of this section is the following.

Theorem 7.10. *Let $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ and $f_k = T_{N_\mathcal{F}} \circ \cdots \circ T_1$ be as in Proposition 7.3. Let $\mathcal{C}_\mathbb{R}$ be as defined in (7.3)-(7.5) and Lemma 7.4, and let $\mathcal{C}_\mathbb{C}$ be its complexification as in Definition 5.3 with associated projective metric $\delta_\mathcal{C}$ as in (5.14).*

There exists $\lambda_0 > 0$ such that for all $k, n \in \mathbb{N}$, $k \le n$, and all $\lambda \in \mathbb{R}$ such that $|\lambda|\sigma_n^{-1} < \lambda_0$, $\mathcal{L}_{k,\lambda}(\mathcal{C}_\mathbb{C}) \subset \mathcal{C}_\mathbb{C}$ and $\operatorname{diam}_{\delta_\mathcal{C}}(\mathcal{L}_{k,\lambda}(\mathcal{C}_\mathbb{C})) =: \Delta_\mathbb{C} < \infty$.

Proof. We will apply Theorem 5.17 to the operators $\mathcal{L}_{k,\lambda}$. First, by Proposition 7.3, the real transfer operators $\mathcal{L}_k$ satisfy $\mathcal{L}_k \in L(\mathcal{B}_\mathbb{R})$ and $\operatorname{diam}_H(\mathcal{L}(\mathcal{C}_\mathbb{R})) := \Delta_\mathbb{R} < \infty$. Moreover, Lemma 7.5 with $g \equiv 1$ implies that $\mathbb{e} = 1$ satisfies (5.3), while (7.14) and Remark 7.6 imply that $\mathbb{m}(\cdot) = \mu_{\mathrm{SRB}}(\cdot)$ satisfies (5.6).

Next, for $h \in \mathcal{B}_\mathbb{R}$,

$$\mathcal{L}_{k,\lambda}h = \mathcal{L}_k(e^{i\lambda\sigma_n^{-1}g_k}h) = \mathcal{L}_k(\cos(\lambda\sigma_n^{-1}g_k)h) + i\mathcal{L}_k(\sin(\lambda\sigma_n^{-1}g_k)h)\,. \tag{7.15}$$

Then temporarily labelling $t = \lambda\sigma_n^{-1}$ for brevity, we can write,

$$\cos(t(\tilde{g}_{k,1} \circ T_1 + \tilde{g}_{k,0})) = \cos(t\tilde{g}_{k,1}) \circ T_1 \cos(t\tilde{g}_{k,0}) + \sin(t\tilde{g}_{k,1}) \circ T_1 \sin(t\tilde{g}_{k,0})\,,$$

so that inductively, using Corollary 7.7 and estimating as in (7.12), it follows that $\mathcal{L}_{k,\lambda}$ defines a bounded operator on $\mathcal{B}_\mathbb{C}$ by Lemma 2.5.

It remains to verify condition (5.23) of Theorem 5.17. We will do this via Lemma 5.18, which implies (5.23). Recalling (7.15), Lemma 5.18 with $a(h) = \mathcal{L}_k(\cos(\lambda\sigma_n^{-1}g_k)h)$ and $b(h) = \mathcal{L}_k(\sin(\lambda\sigma_n^{-1}g_k)h)$ follows immediately from the following lemma.

Lemma 7.11. *Let g_k be as in (7.10). For all $\varepsilon > 0$ there exists $t_0 > 0$ such that for all $h \in \mathcal{C}_\mathbb{R}$, all f_k and all $t \in (-t_0, t_0)$,*

$$\mathcal{L}_k[\varepsilon h \pm 2h(1 - \cos(tg_k))] \in \mathcal{C}_\mathbb{R}\,,$$
$$\mathcal{L}_k[\varepsilon h \pm 2h\sin(tg_k)] \in \mathcal{C}_\mathbb{R}\,.$$

Proof of Lemma 7.11. Let $\psi_t = 2(1 - \cos(tg_k))$ and let $h \in \mathcal{C}_\mathbb{R}$. First we claim that

$$\mathcal{L}_k(\psi_t h) \in \mathcal{B}_\mathbb{R} \quad \text{and} \quad \|\mathcal{L}_k(\psi_t h)\| \le C|t|\|h\|\,, \tag{7.16}$$

for some $C > 0$ independent of t, k and h.

To prove the claim, expand $\cos(tg_k)$ using (7.10) to obtain,

$$\cos(tg_k) = \prod_{j=0}^{N_\mathcal{F}-1} \cos(t\tilde{g}_{k,j} \circ T_j \circ \cdots \circ T_1) + P_k(t)\,, \tag{7.17}$$

where $P_k(t)$ is a polynomial of degree $N_\mathcal{F}$ in terms involving $\cos(t\tilde{g}_{k,j})$ and $\sin(t\tilde{g}_{k,j})$. Indeed, every term in $P_k(t)$ has at least one factor of the form $\sin(t\tilde{g}_{k,j} \circ T_j \circ \cdots \circ T_1)$.

First we estimate the leading term of $\mathcal{L}_k(\psi_t h)$,

$$\begin{aligned}\mathcal{L}_k\Big(\Big(1 - \prod_{j=0}^{N_\mathcal{F}-1} \cos(t\tilde{g}_{k,j} \circ T_j \circ \cdots \circ T_1)\Big)h\Big) &= \mathcal{L}_k\big((1 - \cos(t\tilde{g}_{k,0}))h\big) \\ + \sum_{\ell=1}^{N_\mathcal{F}-1} \mathcal{L}_k\Big(\prod_{j=0}^{\ell-1} \cos(t\tilde{g}_{k,j} \circ T_j \circ \cdots \circ T_1) &- \prod_{j=0}^{\ell} \cos(t\tilde{g}_{k,j} \circ T_j \circ \cdots \circ T_1)\Big)h\Big).\end{aligned} \tag{7.18}$$

By Lemma 7.5 and Proposition (7.9),

$$\|\mathcal{L}_k\big((1 - \cos(t\tilde{g}_{k,0}))h\big)\| \le C_L^{N_\mathcal{F}} C_0 |1 - \cos(t\tilde{g}_{k,0})|_{C^\alpha}\|h\| \le C'|t|\|h\|\,.$$

[28] Recall that $\mathcal{L}_k = \mathcal{L}_{T_{N_\mathcal{F}}} \cdots \mathcal{L}_{T_1}$.

Next, for each ℓ, using that $\mathcal{L}_{T_j}1 = 1$,

$$
\begin{aligned}
&\mathcal{L}_k\Big(\prod_{j=0}^{\ell-1}\cos(t\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1) - \prod_{j=0}^{\ell}\cos(t\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1)\Big)h\Big)\\
&= \mathcal{L}_k\Big(\big(1-\cos(t\tilde{g}_{k,\ell}\circ T_\ell\circ\cdots\circ T_1)\prod_{j=0}^{\ell-1}\cos(t\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1)h\Big)\\
&= \mathcal{L}_{T_{N_{\mathcal{F}}}}\cdots\mathcal{L}_{T_{\ell+1}}\Big(\big(1-\cos(t\tilde{g}_{k,\ell})\big)\mathcal{L}_{T_\ell}\big(\cos(t\tilde{g}_{k,\ell-1})\\
&\quad\times\mathcal{L}_{T_{\ell-1}}\big(\cos(t\tilde{g}_{k,\ell-2})\cdots\mathcal{L}_{T_1}\big(\cos(t\tilde{g}_{k,0})h\big)\big)\big)\Big).
\end{aligned}
\tag{7.19}
$$

Again using Lemma 7.5 and Proposition (7.9), the norm of this expression is bounded by

$$
\begin{aligned}
&C_L^{N_{\mathcal{F}}-\ell-1}C_0|1-\cos(t\tilde{g}_{k,\ell})|_{C^\alpha}\left\|\mathcal{L}_{T_\ell}\left(\cos(t\tilde{g}_{k,\ell-1})\mathcal{L}_{T_{\ell-1}}\big(\cdots\mathcal{L}_{T_1}\big(\cos(t\tilde{g}_{k,0})h\big)\big)\right)\right\|\\
&\qquad\le C_L^{N_{\mathcal{F}}}C_0^{\ell+1}\tilde{K}^\ell|1-\cos(t\tilde{g}_{k,\ell})|_{C^\alpha}\|h\| \le C''|t|\|h\|\,.
\end{aligned}
$$

Combining these estimates in (7.18) and using the triangle inequality proves the required bound for the leading term of $\mathcal{L}_k(\psi_t h)$ from (7.17). According to (7.17), the remaining terms of $\mathcal{L}_k(\psi_t h)$ are given by $\mathcal{L}_k(P_k(t)h)$. These terms can be estimated in a similar manner as the product of cosines. Since each term of $\mathcal{L}_k(P_k(t)h)$ contains one factor of the form $\sin(t\tilde{g}_{k,j})$ and $|\sin(t\tilde{g}_{k,j})|_{C^\alpha} = \mathcal{O}(|t|)$, the claim (7.16) is proved.

Since $\mathcal{L}_k(\psi_t h)\in\mathcal{B}_{\mathbb{R}}$, there exists $E_t\in\mathbb{R}^+$ with $E_t\le C|t|\|h\|_-$ such that $E_t\pm\mathcal{L}_k(\psi_t h)\in\mathcal{C}_{\mathbb{R}}$. Then,

$$
\mathcal{L}_k(\varepsilon h\pm h\psi_t) = \mathcal{L}_k(\varepsilon h - E_t) + E_t \pm \mathcal{L}_k(h\psi_t) = \varepsilon(\mathcal{L}_k h - \varepsilon^{-1}E_t) + E_t \pm \mathcal{L}_k(h\psi_t)\,,
$$

where we have used that $\mathcal{L}_k 1 = 1$. The second term is an element of $\mathcal{C}_{\mathbb{R}}$ by choice of E_t.

For the first term, we use a stronger fact: that $\operatorname{diam}_H(\mathcal{L}_k(\mathcal{C}_{\mathbb{R}}))\le\Delta_{\mathbb{R}}<\infty$. In fact, by [9, Prop. 6.13] for $h\in\mathcal{C}_{\mathbb{R}}$, we have $\alpha(1,\mathcal{L}_k h)\ge\frac{(1-\chi)A}{A+1}\|\mathcal{L}_k h\|_-$, where A is the cone parameter from (7.4) and $\chi<1$ is the contraction in cone parameters.

It follows that

$$
\mathcal{L}_k h - \frac{E_t}{\varepsilon}\in\mathcal{C}_{\mathbb{R}}\quad\text{if}\quad\frac{E_t}{\varepsilon}<\alpha(1,\mathcal{L}_k h),\quad\text{which is implied by}\quad\frac{E_t}{\varepsilon}<\frac{(1-\chi)A}{A+1}\|\mathcal{L}_k h\|_-\,.
$$

By choice of E_t and [9, Lemma 5.4], $E_t\le C|t|\|h\|_-\le 2C|t|\|\mathcal{L}_k h\|_-$, so the above condition is satisfied if

$$
|t|<\varepsilon\frac{(1-\chi)A}{2C(A+1)}\,.
$$

Taking this as our value of t_0 proves the first statement of the lemma.

