Solutions to Selected Exercises from Problem Set 4
Introduction to Stochastic Processes, MA 391
Spring 2018

Chapter 4

20. A transition probability matrix is said to be doubly stochastic if the sum over columns equals 1. If such a chain is irreducible and aperiodic and consists of $M+1$ states, $0, 1, \ldots, M$, show that the limiting probabilities are given by $\pi_j = \frac{1}{M+1}$, $j = 0, 1, \ldots, M$.

Solution. First we show by induction that if P is doubly stochastic, then so is P^n for each n. The base case $n = 1$ is given. Now fix n and assume that P^n is doubly stochastic and show that P^{n+1} is doubly stochastic. Since $P^{n+1} = P^n P$, by the definition of matrix multiplication, we have,

$$P_{i,j}^{n+1} = \sum_{k=0}^{M} P_{i,k}^n P_{k,j}, \qquad \text{for each } i \text{ and } j.$$

Summing this equation over i, we get

$$\begin{aligned}
\sum_{i=0}^{M} P_{i,j}^{n+1} &= \sum_{i=0}^{M} \sum_{k=0}^{M} P_{i,k}^n P_{k,j} \qquad \text{now reverse order of summation} \\
&= \sum_{k=0}^{M} \sum_{i=0}^{M} P_{i,k}^n P_{k,j} \qquad \text{sum over } i \text{ using the fact that } P^n \text{ is doubly stochastic} \\
&= \sum_{k=0} P_{k,j} = 1 \qquad \text{summing over } k \text{ using the fact that } P \text{ is doubly stochastic.}
\end{aligned}$$

Since this is true for each j, P^{n+1} is doubly stochastic. By induction, we conclude that P^n is doubly stochastic for each $n \in \mathbb{N}$.

Since the chain is irreducible and aperiodic, we know by our convergence theorem that

$$\lim_{n\to\infty} \vec{v}_0 P^n = \vec{\pi} \qquad \text{for any probability vector } \vec{v}_0.$$

If we choose $\vec{v}_0$ to be the vector of all zeros except for a 1 in the ith entry, then $\vec{v}_0 P^n$ is simply the ith row of P^n. This means that the ith row of P^n converges to $\vec{\pi}$ and this is true for each $i = 0, 1, \ldots, M$. Thus,

$$\lim_{n\to\infty} P_{i,j}^n = \pi_j \qquad \text{for each } i, j = 0, 1, \ldots, M.$$

Now let's fix j and sum this equation over i.

$$\lim_{n\to\infty} \sum_{i=0}^{M} P_{i,j}^n = \sum_{i=0}^{M} \pi_j.$$

Since P^n is doubly stochastic, the sum over i on the left hand side of the equation is simply 1. Notice that on the right hand side, π_j is independent of i so we are just adding the constant π_j to itself $M+1$ times. So,

$$1 = \sum_{i=0}^{M} \pi_j = (M+1)\pi_j \quad \Longrightarrow \quad \pi_j = \frac{1}{M+1}.$$

Since this is true for each j, we are done. □

Alternate Solution. Since the Markov chain is irreducible and aperiodic, we know that we have a unique stationary distribution vector $\vec{\pi}$ that satisfies $\vec{\pi} = \vec{\pi} P$ and represents the limiting probabilities of the chain.

Since we are given that the vector should be $\pi_j = \frac{1}{M+1}$, $j = 0, 1, \ldots, M$, we can just plug this into the equation $\vec{\pi} = \vec{\pi}P$. If it holds true, then this must be the stationary distribution.

For each j, we have

$$\pi_j = \sum_{i=0}^{M+1} P_{i,j}\pi_i \quad \text{and if } \pi_i = \tfrac{1}{M+1} \text{ for each } i\text{, then}$$
$$\pi_j = \sum_{i=0}^{M+1} P_{i,j}\frac{1}{M+1} = \frac{1}{M+1}\sum_{i=0}^{M+1} P_{i,j} = \frac{1}{M+1},$$

where in the last line we used the fact that P is doubly stochastic to sum over i. □

23. In a good weather year, the number of storms is Poisson distributed with mean 1; in a bad weather year, the number of storms is Poisson distributed with mean 3. A good year is equally likely to be followed by a good or bad year. A bad year is twice as likely to be followed by a bad year as a good year. Suppose year 0 was a good year.

a) Find the expected number of storms in years 1 and 2.
b) Find the probability that there are no storms in year 3.
c) Find the long-run average of the number of storms per year.

