Solutions to Selected Exercises from Problem Set 5
Introduction to Stochastic Processes, MA 391
Spring 2018

Chapter 4

46. The umbrella problem.

Solution. (a) Let $X_n = \#$ umbrellas at home at the beginning of day n. The possible values of X_n are $0, 1, \ldots, r$. These are the $r+1$ states of the Markov chain. To compute the transition probabilities, first let $i = 1, \ldots r-1$ and consider how we can go from i umbrellas at home at the beginning of day n to $i-1$ umbrellas at home at the beginning of day $n+1$. Two independent events must occur: it must be raining in the morning (so the man takes an umbrella to work) AND it must be not raining in the evening (so the man does not take an umbrella home). Thus

$$P_{i,i-1} = p(1-p) \qquad \text{for } i = 1, \ldots, r-1.$$

Similarly, to gain an umbrella at home by the next day, it must be not raining in the morning AND it must be raining in the evening. For the number of umbrellas at home to remain the same, it is either raining both in the morning and in the evening or it is raining neither in the morning nor in the evening. Thus

$$P_{i,i+1} = p(1-p) \quad \text{and} \quad P_{i,i} = p^2 + (1-p)^2 \quad \text{for } i = 1, \ldots, r-1.$$

Now the two boundary cases are $i = 0$ and $i = r$. For $i = 0$, we have,

$$P_{0,0} = \text{Prob(no rain in evening)} = 1-p \quad \text{and} \quad P_{0,1} = \text{Prob(rain in the evening)} = p$$

since it doesn't matter what happens in the morning since he cannot take an umbrella either way. Similarly, for $i = r$, we have

$$P_{r,r} = \text{Prob(no rain in morning)} + \text{Prob(rain in morning and rain in evening)} = 1 - p + p^2$$
$$\text{and} \quad P_{r,r-1} = \text{Prob(rain in the morning and no rain in evening)} = p(1-p).$$

(b) To find the invariant vector $\vec{\pi}$, we set up the transition matrix.

$$P = \begin{bmatrix} 1-p & p & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ p(1-p) & p^2+(1-p)^2 & p(1-p) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & p(1-p) & p^2+(1-p)^2 & p(1-p) & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & p(1-p) & p^2+(1-p)^2 & p(1-p) & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & p(1-p) & p^2+(1-p)^2 & p(1-p) & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & p(1-p) & p^2+(1-p)^2 & p(1-p) \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & p(1-p) & 1-p+p^2 \end{bmatrix}$$

We solve for $\vec{\pi}$ using the equation $\vec{\pi} = \vec{\pi}P$. For the first component,

$$\pi_0 = \pi_0(1-p) + \pi_1 p(1-p) \quad \text{which yields} \quad \pi_0 = (1-p)\pi_1.$$

For the second component,

$$\pi_1 = p\pi_0 + (p^2 + (1-p)^2)\pi_1 + p(1-p)\pi_2\ , \quad \text{now substitute } \pi_0 = (1-p)\pi_1,$$
$$\Longrightarrow \pi_1 = p(1-p)\pi_1 + (p^2 + (1-p)^2)\pi_1 + p(1-p)\pi_2 \quad \Longrightarrow \quad \pi_1 = \pi_2.$$

Now writing the equation for the third component, π_2, we have

$$\pi_2 = p(1-p)\pi_1 + (p^2 + (1-p)^2)\pi_2 + p(1-p)\pi_3 \quad \Longrightarrow \quad \pi_2 = \pi_3 \text{ after substituting } \pi_1 = \pi_2.$$

Note that the rest of the columns up to $r-1$ have the same entries, just shifted by one position, so that we get $\pi_1 = \pi_2 = \pi_3 = \cdots = \pi_{r-1}$. For the last column, we have

$$\pi_r = p(1-p)\pi_{r-1} + (1-p+p^2)\pi_r \quad \Longrightarrow \quad \pi_r = \pi_{r-1} = \pi_1 \text{ as well.}$$

Now we use the fact that all entries of $\vec{\pi}$ add to 1,

$$1 = \pi_0 + \pi_1 + \pi_2 + \ldots + \pi_r = (1-p)\pi_1 + r\pi_1$$
$$\Longrightarrow \pi_1 = \frac{1}{r+1-p} \quad \text{and} \quad \pi_0 = \frac{1-p}{r+1-p},$$

which proves (b) since $\pi_1 = \pi_2 = \cdots = \pi_r$.