Similarly, setting $\tilde{\psi}_t = 2\sin(tg_k)$, we can expand $2\sin(tg_k) = \tilde{P}_k(t)$ where $\tilde{P}_k(t)$ is a polynomial in $\sin(t\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1)$ and $\cos(t\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1)$ and each term of $\tilde{P}_k(t)$ has a factor of the form $\sin(t\tilde{g}_{k,j}\circ T_j\circ\cdots\circ T_1)$. Then estimating as in (7.19), and using the fact that $|\sin(t\tilde{g}_{k,j})|_{C^\alpha} = \mathcal{O}(|t|)$, we conclude,

$$
\mathcal{L}_k(\tilde{\psi}_t h)\in\mathcal{B}_{\mathbb{R}}\quad\text{and}\quad\|\mathcal{L}_k(\tilde{\psi}_t h)\|\le\tilde{C}|t|\|h\|\,,
$$

for some $\tilde{C}>0$ independent of t, k and h. From this point, the same proof with $\tilde{\psi}_t$ replacing ψ_t yields the second statement of the lemma. □

Fixing $\varepsilon<\frac{\kappa^2}{48\sqrt{2}}e^{-2\Delta_{\mathbb{R}}}$, let $t_0>0$ be from Lemma 7.11. Set $\lambda_0 = t_0$. Then for all $\lambda\in\mathbb{R}$ such that $|\lambda|\sigma_n^{-1}<\lambda_0$, setting $t=\lambda\sigma_n^{-1}$, we have $|t|<t_0$ so that Lemma 7.11 implies that Lemma 5.18 and therefore (5.23) holds for $\mathcal{L}_{k,\lambda}$. This completes the proof of the theorem. □

7.7. **Verification of (O-2) and (O-3).**

Throughout this section, we assume $|\lambda|<\lambda_0\sigma_n$, where λ_0 is from Theorem 7.10. From our choices of $\mathbb{e}_k = 1$, $\mathbb{m}_k(h) = \int_M h\,d\mu_{\mathrm{SRB}}$, property (a) of Section 5.3 holds with $\bar{\kappa}=\kappa = C_0^{-1}\bar{C}^{-1}$ from Section 7.6. Moreover, property (b) of Section 5.3 is trivially satisfied with $K=1$, and property (c) follows from Theorem 7.10 with $\mathbb{L}_k = \mathcal{L}_{k,\lambda,n}$.

Hence setting $\alpha_{k,j,\lambda} = \boldsymbol{\alpha}_{k,j}$, $h_{k,j,\lambda} = \boldsymbol{h}_{k,j}$ and $\ell_{k,j,\lambda} = \boldsymbol{\ell}_{k,j}$, Lemma 5.19 implies

$$
\|h_{k,j,\lambda}\|\le\frac{\sqrt{2}}{\kappa},\quad\|\ell_{k,j,\lambda}\|'\le\sqrt{2}
$$

while $|\alpha_{k,j,\lambda}| \leq 1$, by the invariance of the SRB measure. This implies,

$$\|\mathcal{L}_{k,j,\lambda} h\| \leq \frac{2}{\kappa}\|h\|,$$

which verifies (O-2) with $K = 2/\kappa$.

Condition (O-3) is verified exactly as in Section 6.4 since Theorem 7.10 implies that Theorem 5.15 applies to the operators $\mathcal{L}_{k,\lambda,n}$.

7.8. **Central Limit Theorem for Sequential Dispersing Billiards.** We collect our previous results on the contraction of the relevant operators in order to state the culminating Central Limit Theorem for sequential dispersing billiards. This is essentially a restatement of Theorem 2.2, but now all the relevant objects have been properly defined.

Theorem 7.12. *Fix $\tau_*, \mathcal{K}_*, E_* > 0$ and let $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ be the associated family of billiard maps. Let $\rho \in C_{\mathbb{R}}$, $\int \rho d\mu_{SRB} = 1$. For a sequence of observables $(\tilde{g}_{k,j})$ as defined in (7.10), choose an $N_{\mathcal{F}}$-admissible sequence $(T_i)_{i\in\mathbb{N}}$ as defined in Definition 7.1, and define the centered observables $\hat{g}_k$ as in (2.8). Then, recalling the distribution function $F_n(x)$ from (2.16), for all $c_\star \in (0,1)$ there exists $\bar{C} > 0$ such that, for all $n \in \mathbb{N}$ such that $\sigma_n \geq \max\{1, c_\star n^{1/3}\ln(n+1)\}$, we have*

$$\left| F_n(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \right| \leq \bar{C}\sigma_n^{-3}(\ln \sigma_n)^2 n.$$

Proof. We have verified hypotheses (C-1)-(C-4) in Section 7.2 and conditions (O-1)-(O-3) in Section 7.7, accordingly the theorem is an imediate application of Theorem 2.7 and Corollary 2.9. □

The remaining issue is to verify the condition for the growth of the variance.
We start with the following general estimate, recalling $h_k = \mathcal{L}_{k-1}\cdots\mathcal{L}_0\rho$,

$$\begin{aligned}
\sigma_n^2 &:= \int [\hat{S}_n](x)^2 \rho(x) d\mu \\
&= \sum_{k=0}^{n-1} \int \hat{g}_k^2 h_k + 2\sum_{k=0}^{n-1}\sum_{j=k+1}^{n-1} \int \hat{g}_j \mathcal{L}_{j-1}\cdots\mathcal{L}_k(\hat{g}_k h_k) \\
&= \sum_{k=0}^{n-1} \int \hat{g}_k^2 h_k + 2\sum_{k=0}^{n-1}\sum_{j=1}^{\infty} \int \hat{g}_{j+k} \mathcal{L}_{j+k-1}\cdots\mathcal{L}_k(\hat{g}_k h_k) + \mathcal{O}\left(\sum_{k=0}^{n-1}\sum_{j=n}^{\infty} \nu^{j-k}\right) \\
&= \sum_{k=0}^{n-1} \left[\int \hat{g}_k^2 h_k + 2\sum_{j=1}^{\infty} \int \hat{g}_{j+k} \mathcal{L}_{j+k-1}\cdots\mathcal{L}_k(\hat{g}_k h_k)\right] + \mathcal{O}(1),
\end{aligned} \tag{7.20}$$

where we have used hypotheses (O-1) together with (2.7), with $\nu = e^{-c}$. Note that (2.7) applies due to the centering given by (2.9). In the case of billiards, choosing $\rho = 1$ yields $h_k = 1$. Also, for simplicity, we set $d\mu = d\mu_{\text{SRB}}$ and choose g_k such that $\int g_k = 0$, thus $g_k = \hat{g}_k$.

Lemma 7.13. *If there exists $C_0 > 0$ such $C_0 \int \hat{g}_k^2 \geq \|g_k\|_{C^\alpha}^2$, then, for ν small enough, we have $\sigma_n^2 \geq \frac{1}{2}\sum_{k=0}^{n-1}\int \hat{g}_k^2$*

Proof. We use condition (O-1) and (2.7) (recall that we have set $e^{-c} = \nu$) to compute

$$\begin{aligned}
&\int \hat{g}_k^2 + 2\sum_{j=1}^{\infty}\int \hat{g}_{j+k}\mathcal{L}_{j+k-1}\cdots\mathcal{L}_k\hat{g}_k = \int \hat{g}_k^2 + 2\sum_{j=1}^{\infty}\int \mathcal{L}_{j+k}\hat{g}_{j+k}\mathcal{L}_{j+k-1}\cdots\mathcal{L}_k\hat{g}_k \\
&\geq \int \hat{g}_k^2 - 2K\sum_{j=1}^{\infty}\|\mathcal{L}_{j+k-1}\cdots\mathcal{L}_k\hat{g}_k\|_{j+k} \geq \int \hat{g}_k^2 - \frac{6K\nu}{1-\nu}\|g_k\|_{C^\alpha}^2 \\
&\geq \left[1 - \frac{6KC_0}{1-\nu}\nu\right]\int \hat{g}_k^2.
\end{aligned}$$

The formula (7.20) implies the Lemma. □

The above Lemma, although effective, is not very satisfactory as it is not easy to check and applies only if ν is small. It would be nice to have the equivalent of Proposition 6.1; unfortunately, it is not clear how to proceed. It seems natural to introduce a stable foliation, as in [32], but there are some technical problems that require extra ideas.

In the literature, there are some relevant results only in the random case [12, 13, 14], described in the next section. For completeness, we show that the random case can be treated also in our setting.

7.9. **Variance Growth in the Random Case.**
Let $\Omega = \mathcal{A}^{\mathbb{N}}$ be a probability space with a measure $\mathbb{P}_e$. Assume that $\mathbb{P}_e$ is shift invariant and ergodic with respect to the shift and that $\mathbb{P}_e(\{\omega_1 = a\}) > 0$ for each $a \in \mathcal{A}$.

Let $\{f_a\}_{a\in\mathcal{A}}$ be a set of maps of a compact Riemannian manifold M satisfying conditions (C-1)-(C-4) of Section 2.3 with constants independent of $a \in \mathcal{A}$. To simplify the exposition for this calculation of the variance, we take the manifold M and cone C to be fixed, while the maps are random. Also, we assume that the maps f_a preserve a common invariant measure $\mu \in \mathcal{B}$. We define $\mathcal{L}_a$ to be the transfer operator of f_a with respect to μ. This implies in particular that $\mathcal{L}_a 1 = 1$.

Let $\{g_a\}_{a\in\mathcal{A}} \subset L^2(M,\mathbb{R}) \cap \mathcal{B}'$ be a set of observables satisfying (O-1) with constants uniform for $a \in \mathcal{A}$. Note that in Section 7.5 it is proven that this is true if the f_a are billiard maps in the family $\mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$. Suppose that, for each $a \in \mathcal{A}$,

$$\int_M g_a d\mu = 0.$$

Let $\mathrm{g}(\omega, x) = g_{\omega_0}(x)$ for each $(\omega, x) \in \Omega \times M$ and consider the dynamics $F : \Omega \times M \to \Omega \times M$ defined by

$$F(\omega, x) = (\tau\omega, f_{\omega_0}(x)). \tag{7.21}$$

For each $\omega \in \Omega$, we have the sequential observables

$$\{\mathrm{g} \circ F^k\} = \{g_{\omega_k} \circ f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_0}\},$$

hence, for a given $\omega \in \Omega$, we can write

$$\hat{S}_n(\omega, x) = \sum_{k=0}^{n-1} g_{\omega_k} \circ f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_0}(x) = \sum_{k=0}^{n-1} \mathrm{g} \circ F^k(\omega, x)$$

$$\sigma_n^2(\omega) = \int_M \hat{S}_n(\omega, x)^2 d\mu.$$

The reader can check [12, 13, 14] and references therein for a discussion of a similar model (but limited to expanding maps) and [2] (for the case of toral automorphisms) using a related approach. The following result is a generalization of [2]; similar ideas can be applied to the case of expanding maps.

Theorem 7.14. *$\mathbb{P}_e$-a.s. the variance σ_n either grows linearly or there exists $\hat{\psi} \in L^2(M)$ such that $g_a = \hat{\psi} - \hat{\psi} \circ f_a$ for all $a \in \mathcal{A}$, and hence σ_n is uniformly bounded.*

Proof. Let $\beta : \Omega \to \mathbb{R}$ be defined by

$$\beta(\omega) = \int_M g_{\omega_0}^2 d\mu + 2\sum_{j=1}^{\infty} \int_M g_{(\tau^j\omega)_0} \mathcal{L}_{(\tau^{j-1}\omega)_0} \cdots \mathcal{L}_{\omega_0} g_{\omega_0} d\mu,$$

where τ is the shift. Then by (7.20) (taking $\rho = 1$ there), applied to the sequence determined by $\omega \in \Omega$,

$$\sigma_n^2(\omega) = \sum_{k=0}^{n} \beta(\tau^k\omega) + \mathcal{O}(1). \tag{7.22}$$

Lemma 7.15. *Either σ_n^2 grows linearly $\mathbb{P}_e$-a.s. or $\mathbb{E}_{\mathbb{P}_e}(\sigma_n^2) = \mathcal{O}(1)$.*

Proof. Integrating (7.22) yields $\mathbb{E}_{\mathbb{P}_e}(\sigma_n^2) = n\mathbb{E}_{\mathbb{P}_e}(\beta) + \mathcal{O}(1)$. It follows, by Birkhoff's ergodic theorem, that $\mathbb{P}_e$ almost surely

$$\lim_{n\to\infty} \frac{1}{n}\sigma_n^2 = \mathbb{E}_{\mathbb{P}_e}(\beta).$$

Thus, σ_n^2 can grow sublinearly only if $\mathbb{E}_{\mathbb{P}_e}(\beta) = 0$, but then $\mathbb{E}_{\mathbb{P}_e}(\sigma_n^2) = \mathcal{O}(1)$. □

Thus, a sublinear growth can happen only if $\mathbb{E}_{\mathbb{P}_e}(\sigma_n^2) = \mathcal{O}(1)$. If so, consider the annealed measure $\mathbb{P} := \mathbb{P}_e \times \mu$ and the related expectation $\mathbb{E}$.