Solution. Let $X_n = 0$ if year n is good and $X_n = 1$ if year n is bad. Let S_n denote the number of storms in year n. The transition matrix for X_n is

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

(a) Let $\vec{v}_0 = \langle 1, 0\rangle$ represent the distribution of X_0, i.e. definitely a good year. Then $\vec{v}_1 = \vec{v}_0 P$ represents the distribution of X_1 and $\vec{v}_2 = \vec{v}_0 P^2$ represents the distribution of X_2. We have

$$P^2 = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{bmatrix} \qquad P^3 = \begin{bmatrix} \frac{29}{72} & \frac{43}{72} \\ \frac{43}{108} & \frac{65}{108} \end{bmatrix} \tag{1}$$

So

$$\vec{v}_1 = \vec{v}_0 P = \langle \tfrac{1}{2}, \tfrac{1}{2}\rangle \qquad \text{and} \qquad \vec{v}_2 = \vec{v}_0 P^2 = \langle \tfrac{5}{12}, \tfrac{7}{12}\rangle.$$

Then recalling that S_n is Poisson with mean 1 in a good year and mean 3 in a bad year, we have

$$E[S_1] = E[S_1|X_1 = 0]P(X_1 = 0) + E[S_1|X_1 = 1]P(X_1 = 1) = (1)(\tfrac{1}{2}) + (3)(\tfrac{1}{2}) = 2,$$
$$E[S_2] = E[S_2|X_2 = 0]P(X_2 = 0) + E[S_2|X_2 = 1]P(X_2 = 1) = (1)(\tfrac{5}{12}) + (3)(\tfrac{7}{12}) = \tfrac{13}{6}.$$

So the expected number of storms in the next two years is $E[S_1] + E[S_2] = 2 + \frac{13}{6} = \frac{25}{6}$.

(b) Using (1), we have that the distribution of X_3 is given by $\vec{v}_3 = \vec{v}_0 P^3 = \langle \frac{29}{72}, \frac{43}{72}\rangle$. Thus conditioning on whether year 3 is good or bad, we have,

$$P(S_3 = 0) = P(S_3 = 0|X_3 = 0)P(X_3 = 0) + P(S_3 = 0|X_3 = 1)P(X_3 = 1)$$
$$= e^{-1}\frac{1^0}{0!}\left(\frac{29}{72}\right) + e^{-3}\frac{3^0}{0!}\left(\frac{43}{72}\right) = \frac{29}{72}\cdot\frac{1}{e} + \frac{43}{72}\cdot\frac{1}{e^3} \approx .18.$$

(c) In the long run, S_n will depend on the stationary distribution of X_n. This means we must find the invariant vector $\vec{\pi}$ of P. We solve for $\vec{\pi}$ using the equation $\vec{\pi} = \vec{\pi}P$. This yields two equations,

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1, \qquad \pi_1 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1.$$

Solving (you can either use elimination or row reduction), we get $\vec{\pi} = \langle \frac{2}{5}, \frac{3}{5}\rangle$. Thus,

$$\lim_{n\to\infty} E[S_n] = \lim_{n\to\infty} E[S_n|X_n = 0]P(X_n = 0) + E[S_n|X_n = 1]P(X_n = 1)$$
$$= \lim_{n\to\infty} E[S_n|X_n = 0]\cdot\pi_0 + E[S_n|X_n = 1]\cdot\pi_1 = 1(\tfrac{2}{5}) + 3(\tfrac{3}{5}) = \tfrac{11}{5} = 2.2. \qquad \square$$

25. Each morning a runner leaves by either the front door or back door and returns by either the front or back door. He takes a pair of shoes from the door he exits and leaves a pair at the door he enters. If he owns a total of k pairs of shoes, what is the proportion of time that he runs barefoot?

