

ON THE DELTA SET AND CATENARY DEGREE OF KRULL MONOIDS WITH INFINITE CYCLIC DIVISOR CLASS GROUP

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ABSTRACT. Let M be a Krull monoid with divisor class group \mathbb{Z} , and let $S \subseteq \mathbb{Z}$ denote the set of divisor classes of M which contain prime divisors. We find conditions on S equivalent to the finiteness of both $\Delta(M)$, the Delta set of M , and $c(M)$, the catenary degree of M . In the finite case, we obtain explicit upper bounds on $\max \Delta(M)$ and $c(M)$. Our methods generalize and complement a previous result concerning the elasticity of M .

1. INTRODUCTION

Because of their relevance to non-unique factorizations in algebraic number theory, Krull monoids M with finite divisor class group have been well-studied. It is known in this case that the sets of lengths of M are almost-arithmetic multiprogressions (see [8] for a study of such sequences). When the divisor class group of M is not finite, we know considerably less about the structure of the length sets. In particular, if each divisor class of M contains a prime divisor, then a result of Kainrath [13] implies that each finite subset of $\{2, 3, 4, \dots\}$ can be obtained as a set of lengths. In [12], Hassler gives conditions on this distribution of divisor classes with primes which yield “thin” sets of lengths. About 10 years ago, a series of papers appeared ([2] and [3]) which considered problems involving the elasticity of Krull monoids with divisor class group \mathbb{Z} . In such a Krull monoid, let S represent the set of divisor classes of M which contain prime divisors and write S as a disjoint union $S^+ \cup S^-$ where $S^+ = \{s \in S \mid s \geq 0\}$ and $S^- = \{s \in S \mid s < 0\}$. The main result of [2] (Theorems 2.1 and 2.3) implies that the elasticity, $\rho(M)$, is finite if and only if either S^+ or S^- is finite. In this note, we show that this result extends to the cardinality of the Delta set of M , $\Delta(M)$, as well as the catenary degree of M , $c(M)$. In particular, the results of the next two sections along with the result mentioned above from [2] will constitute a proof of the following.

Theorem 1.1. *Let M be a Krull monoid with divisor class group \mathbb{Z} where $S = S^+ \cup S^-$ corresponds to the set of divisor classes which contain prime divisors. The following are equivalent:*

- (1) $\rho(M)$ is finite,

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- (2) $\Delta(M)$ is finite,
- (3) $c(M)$ is finite,
- (4) Either S^+ or S^- is finite.

By [11, Theorem 1.6.3], we have for all atomic monoids that (3) \Rightarrow (2). Few other general relationships exist between (1), (2), and (3). For instance, Example 4.8.11 in [11] yields a monoid M with $\rho(M) < \infty$ and $|\Delta(M)| = \infty$ (other pathological type examples are found in [11] in Examples 1.6.11, 1.6.3.3, and Theorem 3.1.5).

We proceed with a brief summary of the necessary background and notation for the remainder of our work. For a commutative cancellative monoid M , let M^\times denote its subgroup of units; M is called *reduced* if $M^\times = \{1\}$. Let $\mathcal{A}(M)$ denote the set of irreducible elements or *atoms* of M . If $x \in M$, an *atomic factorization* is an expression of the form $F : x = \alpha_1 \cdots \alpha_n$ where $\alpha_1, \dots, \alpha_n \in \mathcal{A}(M)$; in this case we say that $|F| = n$ is the *length* of this atomic factorization. If $x \in M \setminus M^\times$ we define

$$\mathcal{L}(x) = \{n \mid x \text{ has an atomic factorization of length } n\}$$

and $\mathfrak{L}(M) = \{\mathcal{L}(x) \mid x \in M \setminus M^\times\}$. M is called *atomic* if every element has an atomic factorization. If M is atomic, $x \in M$, and $\alpha \in \mathcal{A}(M)$ divides x , then there is an atomic factorization of x in which α appears.

For any $x \in M \setminus M^\times$ the ratio $\sup \mathcal{L}(x) / \min \mathcal{L}(x)$ is called the *elasticity* of x , denoted $\rho(x)$. If M is not a group, we define the elasticity of the monoid M by

$$\rho(M) = \sup\{\rho(x) \mid x \in M \setminus M^\times\}.$$

A review of the main facts concerning the elasticity can be found in [11, Chapter 1.4]. Given $x \in M \setminus M^\times$, write its length set in the form $\mathcal{L}(x) = \{n_1, n_2, \dots, n_k\}$ where $n_i < n_{i+1}$ for $1 \leq i \leq k-1$. The *Delta set* of x is then defined by

$$\Delta(x) = \{n_{i+1} - n_i \mid 1 \leq i < k\}$$

and the Delta set of M by

$$\Delta(M) = \bigcup_{x \in M \setminus M^\times} \Delta(x)$$

(see again [11, Chapter 1.4]). If $\Delta(M)$ is nonempty, $\min \Delta(M) = \gcd \Delta(M)$ (see [11, Proposition 1.4.4]). Computations of Delta sets for various types of monoids can be found in [4] and [5].

Suppose that M is reduced, $x \in M$ is not the identity, and that

$$F : x = \alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_s \text{ and } F' : x = \alpha_1 \cdots \alpha_n \gamma_1 \cdots \gamma_t$$

are distinct atomic factorizations such that $\beta_i \neq \gamma_j$ for all i, j . With notation as above, we define $\gcd(F, F') = \alpha_1 \cdots \alpha_n$ and the *distance* between F and F' by $d(F, F') = \max\{s, t\}$. Extend d to all pairs of factorizations by $d(F, F) = 0$. The basic properties of the factorization distance function can be found in [11, Proposition 1.2.5].