The sequence $\hat{S}_n$ is then uniformly bounded in $L^2(\Omega \times M, \mathbb{P})$, hence it is weakly-compact. Let $\psi \in L^2(\Omega \times M, \mathbb{P})$ be an accumulation point. Then, for each $\varphi \in L^2(\Omega \times M, \mathbb{P})$, we have

$$\begin{aligned}\mathbb{E}(\varphi\psi) &= \lim_{j\to\infty} \mathbb{E}\Big(\varphi \sum_{k=0}^{n_j-1} g_{\omega_k} \circ f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_0}\Big)\\ &= \lim_{j\to\infty} \Big(\mathcal{L}_{\omega_0}\varphi \sum_{k=1}^{n_j-1} g_{\omega_k} \circ f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_1}\Big) + \mathbb{E}(\varphi g_{\omega_0})\\ &= \mathbb{E}\left(\mathcal{L}_{\omega_0}\varphi \cdot \psi \circ \tau\right) + \mathbb{E}(\varphi g_{\omega_0}) = \mathbb{E}(\varphi\psi \circ F) + \mathbb{E}(\varphi g_{\omega_0}).\end{aligned}$$

It follows that $g_{\omega_0}(x) = \psi(\omega, x) - \psi \circ F(\omega, x)$.

Next, let us define $\hat{\psi}(x) = \mathbb{E}_{\mathbb{P}_e}(\psi)$, $\bar{g} = \mathbb{E}_{\mathbb{P}_e}(\mathrm{g})$ and note that

$$\bar{g}(x) = \hat{\psi} - \mathbb{E}_{\mathbb{P}_e}(\hat{\psi} \circ f_{\omega_0}).$$

Furthermore, define

$$\mathrm{g} \circ F^k = g_{\omega_k} \circ f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_0}(x) = \hat{\psi} \circ F^k - \hat{\psi} \circ F^{k+1} + Z_k.$$

Note that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_e}(Z_k \mid \omega_0, \ldots, \omega_k) &= Z_k\\ \mathbb{E}_{\mathbb{P}_e}(Z_k \mid \omega_0, \ldots, \omega_{k-1}) &= 0,\end{aligned}$$

Hence, $M_n = \sum_{k=0}^{n-1} Z_k$ is a reverse martingale. It follows that

$$\mathbb{E}(\hat{S}_n^2) = \mathbb{E}\left([\hat{\psi} \circ F^n - \hat{\psi} + M_n]^2\right) \geq \mathbb{E}(M_n^2) - 2\mathbb{E}([\hat{\psi} \circ F^n - \hat{\psi}]^2)^{\frac{1}{2}}\mathbb{E}(M_n^2)^{\frac{1}{2}}$$

Since $\psi \in L^2$ it follows that there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}$,

$$C \geq \mathbb{E}(M_n^2) = \sum_{k=0}^{n-1} \mathbb{E}(Z_k^2) = n\mathbb{E}(Z_0^2).$$

Thus, it must be that

$$0 = \mathbb{E}(Z_0^2) = \mathbb{E}([\mathrm{g} + \hat{\psi} \circ F - \hat{\psi}]^2).$$

That is, $g_a + \hat{\psi} - \hat{\psi} \circ f_a = 0$ for each $a \in \mathcal{A}$. □

Remark 7.16. *Note that if the f_a are smooth Anosov maps and the observables g_a are smooth, then by Livšic's theory, see [6], we have that $\hat{\psi} \in \mathcal{C}^0$. Hence, the g_a must be continuous coboundaries, a condition that is checkable. On the contrary, it is not obvious how to check that the g_a are not L^2 coboundaries, so although the present section applies to billiards, the result is still not totally satisfactory. Nevertheless, our result is much sharper than the usual one, e.g. see [26,* Equation (2.10)*] where one has an L^2 coboundary only with respect to the map (7.21). A similar result is proven in [2], but limited to Anosov automorphisms.*

7.10. **Proof of Lemma 7.5.**
To estimate $\|gh\|$, we will find $\lambda > 0$ such that $\lambda \pm gh \in \mathcal{C}_{\mathbb{R}}$. To this end, it will be convenient to estimate the quantity $\frac{\int_W hg\psi\, dm_W}{\int_W \psi\, dm_W}$ from above and below, where $W \in \mathcal{W}^s(\delta)$ and $\psi \in \mathcal{D}_{a,\beta}(W)$. For brevity, we will omit dm_W when writing the integral.

For $g \in C^\alpha(M)$ and $\psi \in \mathcal{D}_{a,\beta}(W)$, we choose $B > 0$ so that $g\psi + B \in \mathcal{D}_{a,\beta}(W)$. Letting $H^\beta(g\psi)$ denote the Hölder constant of $g\psi$ with exponent β, let $B > a^{-1}H^\beta(g\psi) + |g\psi|_\infty$. It follows that

$$\frac{g(x)\psi(x) + B}{g(y)\psi(y) + B} \leq \frac{|g(x)\psi(x) - g(y)\psi(y)|}{g(y)\psi(y) + B} + 1 \leq e^{ad(x,y)^\beta}$$

for all $x, y \in M$, so that in particular, $g\psi + B \in \mathcal{D}_{a,\beta}(W)$. Note that $\psi \in \mathcal{D}_{a,\beta}(W)$ implies, for $x, y \in W$,

$$\left|\frac{\psi(x)}{\psi(y)} - 1\right| \leq e^{ad(x,y)^\beta} - 1 \leq ad(x,y)^\beta e^{a(2\delta)^\beta} \leq 2ad(x,y)^\beta\,, \tag{7.23}$$

so that $H^\beta(\psi) \le 2a|\psi|_{C^0}$, where we have used $e^{a(2\delta)^\beta} \le 2$ by [9, eq. (4.9)]. Since $a > 1$, we may choose

$$B \le |g|_{C^\beta}|\psi|_{C^\beta} \le |g|_{C^\alpha}(1+2a)|\psi|_{C^0}\,. \tag{7.24}$$

Now, since $|\psi|_{C^0} \le 2\psi$,

$$\begin{aligned}\frac{\int_W hg\,\psi}{\int_W \psi} &= \frac{\int_W h(g\psi+B)}{\int_W \psi} - B\frac{\int_W h}{\int_W \psi}\\ &\le \frac{\int_W h(g\psi+B)}{\int_W g\psi+B}\frac{\int_W g\psi+B}{\int_W \psi}\\ &\le 3(1+2a)|g|_{C^\alpha}|||h|||_+ \le 3(1+2a)L|g|_{C^\alpha}|||h|||_-\,,\end{aligned} \tag{7.25}$$

where we have dropped the second term since $|||h|||_- > 0$ for $h \in \mathcal{C}_\mathbb{R}$. Similarly, we estimate the lower bound, dropping the first term, which is positive,

$$\begin{aligned}\frac{\int_W hg\,\psi}{\int_W \psi} &= \frac{\int_W h(g\psi+B)}{\int_W \psi} - \frac{\int_W hB}{\int_W \psi}\\ &\ge -2(1+2a)|g|_{C^\alpha}|||h|||_+ \ge -2(1+2a)L|g|_{C^\alpha}|||h|||_-\,.\end{aligned} \tag{7.26}$$

Next we prove that if λ satisfies (7.11), then $\lambda + gh \in \mathcal{C}_\mathbb{R}$. In order for $\lambda + gh$ to satisfy the first cone condition (7.3), we need

$$\sup_{W,\psi}\frac{\int_W(\lambda+gh)\psi}{\int_W\psi} \le L\inf_{W,\psi}\frac{\int_W(\lambda+gh)\psi}{\int_W\psi}\,.$$

This is equivalent to

$$\lambda + \sup_{W,\psi}\frac{\int_W gh\psi}{\int_W\psi} \le L\Big(\lambda + \inf_{W,\psi}\frac{\int_W gh\psi}{\int_W\psi}\Big)\,,$$

which according to (7.25) and (7.26) is satisfied if

$$\lambda + 3(1+2a)L|g|_{C^\alpha}|||h|||_- \le L(\lambda - 2(1+2a)L|g|_{C^\alpha}|||h|||_-)\,.$$

Thus it suffices to choose λ so that

$$\lambda \ge \frac{3+2L}{L-1}L(1+2a)|g|_{C^\alpha}|||h|||_-\,. \tag{7.27}$$

In order to verify the second cone condition (7.4), $\lambda > 0$ should satisfy,

$$|W|^{-q}\frac{|\int_W(\lambda+gh)\psi|}{⨍_W\psi} \le A\delta^{1-q}|||\lambda+gh|||_-\,,$$

for all $W \in \mathcal{W}^s_-(\delta)$, $\psi \in \mathcal{D}_{a,\beta}(W)$.

The left hand side is bounded above by

$$\begin{aligned}&|W|^{1-q}\lambda + |W|^{-q}\frac{|\int_W h(g\psi+B)|}{⨍_W g\psi+B}\frac{\int_W g\psi+B}{\int_W\psi} + |W|^{-q}\frac{|\int_W hB|}{⨍_W\psi}\\ &\le 2^{1-q}\delta^{1-q}\lambda + 5(1+2a)|g|_{C^\alpha}A\delta^{1-q}|||h|||_-\,,\end{aligned} \tag{7.28}$$

while the right hand side is bounded below using (7.26),

$$A\delta^{1-q}|||\lambda+gh|||_- \ge A\delta^{1-q}(\lambda - 2(1+2a)L|g|_{C^\alpha}|||h|||_-)\,.$$

Putting these estimates together, it suffices to choose $\lambda > 0$ so that

$$2^{1-q}\delta^{1-q}\lambda + 5(1+2a)|g|_{C^\alpha}A\delta^{1-q}|||h|||_- \le A\delta^{1-q}(\lambda - 2(1+2a)L|g|_{C^\alpha}|||h|||_-)$$

This holds when

$$\lambda \ge \frac{(5+2L)A}{A-2^{1-q}}(1+2a)|g|_{C^\alpha}|||h|||_-\,. \tag{7.29}$$

Finally, we find λ to satisfy the third cone condition (7.5),

$$\left|\frac{\int_{W^1}(\lambda+gh)\psi_1}{⨍_{W^1}\psi_1} - \frac{\int_{W^2}(\lambda+gh)\psi_2}{⨍_{W^2}\psi_2}\right| \le d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}cA|||\lambda+gh|||_-\,, \tag{7.30}$$

for $W^1, W^2 \in \mathcal{W}^s_-(\delta)$ with $d_{\mathcal{W}^s}(W^1,W^2) \le \delta$ and $\psi_i \in \mathcal{D}_{a,\alpha}(W^i)$ satisfying $d_*(\psi_1,\psi_2) = 0$.

Without loss of generality, we may assume $|W^2| \geq |W^1|$ and $\fint_{W^1} \psi_1 = 1$. Also, we assume

$$|W^2|^q \geq \delta^{q-\gamma} d_{\mathcal{W}^s}(W^1, W^2)^\gamma \tfrac{c}{2}\,, \tag{7.31}$$

otherwise, applying the second cone condition to $\lambda + gh$ with λ satisfying (7.29) yields,

$$\begin{aligned} \left| \frac{\int_{W^1} (\lambda + gh)\psi_1}{\fint_{W^1} \psi_1} - \frac{\int_{W^2} (\lambda + gh)\psi_2}{\fint_{W^2} \psi_2} \right| &\leq (|W^1|^q + |W^2|^q) A\delta^{1-q} |||\lambda + gh|||_- \\ &\leq 2A\delta^{1-\gamma} d_{\mathcal{W}^s}(W^1, W^2)^\gamma \tfrac{c}{2} |||\lambda + gh|||_-\,, \end{aligned} \tag{7.32}$$

which is precisely the needed estimate with no additional condition on λ.