Solution. Our runner goes running barefoot if two (independent) events occur: there are zero shoes at either of the two doors AND he chooses that door. Let's focus first on computing the long-run proportion of time that there are zero shoes at either of the two doors. Let X_n denote the number of shoes at the front door at the beginning of day n's run. We compute the following transition probabilities,

$$\begin{aligned} P_{i,i+1} &= P(\text{leave by back door})P(\text{return by front door}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \\ P_{i,i} &= P(\text{he enters and returns by the same door}) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2} \\ P_{i,i-1} &= P(\text{leave by front door})P(\text{return by back door}) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \end{aligned}$$

These probabilities are valid for $i = 1, 2, \dots k-1$. For the two boundary cases, $i = 0, k$, we have,

$$\begin{aligned} P_{0,0} &= P(\text{leave by front door}) + P(\text{leave by back door AND return by back door}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ P_{0,1} &= P(\text{leave by back door AND return by front door}) = \frac{1}{4} \\ P_{k,k} &= P(\text{leave by back door}) + P(\text{leave by front door AND return by front door}) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ P_{k,k-1} &= P(\text{leave by front door AND return by back door}) = \frac{1}{4} \end{aligned}$$

The probability transition matrix is therefore given by

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

To find the long-run proportion of time that the chain spends at either of the two states 0 and k, we need to find the stationary distribution $\vec{\pi}$. This can be done in one of two ways. Either (1) solve the equation $\vec{\pi} = \vec{\pi}P$ starting with the first component and solving for each component of $\vec{\pi}$ in terms of π_0; or (2) note that the matrix is doubly stochastic (all columns sum to 1) and use exercise 20 to conclude that all entries of $\vec{\pi}$ must be equal. Since $\vec{\pi}$ has $k+1$ entries, this means $\pi_i = \frac{1}{k+1}$ for each $i = 0, 1, \dots, k$.

Now as stated at the beginning of the problem, our runner runs barefoot if two (independent) events occur: there are zero shoes at either of the two doors AND he chooses that door. Notice that there are 0 shoes at the back door if there are k shoes at the front door. So (all the probabilities below refer to long run probabilities),

$$\begin{aligned} P(\text{runs barefoot}) &= P(\text{chooses front door}|0\text{ shoes at front door})P(0\text{ shoes at front door}) \\ &\quad + P(\text{chooses back door}|0\text{ shoes at back door})P(0\text{ shoes at back door}) \\ &= \tfrac{1}{2}\pi_0 + \tfrac{1}{2}\pi_k = \tfrac{1}{k+1}. \end{aligned}$$

□

34. A flea jumps around 3 vertices of a triangle with probability p_i of going clockwise and $q_i = 1 - p_i$ of going counterclockwise, $i = 1, 2, 3$.

a) Find the proportion of time that the flea is at each vertex.

b) How often does the flea make a counterclockwise move followed by 5 consecutive clockwise moves?

Solution. If we arrange the vertices 1, 2, 3 in clockwise order, the transition matrix is given by

$$P = \begin{bmatrix} 0 & p_1 & 1-p_1 \\ 1-p_2 & 0 & p_2 \\ p_3 & 1-p_3 & 0 \end{bmatrix}.$$

To answer (a), we need to find the stationary distribution $\vec{\pi} = [\pi_1, \pi_2, \pi_3]$. We do this by solving $\vec{\pi} = \vec{\pi}P$, or equivalently, $(P^T - I)\vec{\pi}^T = \vec{0}$. We row reduce, assuming $p_1 q_2 - 1 \neq 0$.