(c) Our man gets wet if either:

A: there are 0 umbrellas at home in the morning and it rains in the morning

B: there are r umbrellas at home in the morning and it does not rain in the morning (so that he does not take an umbrella to work) and it does rain in the evening.

Thus

$$P\left(\begin{array}{c}\text{man gets}\\ \text{wet per day}\end{array}\right) = P(A) + P(B) = \pi_0 p + \pi_r(1-p)p = \frac{p(1-p)}{r+1-p} + \frac{p(1-p)}{r+1-p} = \frac{2p(1-p)}{r+1-p}.$$

If we want the frequency with which the man gets wet per trip, we divide this by 2 (since there are 2 trips per day):

$$P\left(\begin{array}{c}\text{man gets}\\ \text{wet per trip}\end{array}\right) = \frac{p(1-p)}{r+1-p}. \tag{1}$$

(d) When $r = 3$, we must maximize the function

$$f(p) = \frac{p-p^2}{4-p} \qquad 0 \le p \le 1.$$

Taking the derivative (using the quotient rule), we have

$$f'(p) = \frac{(4-p)(1-2p) - (p-p^2)(-1)}{(4-p)^2} = \frac{4-8p+p^2}{(4-p)^2}.$$

Setting this fraction equal to 0, we find the two roots of the numerator are $p = 4 \pm \sqrt{12}$. The only one of these roots that is between 0 and 1 is $4 - \sqrt{12}$. Notice also that $f(0) = f(1) = 0$ so that the probability is not maximized at the endpoints. Thus $p = 4 - \sqrt{12} \approx .536$ maximizes the fraction of time that the man gets wet. □

Alternate Solution. Alternatively, you can define the Markov chain as $X_n = \#$ umbrellas at current location after n trips (between home and office). Now if we have i umbrellas at the current location, there are $r-i$ at the other location. So if it is not raining and $X_n = i$, then he does not take an umbrella and arrives at the next location with $X_{n+1} = r - i$; this occurs with probability $1-p$. On the other hand, if it does rain, then he takes an umbrella and the new location gains one umbrella: $X_{n+1} = r - i + 1$ with probability p. Thus the transition matrix is

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & q & p \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & q & p & 0 \\ \vdots & & & & & \vdots & & & & \vdots \\ 0 & 0 & q & p & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & q & p & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ q & p & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

Similar algebra as in part (b) above solves for the entries of the vector $\vec{\pi}$. (Interesting fact: although the matrices are quite different, we get the same values for $\pi_0, \pi_1, \ldots, \pi_r$ using either method.) Now we compute the probability of the man getting wet per trip on average as simply $p\pi_0$, which is the same as (1). The calculations for part (d) are also the same. □

52. Find the taxi driver's average profit per trip.

Solution. First notice that the information given in the problem is sufficient to find the expected profit of a trip conditioned on starting in zone A or B. Let $A \to B$ denote the event that a fare picked up in zone A will have a destination in zone B with analogous definitions for the events $A \to A$, $B \to A$ and $B \to B$. Then we can write

$$\begin{aligned} \mathbb{E}[\text{Profit} \mid \text{pickup in } A] &= \mathbb{E}[\text{Profit} \mid \text{pickup in } A, A \to A] \cdot P(A \to A \mid \text{pickup in } A) \\ &\quad + \mathbb{E}[\text{Profit} \mid \text{pickup in } A, A \to B] \cdot P(A \to B \mid \text{pickup in } A) \qquad (2) \\ &= 6 \cdot (.6) + 12 \cdot (.4) = 3.6 + 4.8 = 8.4. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}[\text{Profit} \mid \text{pickup in } B] &= \mathbb{E}[\text{Profit} \mid \text{pickup in } B, B \to B] \cdot P(B \to B \mid \text{pickup in } B) \\ &\quad + \mathbb{E}[\text{Profit} \mid \text{pickup in } B, B \to A] \cdot P(B \to B \mid \text{pickup in } B) \qquad (3) \\ &= 8 \cdot (.7) + 12 \cdot (.3) = 5.6 + 3.6 = 9.2. \end{aligned}$$

In order to compute the average profit per trip, we need the long run frequency with which the taxi driver picks up fares in zones A and B. Letting zone A be state 1 and zone B be state 2, we can use the information in the problem to write the following transition matrix describing the probabilities of moving from one zone to the next with each new fare. The Markov chain X_n describes the zone in which the nth fare is picked up.