We proceed to estimate both sides of (7.30) under the assumption (7.31). As before, the right-hand side of (7.30) is bounded below by,

$$d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} cA(\lambda - 2(1+2a)L|g|_{C^\alpha} |||h|||_-)\,. \tag{7.33}$$

To bound the left-hand side of (7.30) from above, we first split up the differences, using that $\fint_{W^1} \psi_1 = 1$,

$$\begin{aligned} &\lambda ||W^1| - W^2|| + \left| \frac{\int_{W^1} hg\psi_1}{\fint_{W^1} \psi_1} - \frac{\int_{W^2} hg\psi_2}{\fint_{W^2} \psi_2} \right| \\ &\leq \lambda ||W^1| - W^2|| + \frac{\left|\int_{W^2} hg\psi_2\right|}{\fint_{W^2} \psi_2} \left| \fint_{W^2} \psi_2 - 1 \right| + \left| \int_{W^1} hg\psi_1 - \int_{W^2} hg\psi_2 \right|\,. \end{aligned} \tag{7.34}$$

The first term above is bounded by [9, eq. (5.8)],

$$\big||W^1| - |W^2|\big| \leq 2C_s d_{\mathcal{W}^s}(W^1, W^2)\,, \tag{7.35}$$

where $C_s = \sqrt{1 + (\mathcal{K}_*^{-1} + \tau_*^{-1})^2}$ denotes the maximum absolute value of the slope of stable curves in $\mathcal{W}^s_-(\delta)$, while the difference in the second term above is bounded by [9, eq. (5.10)],

$$\left| \fint_{W^2} \psi_2 - 1 \right| = |W^2|^{-1} \left| \int_{W^2} \psi_2 - |W^2| \right| \leq \frac{6C_s d_{\mathcal{W}^s}(W^1, W^2)}{|W^2|}\,. \tag{7.36}$$

Moreover, by (7.28),

$$\frac{\left|\int_{W^2} hg\psi_2\right|}{\fint_{W^2} \psi_2} \leq 5(1+2a)|W^2|^q |g|_{C^\alpha} A\delta^{1-q} |||h|||_-\,.$$

Combining these estimates in (7.34) yields the following upper bound for the left side of (7.30)

$$\begin{aligned} &\left| \frac{\int_{W^1} (\lambda + gh)\psi_1}{\fint_{W^1} \psi_1} - \frac{\int_{W^2} (\lambda + gh)\psi_2}{\fint_{W^2} \psi_2} \right| \leq \lambda 2C_s d_{\mathcal{W}^s}(W^1, W^2) \\ &+ \frac{\delta^{1-q} d_{\mathcal{W}^s}(W^1, W^2)}{|W^2|^{1-q}} 5(1+2a) 6C_s A|g|_{C^\alpha} |||h|||_- + \left| \int_{W^1} hg\psi_1 - \int_{W^2} hg\psi_2 \right| \\ &\leq \lambda 2C_s d_{\mathcal{W}^s}(W^1, W^2) + 4C_s^q (1+2a) A\delta^{1-\gamma} d_{\mathcal{W}^s}(W^1, W^2)^\gamma |g|_{C^\alpha} |||h|||_- \\ &+ \left| \int_{W^1} hg\psi_1 - \int_{W^2} hg\psi_2 \right|\,, \end{aligned} \tag{7.37}$$

where we have used (7.31) together with $d_{\mathcal{W}^s}(W^1, W^2) \leq \delta$, $\gamma \leq q < 1/2$ and the constraint $c \geq 16C_s^q$ from [9, eq. (5.7)].

To estimate the last term in (7.37), we recall the notation of matched and unmatched pieces. Recall that each curve W^i is expressed as the graph of a function over an arclength interval I_i,

$$W^i = \{G_i(r) = (r, \varphi_i(r) : r \in I_i\}$$

Since $d_{\mathcal{W}^s}(W^1, W^1) \leq \delta$, we have $|I_1 \cap I_2| > 0$. Let $U^i = \{G_i(r) : r \in I_1 \cap I_2\}$ denote the matched pieces and let V^i denote the at most two unmatched pieces. Remark that $|V^i| \leq C_s d_{\mathcal{W}^s}(W^1, W^2)$. Choose $B > 0$ such

that $g\psi_2 + B \in \mathcal{D}_{\frac{a}{2},\alpha}(U^2)$. Following (7.24) we may choose $B \le 2(1+2a)|g|_{C^\alpha}|\psi_2|_{C^0}$. Define the following functions on U^1,

$$\tilde g = g \circ G_2 \circ G_1^{-1} \cdot \frac{\|G_2'\|}{\|G_1'\|} \circ G_1^{-1}\,; \quad \tilde\psi_2 = \psi \circ G_2 \circ G_1^{-1}$$
$$\tilde B = B \cdot \frac{\|G_2'\|}{\|G_1'\|} \circ G_1^{-1}\,.$$

Then by definition, $d_*(\tilde g\tilde\psi_2 + \tilde B, g\psi_2 + B) = 0$. Moreover, in the proof of [9, Lemma 5.5(c)], it is shown that for $r, s \in I_1 \cap I_2$, $x = G_k(r)$, $y = G_k(s)$,

$$\frac{\|G_k'(r)\|}{\|G_k'(s)\|} \le e^{B_* d(x,y)} \quad \text{and} \quad \frac{\|G_2'(r)\|}{\|G_1'(r)\|} \le e^{d_{\mathcal{W}^s}(U^1,U^2)} \le e^\delta \le 2\,, \tag{7.38}$$

for some constant B_* depending only on the maximum curvature of curves in $\mathcal{W}^s_-(\delta)$. This implies in particular that $\tilde B \le 2B$. and that $\tilde g\tilde\psi_2 + \tilde B \in \mathcal{D}_{a,\alpha}(U^2)$ by [9, eq. (5.31)], as long as $2B_*(2\delta)^{1-\alpha} \le \frac{a}{2}$, which is compatible with the restriction on δ from [9, eq. (5.32)].

With these preparations, we are ready to estimate the difference of integrals in (7.37) by splitting into matched and unmatched pieces,

$$\begin{aligned}\Big|\int_{W^1} hg\psi_1 - \int_{W^2} hg\psi_2\Big| &\le \sum_{i=1}^2 \Big|\int_{V^i} hg\psi_i\Big| + \Big|\int_{U^1} hg\psi_1 - \int_{U^2} hg\psi_2\Big| \\ &\le \sum_{i=1}^2 \Big|\int_{V^i} hg\psi_i\Big| + \Big|\int_{U^1} h(g\psi_1 - \tilde g\tilde\psi_2)\Big| + \Big|\int_{U^1} h\tilde g\tilde\psi_2 - \int_{U^2} hg\psi_2\Big|\,.\end{aligned} \tag{7.39}$$

The following sublemma allows us to estimate the principal differences in (7.39).

Sublemma 7.17. *The following estimates hold true*

a) $\displaystyle\Big|\int_{U^1} h(g\psi_1 - \tilde g\tilde\psi_2)\Big| \le 12(3+2a)d_{\mathcal{W}^s}(W^1,W^2)^{\alpha-\beta}\delta A|g|_{C^\alpha}|||h|||_-$.

b) $\displaystyle\Big|\int_{U^1} h\tilde g\tilde\psi_2 - \int_{U^2} hg\psi_2\Big| \le 20(1+2a)(c+3\delta)Ad_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}|g|_{C^\alpha}|||h|||_-$.

Postponing the proof of the sublemma, we use it to complete the estimate for the third cone condition. Using the sublemma, together with (7.28) to bound the integrals on unmatched pieces, and recalling that $|V^i| \le C_s d_{\mathcal{W}^s}(W^1,W^2)$, we bound (7.39) by

$$\begin{aligned}\Big|\int_{W^1} hg\psi_1 - \int_{W^2} hg\psi_2\Big| &\le 40(1+2a)d_{\mathcal{W}^s}(W^1,W^2)^q C_s^q A\delta^{1-q}|g|_{C^\alpha}|||h|||_- \\ &+ 12(3+2a)d_{\mathcal{W}^s}(W^1,W^2)^{\alpha-\beta}\delta A|g|_{C^\alpha}|||h|||_- \\ &+ 20(1+2a)(c+3\delta)Ad_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}|g|_{C^\alpha}|||h|||_- \\ &\le 4(1+2a)(10C_s^q + 9\delta^\gamma + 5c + 15\delta)d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}A|g|_{C^\alpha}|||h|||_-\,,\end{aligned}$$

where we have used that $d_{\mathcal{W}^s}(W^1,W^2) \le \delta$ and $\gamma \le \min\{q, \alpha-\beta\}$. This estimate combined with (7.37) yields our final upper bound for the left hand side of (7.30),

$$\begin{aligned}\left|\frac{\int_{W^1}(\lambda+gh)\psi_1}{\fint_{W^1}\psi_1} - \frac{\int_{W^2}(\lambda+gh)\psi_2}{\fint_{W^2}\psi_2}\right| \le& 2\lambda C_s d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma} \\ &+ C_\star d_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}|g|_{C^\alpha}|||h|||_-\,,\end{aligned}$$

where $C_\star = 4A(1+2a)(14C_s^q + 9\delta^\gamma + 5c + 15\delta)$. Combining this with the lower bound (7.33), we see that (7.30) will be satisfied if

$$\lambda 2C_s + C_\star|g|_{C^\alpha}|||h|||_- \le cA(\lambda - 2(1+2a)L|g|_{C^\alpha}|||h|||_-)\,.$$

Since $cA > 2C_s$ by [9, eq. (5.36)], this in turn is satisfied if

$$\lambda \ge \frac{C_\star + 2cA(1+2a)L}{cA - 2C_s}|g|_{C^\alpha}|||h|||_-\,. \tag{7.40}$$

Now taking together (7.27), (7.29) and (7.40), we conclude that $\lambda + gh \in \mathcal{C}_{\mathbb{R}}$ if λ satisfies (7.11).

We claim that the same value of λ implies that $\lambda - gh \in \mathcal{C}_{\mathbb{R}}$. To see this, write $\lambda - gh = \lambda + (-g)h$, and notice that $-g \in C^\alpha$ with $|-g|_{C^\alpha} = |g|_{C^\alpha}$. Applying the previous argument to $-gh$, we see that $\lambda + (-g)h \in \mathcal{C}_{\mathbb{R}}$ when λ satisfies (7.11). This completes the proof of the lemma.

It remains to prove the sublemma used during the argument.

Proof of Sublemma 7.17. We prove the two statements one at a time:
a) We would like to apply the second cone condition (7.4) to the integral; however, $g\psi_1 - \tilde{g}\tilde{\psi}_2$ may not be in $\mathcal{D}_{a,\beta}(U^1)$. To remedy this, first note that for $x = G_1(r) \in U^1$, using the fact that $d_*(\psi_1, \psi_2) = 0$,

$$\begin{aligned}|g(x)\psi_1(x) - \tilde{g}\tilde{\psi}_2(x)| &= \tfrac{1}{\|G_1'\|}|g\psi_1(G_1(r)) \cdot \|G_1'(r)\| - g\psi_2(G_2(r)) \cdot \|G_2'\||\\ &\le |\psi_1(G_1(r))||g(G_1(r)) - g(G_2(r))|\\ &\le 2H^\alpha(g) d_{\mathcal{W}^s}(W^1, W^2)^\alpha\,,\end{aligned}$$

where we have used the bound $|\psi_1| \le e^{a(2\delta)^\alpha} \fint_{W^1} \psi_1 \le 2$.