$$\left[\begin{array}{ccc|c} -1 & 1-p_2 & p_3 & 0 \\ p_1 & -1 & 1-p_3 & 0 \\ 1-p_1 & p_2 & -1 & 0 \end{array}\right] \xrightarrow{r_3 \to r_3 + r_2 + r_1} \left[\begin{array}{ccc|c} -1 & 1-p_2 & p_3 & 0 \\ p_1 & -1 & 1-p_3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_2 \to r_2 + p_1 r_1} \left[\begin{array}{ccc|c} -1 & 1-p_2 & p_3 & 0 \\ 0 & p_1 q_2 - 1 & 1-p_3+p_1 p_3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \to r_2/(p_1 q_2 - 1)} \left[\begin{array}{ccc|c} -1 & 1-p_2 & p_3 & 0 \\ 0 & 1 & \frac{1-p_3 q_1}{p_1 q_2 - 1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_1 \to -r_1 + q_2 r_2} \left[\begin{array}{ccc|c} 1 & 0 & -p_3 + q_2 \frac{1-p_3 q_1}{p_1 q_2 - 1} & 0 \\ 0 & 1 & \frac{1-p_3 q_1}{p_1 q_2 - 1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is reduced row echelon form. We now simplify the $(1,3)$ entry of the matrix. Some algebra yields,

$$-p_3 + q_2 \frac{1-p_3 q_1}{p_1 q_2 - 1} = \frac{-p_3(p_1 q_2 - 1) + q_2(1 - p_3 q_1)}{p_1 q_2 - 1} = \frac{1 - p_2 q_3}{p_1 q_2 - 1}.$$

Translating the augmented matrix into equations for $\vec{\pi}$ yields,

$$\pi_1 = \frac{1 - p_2 q_3}{1 - p_1 q_2} \pi_3 \qquad \text{and} \qquad \pi_2 = \frac{1 - p_3 q_1}{1 - p_1 q_2} \pi_3.$$

Now using the fact that $\vec{\pi}$ is a probability vector,

$$1 = \pi_1 + \pi_2 + \pi_3 = \frac{1 - p_2 q_3}{1 - p_1 q_2}\pi_3 + \frac{1 - p_3 q_1}{1 - p_1 q_2}\pi_3 + \pi_3 = \frac{3 - (p_1 q_2 + p_2 q_3 + p_3 q_1)}{1 - p_1 q_2}\pi_3.$$

This implies that,

$$\begin{aligned} \pi_1 &= \frac{1 - p_2 q_3}{3 - (p_1 q_2 + p_2 q_3 + p_3 q_1)}, \quad \pi_2 = \frac{1 - p_3 q_1}{3 - (p_1 q_2 + p_2 q_3 + p_3 q_1)}, \\ \pi_3 &= \frac{1 - p_1 q_2}{3 - (p_1 q_2 + p_2 q_3 + p_3 q_1)}. \end{aligned} \tag{2}$$

Notice that the denominator is zero only if $p_1 q_2 + p_2 q_3 + p_3 q_1 = 3$. This can happen only if each term equals 1. But this would require both p_1 and q_1 to be 1, which is impossible.

We have derived these stationary probabilities **assuming that** $1 - p_1 q_2 \neq 0$. But if $1 - p_1 q_2 = 0$, we must have $p_1 = q_2 = 1$. This immediately implies $q_1 = p_2 = 0$ and a quick look at the diagram shows that in this case, the flea never jumps to vertex 3 and spends time evenly in 1 and 2. Thus, $\pi_1 = \pi_2 = \frac{1}{2}$ and $\pi_3 = 0$. This agrees with the formulas in (2) so that these formulas hold in all cases.

To answer (b), we condition on the flea being in each one of the 3 vertices, distributed according to the vector $\vec{\pi}$. For brevity, let E be the event that the flea makes the required sequence of moves and let F be the position of the flea.

$$\begin{aligned} P(E) &= P(E \mid F = 1)P(F = 1) + P(E \mid F = 2)P(F = 2) + P(E \mid F = 3)P(F = 3) \\ &= q_1 p_3 p_1 p_2 p_3 p_1 \pi_1 + q_2 p_1 p_2 p_3 p_1 p_2 \pi_2 + q_3 p_2 p_3 p_1 p_2 p_3 \pi_3 \\ &= p_1 p_2 p_3 (q_1 p_1 p_3 \pi_1 + q_2 p_1 p_2 \pi_2 + q_3 p_2 p_3 \pi_3). \end{aligned}$$

□