$$P = \begin{bmatrix} .6 & .4 \\ .3 & .7 \end{bmatrix}$$

We find the invariant vector $\vec{\pi}$ by solving $\vec{\pi} = \vec{\pi}P$ or equivalently, $(P^T - I)\vec{\pi}^T = \vec{0}$. We row reduce the augmented matrix $[P^T - I \mid \vec{0}]$.

$$\left[\begin{array}{rr|r} -.4 & .3 & 0 \\ .4 & -.3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{rr|r} -.4 & .3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This tells us that $(-.4)\pi_1 + (.3)\pi_2 = 0$ so that $\pi_2 = \frac{4}{3}\pi_1$. Then using the fact that $\vec{\pi}$ is a probability vector,

$$1 = \pi_1 + \pi_2 = \pi_1 + \tfrac{4}{3}\pi_1 = \tfrac{7}{3}\pi_1 \quad \Longrightarrow \quad \pi_1 = \tfrac{3}{7} \text{ and } \pi_2 = \tfrac{4}{7}.$$

Now using $\vec{\pi}$, we can compute the taxi driver's average profit per trip as the expected value of the profit based on the long run frequency of fares picked up in each zone. We condition on picking up a fare in zone A or B and use (2) and (3).

$$\begin{aligned} \mathbb{E}[\text{Profit}] &= \mathbb{E}[\text{Profit} \mid \text{pickup in } A] \cdot P(\text{pick up in } A) \\ &\quad + \mathbb{E}[\text{Profit} \mid \text{pickup in } B] \cdot P(\text{pick up in } B) \\ &= 8.4\pi_1 + 9.2\pi_2 = 8.4(\tfrac{3}{7}) + 9.2(\tfrac{4}{7}) = \tfrac{62}{7} = 8\tfrac{6}{7}. \end{aligned}$$

□

57. A particle moves among $n+1$ vertices that are ordered clockwise around a circle. If the particle starts in vertex 0, find the probability that it visits every vertex at least once before returning to 0.

Solution. The particle moves one step clockwise with probability p and one step counterclockwise with probability $1-p$. Picture the vertices arranged around a circle like a clock, with 0 at the 12 o'clock position and the other vertices numbered clockwise so that vertex n is near the top in the 11 o'clock position.

Suppose the first step the particle takes is clockwise to vertex 1. Now starting at vertex 1, we must compute the probability that the particle visits vertices 2 through n before returning to 0. Following the hint in the text, we frame this question in terms of the gambler's ruin problem. In the gambler's ruin problem, the gambler starts with i units (dollars) and with each bet, wins a dollar with probability p and loses a dollar with probability $1-p$. The probability that the gambler starting with i dollars reaches $N > i$ dollars before reaching 0 dollars (going broke) is given on page 232 as

$$P_i = \begin{cases} \dfrac{1-(\frac{1-p}{p})^i}{1-(\frac{1-p}{p})^N}, & \text{if } p \neq \frac{1}{2} \\ \dfrac{i}{N}, & \text{if } p = \frac{1}{2} \end{cases}. \tag{4}$$

We interpret this in terms of our particle moving around the circle by identifying vertex i with having i dollars. Winning a bet corresponds to taking one step clockwise (with probability p) and losing a bet corresponds with taking one step counterclockwise (with probability $1-p$. We have to reach vertex n before reaching vertex 0 so that our n is the N in the gambler's ruin. We start from vertex 1, so $i = 1$. Thus using (4), the probability that starting from vertex 1, we reach vertex n before reaching vertex 0 is

$$P_1 = \begin{cases} \dfrac{1-(\frac{1-p}{p})}{1-(\frac{1-p}{p})^n}, & \text{if } p \neq \frac{1}{2} \\ \dfrac{1}{n}, & \text{if } p = \frac{1}{2} \end{cases}.$$

Next we suppose the first step the particle takes is counterclockwise to vertex n. We want to calculate the probability of visiting vertices $n-1$ down to 1 before returning to vertex 0. This is really the same calculation as above with the roles of p and $1-p$ interchanged since the steps we want to take are oriented counterclockwise from vertex n instead of clockwise. Using this symmetry and interchanging the roles of p and $1-p$, we find that the probability that starting from vertex n, we reach vertex 1 before reaching vertex 0 is

$$Q_1 = \begin{cases} \dfrac{1-(\frac{p}{1-p})}{1-(\frac{p}{1-p})^n}, & \text{if } p \neq \frac{1}{2} \\ \dfrac{1}{n}, & \text{if } p = \frac{1}{2} \end{cases}.$$