So on the one hand,

$$|(g\psi_1 - \tilde{g}\tilde{\psi}_2)(x) - (g\psi_1 - \tilde{g}\tilde{\psi}_2)(y)| \le 4H^\alpha(g) d_{\mathcal{W}^s}(W^1, W^2)^\alpha\,, \text{ for } x, y \in U^1\,.$$

On the other hand, for $x = G_1(r) \in U^1$, let $\tilde{x} = G_2(r) \in U^2$ denote the corresponding point in U^2. Then,

$$\begin{aligned}d(\tilde{x}, \tilde{y}) &= \int_{r_1}^{r_2} \sqrt{1 + (d\varphi_2/dr)^2}\, dr \le \sup_{I_1 \cap I_2} \sqrt{\frac{1 + (d\varphi_2/dr)^2}{1 + (d\varphi_1/dr)^2}} \int_{r_1}^{r_2} \sqrt{1 + (d\varphi_1/dr)^2}\, dr\\ &\le 2d(x, y)\,,\end{aligned}$$

where we have used the estimate before [9, eq. (5.9)] to estimate the ratio of Jacobians. Then recalling (7.24),

$$\begin{aligned}|(g\psi_1 - \tilde{g}\tilde{\psi}_2)(x) - (g\psi_1 - \tilde{g}\tilde{\psi}_2)(y)| &\le |g\psi_1(x) - g\psi_1(y)| + |g(\tilde{x})\psi_1(x) - g(\tilde{y})\psi_1(y)|\\ &\le |g|_{C^\alpha}|\psi_1|_{C^\alpha}(d(x, y)^\alpha + d(\tilde{x}, \tilde{y})^\alpha)\\ &\le 6(1 + 2a)|g|_{C^\alpha} d(x, y)^\alpha\,.\end{aligned}$$

Putting these estimates together, we see that

$$\frac{|(g\psi_1 - \tilde{g}\tilde{\psi}_2)(x) - (g\psi_1 - \tilde{g}\tilde{\psi}_2)(y)|}{d(x, y)^\beta} \le \frac{|g|_{C^\alpha}}{d(x, y)^\beta} \min\{4d_{\mathcal{W}^s}(W^1, W^2)^\alpha, 6(1 + 2a)d(x, y)^\alpha\}\,,$$

and the expression is maximized when the two quantities are equal, i.e. when $d_{\mathcal{W}^s}(W^1, W^2) = (3(1 + 2a)/2)^{1/\alpha} d(x, y)$. Thus,

$$H^\beta_{U^1}(g\psi_1 - \tilde{g}) \le 6(1 + 2a)|g|_{C^\alpha} d_{\mathcal{W}^s}(W^1, W^2)^{\alpha-\beta}\,.$$

So we choose a constant $\Delta \ge a^{-1}H^\beta_{U^1}(g\psi_1 - \tilde{g}\tilde{\psi}_2) + |g\psi_1 - \tilde{g}\tilde{\psi}_2|_{C^0(U^1)}$, so that $g\psi_1 - \tilde{g}\tilde{\psi}_2 + \Delta \in \mathcal{D}_{a,\beta}(U^1)$, and $\Delta \le |g\psi_1 - \tilde{g}\tilde{\psi}_2|_{C^\beta(U^1)} \le (8 + 6a)|g|_{C^\alpha} d_{\mathcal{W}^s}(W^1, W^2)^{\alpha-\beta}$.

Now applying (7.4) completes the proof of statement (a) of the sublemma,

$$\begin{aligned}\left|\int_{U^1} h(g\psi_1 - \tilde{g}\tilde{\psi}_2)\right| &\le \left|\int_{U^1} h(g\psi_1 - \tilde{g}\tilde{\psi}_2 + \Delta)\right| + \left|\int_{U^1} h\Delta\right|\\ &\le |U^1|^q \fint_{U^1} (g\psi_1 - \tilde{g}\tilde{\psi}_2 + 2\Delta) A\delta^{1-q}|||h|||_-\\ &\le 2(18 + 12a) d_{\mathcal{W}^s}(W^1, W^2)^{\alpha-\beta}\delta A|g|_{C^\alpha}|||h|||_-\,.\end{aligned}$$

b) With B and $\tilde{B}$ as chosen before the statement of the sublemma, write,

$$\begin{aligned}\left|\int_{U^1} h\tilde{g}\tilde{\psi}_2 - \int_{U^2} hg\psi_2\right| &\le \left|\frac{\int_{U^1} h(\tilde{g}\tilde{\psi}_2 + \tilde{B})}{\fint_{U^1} \tilde{g}\tilde{\psi}_2 + \tilde{B}} - \frac{\int_{U^2} h(g\psi_2 + B)}{\fint_{U^2} g\psi_2 + B}\right| \fint_{U^2} (g\psi_2 + B)\\ &\quad + \left|\int_{U^1} h(\tilde{g}\tilde{\psi}_2 + \tilde{B})\right| \left|1 - \frac{|U^1|}{|U^2|}\right| + \left|\int_{U^1} h\tilde{B} - \int_{U^2} hB\right|.\end{aligned} \tag{7.41}$$

The first term of (7.41) can be estimated using (7.5) since $d_*(\tilde{g}\tilde{\psi}_2+\tilde{B}, g\psi_2+B)=0$ and both are valid test functions by choice of B. For the second term of (7.41), we use [9, eq. (5.24)] to estimate

$$\left|1-\frac{|U^2|}{|U^1|}\right| \le d_{\mathcal{W}^s}(U^1,U^2) \le d_{\mathcal{W}^s}(W^1,W^2)\,. \tag{7.42}$$

Using this together with (7.4) yields the bound,

$$\begin{aligned}&\left|\frac{\int_{U^1} h(\tilde{g}\tilde{\psi}_2+\tilde{B})}{\fint_{U^1}\tilde{g}\tilde{\psi}_2+\tilde{B}}-\frac{\int_{U^2} h(g\psi_2+B)}{\fint_{U^2} g\psi_2+B}\right|\fint_{U^2}(g\psi_2+B)+\left|\int_{U^1} h(\tilde{g}\tilde{\psi}_2+\tilde{B})\right|\left|1-\frac{|U^1|}{|U^2|}\right|\\ &\le cAd_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}|||h|||_- 12(1+2a)|g|_{C^\alpha}+36(1+2a)A\delta|g|_{C^\alpha}|||h|||_- d_{\mathcal{W}^s}(W^1,W^2)\,.\end{aligned}$$

Moreover, it is clear that the final term in (7.41) can be estimated in an analogous way, with B and $\tilde{B}$ replacing the test functions $g\psi_2+B$ and $\tilde{g}\tilde{\psi}_2+\tilde{B}$. Thus,

$$\begin{aligned}\left|\int_{U^1} h\tilde{B}-\int_{U^2} hB\right| &\le BcAd_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}|||h|||_- + 3Bd_{\mathcal{W}^s}(W^1,W^2)\delta A|||h|||_-\\ &\le 8(1+2a)Ad_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}|g|_{C^\alpha}|||h|||_-(3\delta+c)\,.\end{aligned}$$

Putting this estimate together with the previous one completes the proof of statement (b), using that $d_{\mathcal{W}^s}(W^1,W^2)\le\delta$. □

7.11. Proof of Proposition 7.9 (Bounding the Norm of $\mathcal{L}_T$).

As mentioned in Remark 7.8, the goal of this section is to obtain a bound on $\|\mathcal{L}_T\|$, that is, for the action of a single operator rather than a sequence of $N_{\mathcal{F}}$ operators. The proof is similar to the proof of Lemma 7.5 in Section 7.11. Unfortunately, the details are different enough that we need to carry it out explicitely.

We will first bound $\mathcal{L}h$ for $h\in\mathcal{C}_{\mathbb{R}}$ and then show how this extends to all $h\in\mathcal{B}_{\mathbb{R}}$ as in the proof of Corollary 7.7.

Let $h\in\mathcal{C}_{\mathbb{R}}$ and $T\in\mathcal{F}(\tau_*,\mathcal{K}_*,E_*)$ with $\mathcal{L}=\mathcal{L}_T$. We must estimate the norm of $\mathcal{L}h$. To this end, we first obtain bounds on $|||\mathcal{L}h|||_+$ and $|||\mathcal{L}h|||_-$. Let $W\in\mathcal{W}^s(\delta)$ and $\psi\in\mathcal{D}_{a,\beta}(W)$. Then,

$$\int_W \mathcal{L}h\,\psi=\sum_{i\in L_1(W)}\int_{W_i} h\,\psi\circ T\,J_{W_i}T+\sum_{i\in S_1(W)}\int_{W_i} h\,\psi\circ T\,J_{W_i}T\,, \tag{7.43}$$

where we have changed variables and denote by $L_1(W)$ the connected homogenous components of $T^{-1}W=\{W_i\}_i$ longer than δ, and by $S_1(W)$ those homogeneous components shorter than δ.

Following (7.24), in order to transform the $\psi\circ TJ_{W_i}T$ into valid test functions, we choose B_i such that

$$\begin{aligned}a^{-1}H^\beta(\psi\circ TJ_{W_i}T)+|\psi\circ TJ_{W_i}T|_{C^0(W_i)} < B_i &\le |J_{W_i}T\psi\circ T|_{C^\beta(W_i)}\\ &\le (1+2aC_1+C_d)|J_{W_i}T|_{C^0(W_i)}|\psi\circ T|_{C^0(W_i)}\,,\end{aligned} \tag{7.44}$$

where $C_d>0$ is a distortion constant for $J_{W^i}T$, uniform for $T\in\mathcal{F}(\tau_*,\mathcal{K}_*,E_*)$ as in [9, eq. (3.5)], and we have used (7.23) to bound the distortion of $\psi\circ T$ together with $d(Tx,Ty)\le C_1d(x,y)$, where $C_1\ge1$ is the minimum hyperbolicity constant[29] from [9, eq. (3.1)]. Then as in Section 7.10, $J_{W_i}T\psi\circ T+B_i\in\mathcal{D}_{a,\beta}(W)$. For ease of notation, let us name the combined distortion constants,

$$C_a:=1+2aC_1+C_d\,.$$

[29]That is, setting $\Lambda=1+2\mathcal{K}_*\tau_*$, $C_1\ge1$ satisfies $\|DT^{-n}(x)v\|\ge C_1^{-1}\Lambda^n\|v\|$ for all $T\in\mathcal{F}(\tau_*,\mathcal{K}_*,E_*)$ and v in the stable cone.

Now we estimate the contribution from long pieces in (7.43) using (7.2) and (7.3), as well as the upper bound on B_i from (7.44),

$$\begin{aligned}
&\sum_{i\in L_1(W)}\int_{W_i} h\,(\psi\circ T\,J_{W_i}T+B_i)-B_i\sum_{i\in L_1(W)}\int_{W_i} h\\
&\qquad\le |||h|||_+\sum_{i\in L_1(W)}\int_{W_i}(\psi\circ T\,J_{W_i}T+2B_i)\\
&\qquad\le \big[1+2C_a e^{C_d(2\delta)^{1/3}+a(2\delta)^\beta}\big]L|||h|||_-\sum_{i\in L_1(W)}\int_{TW_i}\psi\\
&\qquad\le \big[1+8C_a\big]L|||h|||_-\int_W\psi\,,
\end{aligned}\tag{7.45}$$

recalling that by [9, Sect. 5.3], δ is small enough that $e^{a(2\delta)^\beta}$ and $e^{C_d(2\delta)^{1/3}}$ are each less than 2.

Next we estimate the contribution from short pieces in (7.43), using (7.4),

$$\begin{aligned}
&\sum_{i\in S_1(W)}\int_{W_i} h\,(\psi\circ T\,J_{W_i}T+B_i)-B_i\sum_{i\in S_1(W)}\int_{W_i} h\\
&\qquad\le |||h|||_-\sum_{i\in S_1(W)}A\delta^{1-q}|W_i|^q\fint_{W_i}(\psi\circ T\,J_{W_i}T+2B_i)\\
&\qquad\le |||h|||_-A\delta\big[1+2C_a)\big]|\psi|_{C^0(W)}\sum_{i\in S_1(W)}|J_{W_i}T|_{C^0(W_i)}.
\end{aligned}\tag{7.46}$$

Since $\psi\in\mathcal{D}_{a,\beta}(W)$ and $|W|\ge\delta$, we estimate $\delta|\psi|_{C^0(W)}\le\delta e^{a|W|^\beta}\fint_W\psi\le 2\int_W\psi$, recalling again that $e^{a(2\delta)^\beta}\le 2$. The sum over Jacobians is bounded by $C_0\theta_0$, which is the uniform one-step expansion estimate stated in [9, eq. (3.4)], where $C_0\ge 1$ and $\theta_0<1$. Putting these bounds together yields,

$$\sum_{i\in S_1(W)}\int_{W_i} h\,\psi\circ T\,J_{W_i}T\le |||h|||_-A\big[1+2C_a\big]2C_0\theta_0\int_W\psi\,.\tag{7.47}$$

Substituting (7.45) and (7.47) into (7.43) and taking the supremum over $W\in\mathcal{W}^s(\delta)$ and $\psi\in\mathcal{D}_{a,\beta}(W)$ yields,

$$|||\mathcal{L}h|||_+\le |||h|||_-\Big(L\big[1+8C_a\big]+A\big[1+2C_a\big]2C_0\theta_0\Big)\,.\tag{7.48}$$

Next, we need the analogous lower bound for $|||\mathcal{L}h|||_-$. Taking $W\in\mathcal{W}^s(\delta)$ and $\psi\in\mathcal{D}_{a,\beta}(W)$, we first change variables as in (7.43). For the estimate on long pieces, we use the fact that integrals of h against valid test functions on curves of length at least δ are positive by the first cone condition, (7.3). Then, using (7.44) and the distortion bounds as before,

$$\begin{aligned}
&\sum_{i\in L_1(W)}\int_{W_i} h\,(\psi\circ T\,J_{W_i}T+B_i)-B_i\sum_{i\in L_1(W)}\int_{W_i} h\\
&\qquad\ge \sum_{i\in L_1(W)}|||h|||_-\int_{W_i}(\psi\circ T\,J_{W_i}T+B_i)-|||h|||_+\int_{W_i}B_i\\
&\qquad\ge \sum_{i\in L_1(W)}|||h|||_-2\int_{TW_i}\psi-L|||h|||_-C_a e^{C_d(2\delta)^{1/3}+a(2\delta)^\beta}\int_{TW_i}\psi\\
&\qquad\ge |||h|||_-\big[2-4LC_a\big]\int_W\psi\,.
\end{aligned}\tag{7.49}$$

The contribution from short pieces is estimated using (7.4) as in (7.47), but with the observation that the integrals of h on short pieces can be negative,

$$\begin{aligned}
&\sum_{i\in S_1(W)} \int_{W_i} h\,\psi\circ T\,J_{W_i}T + B_i) - B_i \sum_{i\in S_1(W)} \int_{W_i} h \\
&\qquad \geq -|||h|||_- \sum_{i\in S_1(W)} A\delta^{1-q}|W_i|^q \fint_{W_i} (\psi\circ T\,J_{W_i}T + 2B_i) \\
&\qquad \geq -|||h|||_- A\big[1+2C_a\big]2C_0\theta_0 \int_W \psi\,.
\end{aligned} \tag{7.50}$$

Putting together (7.49) and (7.50) and taking the appropriate infima yields,

$$|||\mathcal{L}h|||_- \geq |||h|||_-\Big(2 - 4LC_a - A\big[1+2C_a\big]2C_0\theta_0\Big)\,. \tag{7.51}$$

We are now in a position to choose $\lambda > 0$ so that $\lambda + \mathcal{L}h$ satisfies the first cone condition (7.3). To simplify notation, define $H_0 = L\big[1+8C_a\big] + A\big[1+2C_a\big]2C_0\theta_0$ and $H_1 = -2 + 4LC_a + A\big[1+2C_a\big]2C_0\theta_0 > 0$. Then using (7.48) and (7.51),

$$\frac{|||\lambda+\mathcal{L}h|||_+}{|||\lambda+\mathcal{L}h|||_-} \leq L \qquad \Longleftarrow \qquad \frac{\lambda + H_0|||h|||_-}{\lambda - H_1|||h|||_-} \leq L\,.$$

This later inequality is satisfied as soon as

$$\lambda \geq |||h|||_- \frac{H_0 + LH_1}{L-1}\,. \tag{7.52}$$

To guarantee that $\lambda + \mathcal{L}h$ satisfies the second cone condition (7.4), we need

$$|W|^{-q}\frac{|\int_W(\lambda+\mathcal{L}h)\psi|}{\fint_W \psi} \leq A\delta^{1-q}|||\lambda+\mathcal{L}h|||_-\,. \tag{7.53}$$

for all $W \in \mathcal{W}^s_-(\delta)$ and $\psi \in \mathcal{D}_{a,\beta}(W)$. Using (7.45) and (7.46), the left hand side is bounded above by

$$|W|^{1-q}\lambda + |W|^{-q}\frac{|\int_W \mathcal{L}h\psi|}{\fint_W \psi}\,. \tag{7.54}$$

Changing variables as in (7.43), we estimate the contribution from long pieces precisely as in (7.45). Yet the contribution from short pieces must be done with care since W itself may be short. As in (7.46), we estimate,

$$\begin{aligned}
&\sum_{i\in S_1(W)} \int_{W_i} h\,(\psi\circ T\,J_{W_i}T + B_i) - B_i \sum_{i\in S_1(W)} \int_{W_i} h \\
&\qquad \leq |||h|||_- \sum_{i\in S_1(W)} A\delta^{1-q}|W_i|^q \fint_{W_i} (\psi\circ T\,J_{W_i}T + 2B_i) \\
&\qquad \leq |||h|||_- A\delta^{1-q}\big[1+2C_a\big] \sum_{i\in S_1(W)} |W_i|^q|\psi|_{C^0}|J_{W_i}T|_{C^0(W_i)}\,.
\end{aligned}$$

Now $|\psi|_{C^0} \leq e^{a(2\delta)^\beta}\fint_W \psi \leq 2\fint_W \psi$ so that,

$$\begin{aligned}
\sum_{i\in S_1(W)} \frac{|\int_{W_i} h\psi\circ TJ_{W_i}T|}{|W|^q \fint_W \psi} &\leq 2|||h|||_- A\frac{\big[1+2C_a\big]}{\delta^{q-1}} \sum_{i\in S_1(W)} \frac{|W_i|^q}{|W|^q}|J_{W_i}T|_{C^0(W_i)} \\
&\leq |||h|||_- 4A\delta^{1-q}\big[1+2C_a\big] \sum_{i\in S_1(W)} \frac{|TW_i|^q}{|W|^q}\frac{|TW_i|^{1-q}}{|W_i|^{1-q}} \\
&\leq |||h|||_- 4A\delta^{1-q}\big[1+2C_a\big] \left(\sum_{i\in S_1(W)} \frac{|TW_i|}{|W_i|}\right)^{1-q} \\
&\leq |||h|||_- 4A\delta^{1-q}\big[1+2C_a\big]\,(C_0\theta_0)^{1-q}
\end{aligned} \tag{7.55}$$

where in the third line we have used the Hölder inequality together with the fact that $\sum_i \frac{|TW_i|}{|W|} \le 1$, and in the last line we have again used the one-step expansion from [9, eq. (3.4)].

Putting this estimate together with the estimate on long pieces yields an upper bound for the left side of (7.53) (using the fact that $|W| \le 2\delta$),

$$\begin{aligned}(2\delta)^{1-q}\lambda + |||h|||_- \Big((2\delta)^{1-q}\big[1+8C_a\big]L + 4A\delta^{1-q}\big[1+2C_a\big](C_0\theta_0)^{1-q}\Big) \\ =: 2^{1-q}\delta^{1-q}\lambda + \delta^{1-q}|||h|||_- H_2\,.\end{aligned} \tag{7.56}$$

On the other hand, we obtain a lower bound on the right side of (7.53) using (7.51)

$$A\delta^{1-q}|||\lambda + \mathcal{L}h|||_- \ge A\delta^{1-q}(\lambda - H_1|||h|||_-)\,.$$

This together with (7.56) implies that (7.53) is satisfied provided

$$2^{1-q}\delta^{1-q}\lambda + \delta^{1-q}|||h|||_- H_2 \le A\delta^{1-q}(\lambda - H_1|||h|||_-)\,.$$

The above holds true provided,

$$\lambda \ge |||h|||_- \frac{H_2 + AH_1}{A - 2^{1-q}}\,. \tag{7.57}$$

Remark 7.18. *Although $\mathcal{L}h$ may not be in the cone, the estimates (7.54) and (7.56) together prove that for all $W \in \mathcal{W}^s_-(\delta)$ and all $\psi \in \mathcal{D}_{a,\beta}(W)$,*

$$\left|\int_W \mathcal{L}h\,\psi\right| \le \fint_W \psi |W|^q \delta^{1-q} H_2 |||h|||_-\,,$$

which will be useful in what follows.

It remains to choose λ large enough to satisfy the third cone condition (7.5). For this we need,

$$\left|\frac{\int_{W^1}(\lambda+\mathcal{L}h)\psi_1}{\fint_{W^1}\psi_1} - \frac{\int_{W^2}(\lambda+\mathcal{L}h)\psi_2}{\fint_{W^2}\psi_2}\right| \le d_{\mathcal{W}^s}(W^1,W^2)^\gamma\,\delta^{1-\gamma}cA|||\lambda+\mathcal{L}h|||_-\,, \tag{7.58}$$

for $W^j \in \mathcal{W}^s_-(\delta)$ and $\psi_j \in \mathcal{D}_{a,\alpha}(W)$ with $d_*(\psi_1,\psi_2) = 0$. As before, using (7.51) the right hand side of (7.58) is bounded below by

$$d_{\mathcal{W}^s}(W^1,W^2)^\gamma \delta^{1-\gamma} cA(\lambda - H_1|||h|||_-)\,. \tag{7.59}$$

We proceed to obtain an upper bound for the left hand side of (7.58). Without loss of generality, we may assume that $\fint_{W^1}\psi_1 = 1$ and $|W^2| \ge |W^1|$. Also, we assume

$$|W^2|^q \ge \tfrac{1}{4}c\delta^{q-\gamma} d_{\mathcal{W}^s}(W^1,W^2)^\gamma\,. \tag{7.60}$$

Otherwise, applying (7.4) to both terms separately and using (7.48), we obtain,

$$\begin{aligned}\left|\frac{\int_{W^1}(\lambda+\mathcal{L}h)\psi_1}{\fint_{W^1}\psi_1} - \frac{\int_{W^2}(\lambda+\mathcal{L}h)\psi_2}{\fint_{W^2}\psi_2}\right| &\le 2|W^2|^q A\delta^{1-q}|||\lambda+\mathcal{L}h|||_- \\ &\le \tfrac{1}{2}cAd_{\mathcal{W}^s}(W^1,W^2)^\gamma\delta^{1-\gamma}(\lambda + H_0|||h|||_-)\,.\end{aligned}$$

This, together with (7.59) implies that (7.58) holds whenever $\lambda \ge |||h|||_-(H_0 + 2H_1)$.

We proceed to prove the upper bound under the assumption (7.60). Now,

$$\begin{aligned}\left|\frac{\int_{W^1}(\lambda+\mathcal{L}h)\psi_1}{\fint_{W^1}\psi_1} - \frac{\int_{W^2}(\lambda+\mathcal{L}h)\psi_2}{\fint_{W^2}\psi_2}\right| &\le \big||W^1| - |W^2|\big|\,\lambda + \left|\frac{\int_{W^1}\mathcal{L}h\,\psi_1}{\fint_{W^1}\psi_1} - \frac{\int_{W^2}\mathcal{L}h\,\psi_2}{\fint_{W^2}\psi_2}\right| \\ &\le \big||W^1| - |W^2|\big|\,\lambda + \frac{\left|\int_{W^2}\mathcal{L}h\,\psi_2\right|}{\fint_{W^2}\psi_2}\left|\fint_{W^2}\psi_2 - 1\right| + \left|\int_{W^1}\mathcal{L}h\,\psi_1 - \int_{W^2}\mathcal{L}h\,\psi_2\right|\,.\end{aligned}$$

Using (7.35), (7.36) and Remark 7.18, we can write

$$
\begin{aligned}
\left|\frac{\int_{W^1}(\lambda+\mathcal{L}h)\psi_1}{\fint_{W^1}\psi_1}-\frac{\int_{W^2}(\lambda+\mathcal{L}h)\psi_2}{\fint_{W^2}\psi_2}\right| &\le \left|\int_{W^1}\mathcal{L}h\,\psi_1-\int_{W^2}\mathcal{L}h\,\psi_2\right| \\
&\quad+\lambda 2C_s d_{\mathcal{W}^s}(W^1,W^2)+\left(\frac{\delta}{|W^2|}\right)^{1-q}6C_sH_2|||h|||_- d_{\mathcal{W}^s}(W^1,W^2) \\
&\le \left|\int_{W^1}\mathcal{L}h\,\psi_1-\int_{W^2}\mathcal{L}h\,\psi_2\right| \\
&\quad+\delta^{1-\gamma}d_{\mathcal{W}^s}(W^1,W^2)^\gamma\left(2C_s^qH_2|||h|||_-+2\lambda C_s\right)\,,
\end{aligned}
\tag{7.61}
$$

where in the last line we have used (7.60) together with $d_{\mathcal{W}^s}(W^1,W^2)\le\delta$, $\gamma\le q<1/2$ and the constraint $c\ge 16C_s^q$ from (7.6) (see [9, eq. (5.7)]for more details).