Let E be the event that every vertex is visited before returning to 0. Then conditioning on the first transition, we have,

$$\begin{aligned} P(E) &= P(E \mid \text{first step is clockwise}) \cdot P(\text{first step is clockwise}) \\ &\quad + P(E \mid \text{first step is counterclockwise}) \cdot P(\text{first step is counterclockwise}) \\ &= P_1 \cdot p + Q_1 \cdot (1-p) \end{aligned}$$

Now if $p \neq \frac{1}{2}$, we use the first formula for P_1 and Q_1,

$$\begin{aligned} P(E) &= \frac{1-(\frac{1-p}{p})}{1-(\frac{1-p}{p})^n} p + \frac{1-(\frac{p}{1-p})}{1-(\frac{p}{1-p})^n}(1-p) = \frac{p-(1-p)}{1-(\frac{1-p}{p})^n} + \frac{1-p-(p)}{1-(\frac{p}{1-p})^n} \\ &= \frac{(2p-1)p^n}{p^n-(1-p)^n} + \frac{(1-2p)(1-p)^n}{(1-p)^n-p^n} = \frac{(2p-1)(p^n+(1-p)^n)}{p^n-(1-p)^n}. \end{aligned}$$

On the other hand, if $p = \frac{1}{2}$, we use the second formula for P_1 and Q_1,

$$P(E) = \frac{1}{n}p + \frac{1}{n}(1-p) = \frac{1}{n}.$$

Chapter 5

20. Suppose we have a two server system with exponentially distributed service times μ_i, $i = 1, 2$. When you arrive, Server 1 is free, person A is in service at Server 2 and person B is waiting in line at Server 2.
a) Find P_A, the probability that A is still in service when you move over to Server 2.
b) Find P_B, the probability that B is *still in the system* when you move over to Server 2.
c) Find $E[T]$, where T is the time you spend in the system.

Solution. a) Due to the memoryless property of the exponential distribution, it does not matter how long A has been at Server 2 when you walk in. The clock "resets" the moment you walk in so that the probability P_A is simply the probability that Server 1 finishes before Server 2. Letting T_1 and T_2 denote the service times of servers 1 and 2, we have

$$P_A = P(T_1 < T_2) = \frac{\mu_1}{\mu_1 + \mu_2}.$$

b) Notice that the question asks you to find the probability that B is *still in the system* when you finish at Server 1. This is different from the probability that B is in service at Server 2. The event that B is still in the system can be split into the union of two disjoint events: either (1) A is still in service at Server 2 so B is still waiting when you move over to Server 2, or (2) A has finished, but B has not finished when you move over to Server 2.

Now notice that for the first event P(1) is simply P_A, the probability from part (a) that you finish at Server 1 before A finishes at Server 2. For the second event to occur, two independent events must occur: A finishes before you and then you finish before B. So using the memoryless property of the exponential,

$$P(2) = \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)\left(\frac{\mu_1}{\mu_1 + \mu_2}\right) = \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2}.$$

Since (1) and (2) are disjoint events, we can add the probabilities,

$$P_B = P(1) + P(2) = \frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2}. \tag{5}$$

Another way to think of P_B is that the event that B is still in service is the complement of the event that B has finished by the time you move over to Server 2. This means both A and B finish before you. Thus,

$$P_B = 1 - \left(\frac{\mu_2}{\mu_1 + \mu_2}\right)^2,$$

which is equivalent to the expression in (5).

(c) We follow the hint in the book and write

$$\begin{aligned} E[T] &= E[S_1] + E[S_2] + E[W_A] + E[W_B] \\ &= E[S_1] + E[S_2] + E[W_A \mid A \text{ still in service}] \cdot P(A \text{ still in service}) \\ &\quad + E[W_B \mid B \text{ still in system}] \cdot P(B \text{ still in system}) \end{aligned}$$

where we have added the conditioning on the expected values of W_A and W_B, since if $T_1 > T_2$, then A finishes before you move over so $W_A = 0$; and if B is not still in service when you move over, then $W_B = 0$. Then using our answers from parts (a) and (b),

$$\begin{aligned} E[T] &= E[S_1] + E[S_2] + E[W_A \mid A \text{ still in service}] \cdot P_A + E[W_B \mid B \text{ still in system}] \cdot P_B \\ &= \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_2}\left(\frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2}\right) \\ &= \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{2\mu_1^2 + 3\mu_1 \mu_2}{\mu_2(\mu_1 + \mu_2)^2}. \end{aligned}$$