It remains to estimate the difference in integrals in (7.61). For this, we change variables as usual and integrate on elements of $\mathcal{G}_1^\delta(W^k)=\{W_i^k\}_i$, the homogeneous connected components of $T^{-1}W^k$, with long pieces subdivided to have length between δ and 2δ. As in Section 7.10 and following [9, Sect. 5.2.3], we subdivide elements of $\mathcal{G}_1(W^k)$ into matched and unmatched pieces by defining a foliation of vertical line segments $\{\ell_x\}_{x\in W_i^1}$ centered at x of length at most $3C_1d_{\mathcal{W}^s}(W^1,W^2)$ such that their images under T either terminate on a singularity curve in $\mathcal{S}_{-1}^{\mathbb{H}}$ or else are unstable curves having length at least $d_{\mathcal{W}^s}(W^1,W^2)$ on either side of $T(x)\in W^1$. In the latter case, either $T(\ell_x)$ intersects W^2, or by the uniform transversality of stable and unstable curves, lies within distance $C_2d_{\mathcal{W}^s}(W^1,W^2)$ of an endpoint of W^1, where C_2 depends only on the minimum angle between stable and unstable curves. When $T(\ell_x)$ intersects W^2, then necessarily ℓ_x intersects an element of $\mathcal{G}_1^\delta(W^2)$. We call subcurves for which all points are connected by such vertical line segments 'matched.' The rest we call 'unmatched.'

With this identification, we may label elements of $\mathcal{G}_1^\delta(W^1)$ and $\mathcal{G}_2^\delta(W^2)$ so that each element of $\mathcal{G}_1^\delta(W^1)$ contains at most one matched subcurve and at most 2 unmatched subcurves. We will use the decomposition $\mathcal{G}_1^\delta(W^k)=(\cup_jU_j^k)\cup(\cup_jV_j^k)$, so that U_j^1 and U_j^2 are matched and so recalling (7.1), are defined as the graphs of functions $G_{U_j^k}$ over the same r-interval I_j for each j. Using this decomposition, we write

$$
\int_{W^k}\mathcal{L}h\,\psi_k=\sum_j\int_{U_j^k}h\,\psi_k\circ T\,J_{U_j^k}T+\sum_j\int_{V_j^k}h\,\psi_k\circ T\,J_{V_j^k}T\,.
$$

We estimate the contribution to (7.61) from unmatched pieces first.

As noted previously, an unmatched curve in V_j^k has image satisfying $|T(V_j^k)|\le C_2d_{\mathcal{W}^s}(W^1,W^2)$. Choosing B_i as in (7.44), we estimate the sum on V_j^k using (7.55) to obtain,

$$
\begin{aligned}
\sum_{k=1}^2\sum_j\int_{V_j^k}h\,\psi_k\circ T\,J_{V_j^k}T &\le |||h|||_-A\delta^{1-q}\left[1+2C_a\right] \\
&\quad\times\sum_{k=1}^2\sum_j|\psi_k|_{C^0}|J_{V_j^k}T|_{C^0(V_j^k)}|V_j^k|^q \\
&\le 6A\delta^{1-q}|||h|||_-C_2^qd_{\mathcal{W}^s}(W^1,W^2)^q\sum_j|J_{V_j^2}T|^{1-q}_{C^0(V_j^2)}\,,
\end{aligned}
\tag{7.62}
$$

where we have used the fact that since $\fint_{W^1}\psi_1=1$, we have $|\psi_1|_{C^0}\le e^{a(2\delta)^\alpha}\le 2$, and also $|\psi_2|_{C^0}\le 4$ as in [9, eq. (5.9)]. Since $q<1/2$ and there are at most 2 curves V_j^k corresponding to each element of $\mathcal{G}_1^\delta(W^k)$ the final sum converges uniformly for $W^k\in\mathcal{W}^s$ and $T\in\mathcal{F}(\tau_*,\mathcal{K}_*,E_*)$ by [10, Lemma 3.4].

Finally, we estimate the contribution to (7.61) from matched pieces U_j^k. For this we will need to change test functions on the matched curves since it may be that $d_*(\psi_1\circ T\,J_{U_j^1}T,\psi_2\circ T\,J_{U_j^2}T)\ne 0$. We define the following functions on U_j^1,

$$
\tilde\psi_2=\psi_2\circ T\circ G_{U_j^2}\circ G_{U_j^1}^{-1}\,;\quad \widetilde{J}_{U_j^2}T=J_{U_j^2}T\circ G_{U_j^2}\circ G_{U_j^1}^{-1}
$$

$$
\widetilde{T}_{U_j^2}(\psi_2)=\tilde\psi_2\cdot\widetilde{J}_{U_j^2}T\frac{\|G'_{U_j^2}\|\circ G_{U_j^1}^{-1}}{\|G'_{U_j^1}\|\circ G_{U_j^1}^{-1}}\,;\quad \widehat{T}_{U_j^k}(\psi_k)=\psi_k\circ T\,J_{U_j^k}T\,.
$$

Since these functions may not belong to the cone of test functions, we choose a constant B_j as in (7.44) so that $\widehat{T}_{U_j^2}(\psi_2) + B_j \in \mathcal{D}_{\frac{a}{2},\alpha}(U_j^2)$.[30] Define

$$\tilde{B}_j = B_j \cdot \frac{\|G'_{U_j^2}\|}{\|G'_{U_j^1}\|} \circ G^{-1}_{U_j^1}.$$

Then by construction, $d_*(\widehat{T}_{U_j^2}(\psi_2) + B_j, \widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j) = 0$. Moreover, the bounds of (7.38) apply to $G'_{U_j^k}$ since in particular, $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le C d_{\mathcal{W}^s}(W^1, W^2)$ for some uniform $C > 0$ by [11, Lemma 3.3]. This implies, as before, that $\tilde{B}_j \le 2B_j$ and that $\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j \in \mathcal{D}_{a,\alpha}(U_j^2)$ by [9, eq. (5.31)].

For each j we split the difference between matched pieces in (7.61) as follows,

$$\begin{aligned}
\left|\int_{U_j^1} h\widehat{T}_{U_j^1}(\psi_1) - \int_{U_j^2} h\widehat{T}_{U_j^2}(\psi_2)\right| &\le \left|\int_{U_j^1} h(\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2))\right| \\
&+ \left|\int_{U_j^1} h\widetilde{T}_{U_j^2}(\psi_2) - \int_{U_j^2} h\widehat{T}_{U_j^2}(\psi_2)\right| \le \left|\int_{U_j^1} h(\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2))\right| \\
&+ \left|\frac{\int_{U_j^1} h(\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j)}{\fint_{U_j^1} \widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j} - \frac{\int_{U_j^2} h(\widehat{T}_{U_j^2}(\psi_2) + B_j)}{\fint_{U_j^2} \widehat{T}_{U_j^2}(\psi_2) + B_j}\right| \fint_{U_j^2} (\widehat{T}_{U_j^2}(\psi_2) + B_j) \\
&+ \frac{\left|\int_{U_j^1} h(\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j)\right|}{\fint_{U_j^1} \widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j} \left|\frac{|U_j^2| - |U_j^1|}{|U_j^1|}\right| \fint_{U_j^2} (\widehat{T}_{U_j^2}(\psi_2) + B_j) \\
&+ \left|\int_{U_j^1} h\tilde{B}_j - \int_{U_j^2} hB_j\right|.
\end{aligned} \tag{7.63}$$

The following sublemma estimates the three most relevant terms of (7.63).

Sublemma 7.19. *There exists $C_3 > 0$, independent of j, W^k and ψ_k, such that,*

a) $\left|\int_{U_j^1} h(\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2))\right| \le C_3 d_{\mathcal{W}^s}(W^1, W^2)^{\alpha-\beta} \delta A |||h|||_- |J_{U_j^1} T|_{C^0(U_j^1)}$

b) $\left|\frac{\int_{U_j^1} h(\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j)}{\fint_{U_j^1} \widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j} - \frac{\int_{U_j^2} h(\widehat{T}_{U_j^2}(\psi_2) + B_j)}{\fint_{U_j^2} \widehat{T}_{U_j^2}(\psi_2) + B_j}\right| \le C_3 d_{\mathcal{W}^s}(W^1, W^2)^{\gamma} \delta^{1-\gamma} cA |||h|||_-$

c) $\frac{\left|\int_{U_j^1} h(\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j)\right|}{\fint_{U_j^1} \widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j} \left|\frac{|U_j^2| - |U_j^1|}{|U_j^1|}\right| \le C_3 d_{\mathcal{W}^s}(W^1, W^2) \delta A |||h|||_-$.

We postpone the proof of the sublemma and use it to complete the proof of the bound on the third cone condition. Using the 3 items of Sublemma 7.19 to bound the corresponding terms of (7.63) yields (recalling again $|\psi_2|_{C^0} \le 4$),

$$\begin{aligned}
\left|\int_{U_j^1} h\widehat{T}_{U_j^1}(\psi_1) \int_{U_j^2} h\widehat{T}_{U_j^2}(\psi_2)\right| &\le \left|\int_{U_j^1} h\tilde{B}_j - \int_{U_j^2} hB_j\right| \\
&+ C_3 d_{\mathcal{W}^s}(W^1, W^2)^{\gamma} \delta^{1-\gamma} A |||h|||_- \left(\delta^{\gamma} + 4(c+\delta)(1 + C_a)\right) |J_{U_j^2} T|_{C^0(U_j^2)}.
\end{aligned} \tag{7.64}$$

Since $d_*(B_j, \tilde{B}_j) = 0$, it is clear that we can use parts (b) and (c) of Sublemma 7.19 to estimate the last difference of integrals with test functions B_j and $\tilde{B}_j$, i.e.

$$\begin{aligned}
\left|\int_{U_j^1} h\tilde{B}_j - \int_{U_j^2} hB_j\right| &\le \left|\frac{\int_{U_j^1} h\tilde{B}_j}{\fint_{U_j^1} \tilde{B}_j} - \frac{\int_{U_j^2} hB_j}{\fint_{U_j^2} B_j}\right| \fint_{U_j^2} B_j + \frac{\left|\int_{U_j^1} h\tilde{B}_j\right|}{\fint_{U_j^1} \tilde{B}_j} \left|\frac{|U_j^2| - |U_j^1|}{|U_j^1|}\right| \fint_{U_j^2} B_j \\
&\le C_3 d_{\mathcal{W}^s}(W^1, W^2)^{\gamma} \delta^{1-\gamma} A |||h|||_- \left(4cC_a + 4\delta C_a\right) |J_{U_j^2} T|_{C^0(U_j^2)}.
\end{aligned}$$

[30] Increasing B_j to $2B_j$ is sufficient to scale a to $a/2$.

Using this estimate together with (7.64) completes our estimate on matched pieces,

$$\sum_j \left| \int_{U_j^1} h\widehat{T}_{U_j^1}(\psi_1) - \int_{U_j^2} h\widehat{T}_{U_j^2}(\psi_2) \right| \le C_0 C_3 d_{\mathcal{W}^s}(W^1, W^2)^\gamma \delta^{1-\gamma} A |||h|||_- \left(\delta^\gamma + 8c(1+4C_a)\right),$$

where we have used $\delta < c$ and the Jacobian to sum over j according to [9, Lemma 3.3]. Combining this estimate with the estimate on unmatched pieces from (7.62) in (7.61) yields finally the upper bound on the left hand side of (7.58)

$$\begin{aligned} &\left| \frac{\int_{W^1}(\lambda + \mathcal{L}h)\psi_1}{\fint_{W^1}\psi_1} - \frac{\int_{W^2}(\lambda+\mathcal{L}h)\psi_2}{\fint_{W^2}\psi_2} \right| \le \frac{2d_{\mathcal{W}^s}(W^1,W^2)^\gamma \left(C_s^q H_2 |||h|||_- + \lambda C_s\right)}{\delta^{\gamma-1}} \\ &+ C_0 C_3 d_{\mathcal{W}^s}(W^1,W^2)^\gamma \delta^{1-\gamma} A |||h|||_- \left(\delta^\gamma + 8c(1+4C_a)\right) \\ &+ 6AC\delta^{1-\gamma} d_{\mathcal{W}^s}(W^1,W^2)^q C_2^q |||h|||_- \\ &=: d_{\mathcal{W}^s}(W^1,W^2)^\gamma \delta^{1-\gamma} cA \left(H_3 |||h|||_- + 2C_s(cA)^{-1}\lambda\right), \end{aligned} \tag{7.65}$$

where we have used the fact that $\gamma \le q$ and $d_{\mathcal{W}^s}(W^1,W^2) \le \delta$.

Using (7.65) as the upper bound on the left hand side of (7.58) and (7.59) as the lower bound on the right hand side of (7.58), we conclude that (7.58) is satisfied if

$$H_3 |||h|||_- + 2C_s(cA)^{-1}\lambda \le \lambda - H_1 |||h|||_- \implies \lambda \ge \frac{H_1 + H_3}{cA - 2C_s} cA |||h|||_- .$$

This is a valid choice since $cA > 2C_s$ by [9, eq. (5.36)].

Since $|||h|||_- \le \|h\|$ by Remark 7.6, this estimate together with (7.52) and (7.57) implies that $\lambda + \mathcal{L}h \in \mathcal{C}_\mathbb{R}$ whenever $h \in \mathcal{C}_\mathbb{R}$ and

$$\lambda \ge \bar{C}_L \|h\| := \max\left\{ \frac{H_0 + LH_1}{L-1}, \frac{H_2 + AH_1}{A - 2^{1-q}}, \frac{H_1 + H_3}{cA - 2C_s} cA \right\} \|h\|.$$

We claim that this suffices to complete the proof of Proposition 7.9.

As in the proof of Corollary 7.7, for $h \in \mathcal{B}_\mathbb{R}$, we have $h + \|h\|, \|h\| \in \mathcal{C}_\mathbb{R} \cup \{0\}$. Thus with λ chosen as above for the element h, and since $\|h + \|h\|\| \le 2\|h\|$, we may write,

$$3\lambda + \mathcal{L}h = 2\lambda + \mathcal{L}(h + \|h\|) + (\lambda - \|h\|),$$

where we have used $\mathcal{L}1 = 1$. Then since both terms on the right belong to $\mathcal{C}_\mathbb{R}$, so does $3\lambda + \mathcal{L}h$. Now replacing h with $-h$, and since $\|-h\| = \|h\|$, we have also that $3\lambda - \mathcal{L}h \in \mathcal{C}_\mathbb{R}$. We conclude that $\|\mathcal{L}h\| \le 3\lambda$, which completes the claim and the proof of Proposition 7.9 with $C_L = 3\bar{C}_L$.

Proof of Sublemma 7.19. (a) We would like to simply apply the second cone condition (7.4) to the integral in part (a), but $\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)$ is not necessarily a valid test function. Following (7.44), it suffices to choose

$$a^{-1}H^\beta(\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)) + |\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)|_{C^0} \le D_j \le |\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)|_{C^\beta(U_j^1)}$$

in order to guarantee that $\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2) + D_j \in \mathcal{D}_{a,\beta}(U_j^1)$. We proceed to estimate $|\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)|_{C^\beta(U_j^1)}$.

First, from [9, eq. (5.18)] it follows that,

$$|\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)|_{C^0(U_j^1)} \le C d_{\mathcal{W}^s}(W^1,W^2)^\alpha |\widehat{T}_{U_j^2}(\psi_2)|_{C^0(U_j^2)} . \tag{7.66}$$

Then by standard estimates (see, for example, the proof of Sublemma 7.17 or [11, Lemma 4.3]), we conclude

$$|\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)|_{C^\beta(U_j^1)} \le 3d_{\mathcal{W}^s}(W^1,W^2)^{\alpha-\beta} \max\{C|\widehat{T}_{U_j^2}(\psi_2)|_{C^0}, H^\beta(\widehat{T}_{U_j^1}(\psi_1) + H^\beta(\widetilde{T}_{U_j^2}(\psi_2))\} .$$

However, since $\psi_k \in \mathcal{D}_{a,\alpha}(W^k)$ and $J_{U_j^k}T$ and $\frac{\|G'_{U_j^1}\|}{\|G'_{U_j^2}\|} \circ G_{U_j^1}^{-1}$ enjoy similar distortion bounds by [9, Lemma 5.5], we have

$$H^\beta(\widehat{T}_{U_j^1}(\psi_1) + H^\beta(\widetilde{T}_{U_j^2}(\psi_2)) \le C'(|\widehat{T}_{U_j^1}(\psi_1)|_{C^0} + |\widetilde{T}_{U_j^2}(\psi_2)|_{C^0}) \le C''|\widehat{T}_{U_j^2}(\psi_2)|_{C^0} .$$

Combining this estimate with (7.66) yields,

$$|\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2)|_{C^\beta(U_j^1)} \le C d_{\mathcal{W}^s}(W^1, W^2)^{\alpha-\beta} |\widehat{T}_{U_j^2}(\psi_2)|_{C^0(U_j^2)}\,, \tag{7.67}$$

for some uniform $C > 0$. We will use this as our upper bound on D_j.

With this bound established, we complete the proof of (a) using the second cone condition (7.4),

$$\begin{aligned}
&\left|\int_{U_j^1} h(\widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2) + D_j)\right| + \left|\int_{U_j^1} hD_j\right| \\
&\le \left(\fint_{U_j^1} \widehat{T}_{U_j^1}(\psi_1) - \widetilde{T}_{U_j^2}(\psi_2) + 2D_j\right) A|U_j^1|^q \delta^{1-q} |||h|||_- \\
&\le 12 C d_{\mathcal{W}^s}(W^1, W^2)^{\alpha-\beta} A 2^{1-q} \delta |||h|||_- |J_{U_j^2} T|_{C^0(U_j^2)}\,,
\end{aligned}$$

where we have used the previous bound $|\psi_2|_{C^0} \le 4$.

(b) Since have already verified that $\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j \in \mathcal{D}_{a,\alpha}(U_j^1)$ and $\widehat{T}_{U_j^2}(\psi_2) + B_j \in \mathcal{D}_{a,\alpha}(U_j^2)$, with $d_*(\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j, \widehat{T}_{U_j^2}(\psi_2) + B_j) = 0$, we may apply the third cone condition (7.5) immediately to obtain,

$$\left|\frac{\int_{U_j^1} h(\widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j)}{\fint_{U_j^1} \widetilde{T}_{U_j^2}(\psi_2) + \tilde{B}_j} - \frac{\int_{U_j^2} h(\widehat{T}_{U_j^2}(\psi_2) + B_j)}{\fint_{U_j^2} \widehat{T}_{U_j^2}(\psi_2) + B_j}\right| \le d_{\mathcal{W}^s}(U_j^1, U_j^2)^\gamma \delta^{1-\gamma} c A |||h|||_-\,.$$

Statement (b) of the lemma follows using again that $d_{\mathcal{W}^s}(U_j^1, U_j^2) \le C d_{\mathcal{W}^s}(W^1, W^2)$ for some uniform $C > 0$ by [11, Lemma 3.3].

(c) Since the U_j^k are vertically aligned, we use (7.42) to bound the difference in curve lengths. Statement (c) follows immediately using the second cone condition (7.4) to estimate the integral since $|U_j^1| \le 2\delta$. □

Appendix A. Complexification

Proof of Lemma 2.5. We follow [40, Section 1 (b)]. We consider the real vector space $\mathcal{B}_k^2$ and, for $(x, y) \in \mathcal{B}_k^2$, define the multiplication by a complex number $a + ib \in \mathbb{C}$ by

$$(a + ib)(x, y) = (ax - by, ay + bx). \tag{A.1}$$

One can check directly that this defines a complex vector space. Next, we define the norms

$$\begin{aligned}
\|(x, y)\|_{r,k} &= \sqrt{\|x\|_k^2 + \|y\|_k^2} \\
\|(x, y)\|_{c,k} &= \sup_{\theta \in [0, 2\pi]} \|e^{i\theta}(x, y)\|_{r,k}.
\end{aligned}$$

Note that, since[31]

$$\begin{aligned}
\|(x, y) + (w, z)\|_{r,k} &\le \sqrt{\|x\|_k^2 + \|w\|_k^2 + \|z\|_k^2 + \|y\|_k^2 + 2\|x\|_k\|w\|_k + 2\|y\|_k\|z\|_k} \\
&\le \|(x, y)\|_{r,k} + \|(w, z)\|_{r,k}
\end{aligned}$$

$\|\cdot\|_{r,k}$ is a norm for $\mathcal{B}_k^2$ and with such a norm $\mathcal{B}_k^2$ is obviously a real Banach space. On the other hand, $\|e^{i\theta}(x, y)\|_{r,k}$ is a continuous function of θ, hence there exists θ_* such that

$$\begin{aligned}
\|(x, y) + (w, z)\|_{c,k} &= \sup_{\theta \in [0, 2\pi]} \|e^{i\theta}[(x, y) + (w, z)]\|_{r,k} = \|e^{i\theta_*}[(x, y) + (w, z)]\|_{r,k} \\
&\le \|e^{i\theta_*}(x, y)\|_{r,k} + \|e^{i\theta_*}(w, z)]\|_{r,k} \le \|(x, y)\|_{c,k} + \|(w, z)\|_{c,k}.
\end{aligned}$$

[31] The second inequality follows from

$$\begin{aligned}
\sqrt{(\|x\|_k^2 + \|y\|_k^2)(\|w\|_k^2 + \|z\|_k^2)} &\ge \sqrt{\|x\|_k^2\|w\|_k^2 + \|y\|_k^2\|z\|_k^2 + 2\|y\|_k\|w\|_k\|x\|_k\|z\|_k} \\
&= \|x\|_k\|w\|_k + \|y\|_k\|z\|_k.
\end{aligned}$$

In addition, for all $\rho \in \mathbb{R}_+, \varphi \in \mathbb{R}$, we have

$$\|\rho e^{i\varphi}(x,y)\|_{c,k} = \sup_\theta \|\rho e^{i\theta}(x,y)\|_{r,k} = \rho\|(x,y)\|_{c,k} = |\rho e^{i\varphi}|\|(x,y)\|_{c,k}.$$

That is $\|\cdot\|_{c,k}$ is a norm for the complex vector space $\mathbb{B}_k$. Note that, by equation (A.1), it is natural to write the elements of $\mathbb{B}_k$ as $x+iy$, $x,y \in \mathcal{B}_k$.

Since the norm $\|\cdot\|_{c,k}$ dominates the real norm, we have that $\mathbb{B}_k$ is a complex Banach space. We have then the canonical extension of $\mathcal{L}_k$ defined as $\mathcal{L}_k(x+iy) = \mathcal{L}_k x + i\mathcal{L}_k y$. In addition,

$$\|x+iy\|_{c,k} \le \|x\|_{c,k} + \|y\|_{c,k} = \|x\|_{r,k} + \|y\|_{r,k} \le \sqrt{2}\|x+iy\|_{r,k}\,,$$

which implies

$$\|h\|_{c,k} \le \sqrt{2}\|h\|_{r,k}. \tag{A.2}$$

To conclude the proof note that

$$\begin{aligned}
\|\mathcal{L}_{k+m}\cdots\mathcal{L}_k(x+iy)\|_{c,k+m+1} &= \|\mathcal{L}_{k+m}\cdots\mathcal{L}_k x + i\mathcal{L}_{k+m}\cdots\mathcal{L}_k y\|_{c,k+m+1}\\
&\le \|\mathcal{L}_{k+m}\cdots\mathcal{L}_k x\|_{c,k+m+1} + \|i\mathcal{L}_{k+m}\cdots\mathcal{L}_k y\|_{c,k+m+1}\\
&= \|\mathcal{L}_{k+m}\cdots\mathcal{L}_k x\|_{k+m+1} + \|\mathcal{L}_{k+m}\cdots\mathcal{L}_k y\|_{k+m+1}\\
&\le C_*(\|x\|_k + \|y\|_k) \le C_*\sqrt{2}\|(x,y)\|_{r,k}\\
&\le C_*\sqrt{2}\|x+iy\|_{c,k}.
\end{aligned}$$

□

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