An *N -chain of factorizations* from F to F' is a sequence F_0, \dots, F_k such that each F_i is a factorization of x , $F_0 = F$ and $F_k = F'$, and $d(F_i, F_{i+1}) \leq N$ for all $i < k$. The *catenary degree* of x , denoted $c(x)$, is the least $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that for any two factorizations

F, F' of x there is an N -chain between F and F' . The catenary degree of the monoid M is defined as

$$c(M) = \sup\{c(x) \mid x \in M \setminus M^\times\}.$$

A review of the known facts concerning the catenary degree can be found in [11, Chapter 3]. An algorithm which computes the catenary degree of a finitely generated monoid can be found in [6] and a more specific version for numerical monoids in [7].

A monoid M is called a *Krull monoid* if there is an injective monoid homomorphism $\varphi : M \rightarrow D$ where D is a free abelian monoid and φ satisfies the following two conditions:

- (1) If $a, b \in M$ and $\varphi(a) \mid \varphi(b)$ in D , then $a \mid b$ in M ,
- (2) For every $\alpha \in D$ there exists $a_1, \dots, a_n \in M$ with $\alpha = \gcd\{\varphi(a_1), \dots, \varphi(a_n)\}$.

The basis elements of D are called the *prime divisors* of M . The above properties guarantee that $\text{Cl}(M) = D/\varphi(M)$ is an abelian group, which we call the *divisor class group* of M (see [11, Section 2.3]). Note that since any Krull monoid is isomorphic to a submonoid of a free abelian monoid, a Krull monoid is commutative, cancellative, and atomic.

Given an abelian group G (written additively), it is easy to explicitly construct a Krull monoid with divisor class group isomorphic to G . Let $\mathcal{F}(G)$ denote the free abelian monoid (written multiplicatively on the set G) and let $\theta : \mathcal{F}(G) \rightarrow G$ be the homomorphism $\prod_{g \in G} g^{a_g} \mapsto \sum_{g \in G} a_g \cdot g$. Then it is easy to check that the kernel of this homomorphism is a Krull monoid, where $D = \mathcal{F}(G)$ and φ is inclusion. We denote this monoid, called the *block monoid on G* by

$$\mathcal{B}(G) = \left\{ \prod_{g \in G} g^{a_g} \in \mathcal{F}(G) \mid \sum_{g \in G} a_g \cdot g = 0 \right\}$$

and we refer to elements of $\mathcal{B}(G)$ as *blocks* or *zero sequences* from G . Note that every divisor class of $\mathcal{B}(G)$ contains a prime divisor: The divisor class corresponding to $g \in G$ contains the basis element g of $\mathcal{F}(G)$.

If D is a free monoid, f is a basis element of D , and $X \in D$, we write $v_f(X)$ for the power of f appearing in X . We say that X is *supported on* some subset S of the basis of D if and only if $v_f(X) = 0$ for all $f \notin S$. For any $S \subseteq G$, the set

$$\mathcal{B}(G, S) = \{B \in \mathcal{B}(G) \mid B \text{ is supported on } S\}$$

forms a submonoid of $\mathcal{B}(G)$ called *the restriction of $\mathcal{B}(G)$ to S* . $\mathcal{B}(G, S)$ is again a Krull monoid, and it is not difficult to check that the divisor class corresponding to some $g \in G$ contains a prime divisor if and only if $g \in S$.

Restricted block monoids play an important role in the theory of non-unique factorizations, as the following proposition indicates.

Proposition 1.2 ([9], Proposition 1 see also [11], Theorem 3.4.10(5)). *Suppose that M is a Krull monoid with divisor class group G and let $S \subseteq G$ denote the set of divisor classes which contain prime divisors. We have the equality*

$$\mathfrak{L}(M) = \mathfrak{L}(\mathcal{B}(G, S))$$

and the inequality

$$c(\mathcal{B}(G, S)) \leq c(M) \leq \max\{c(\mathcal{B}(G, S)), 2\}.$$

In particular, $\Delta(M) = \Delta(\mathcal{B}(G, S))$. Moreover, if $c(\mathcal{B}(G, S)) \geq 2$, then $c(M) = c(\mathcal{B}(G, S))$ and if $c(\mathcal{B}(G, S)) = 0$, then $c(M) = 0$ or 2 .

Due to Proposition 1.2 and [1, Lemma 3.3], to prove Theorem 1.1, it suffices to consider block monoids of the form $\mathcal{B}(\mathbb{Z}, S)$, where $S = S^+ \cup S^-$, and both S^+ and S^- are nonempty. Moreover, if $0 \in S^+$, then the irreducible block represented by 0 is prime in $\mathcal{B}(\mathbb{Z}, S)$ and does not effect any factorization properties, since it must appear in every factorization of a block that contains 0 .

2. THE UNBOUNDED CASE

Theorem 2.1. *If S^+ and S^- are infinite, then $\Delta(\mathcal{B}(\mathbb{Z}, S))$ and $c(\mathcal{B}(\mathbb{Z}, S))$ are infinite.*

Proof. Assume hereafter that S^+ and S^- are infinite.

- Fix an integer $j \geq 2$.
- Let $-m \in S^-$ (so $m > 0$).
- Since S^+ is infinite, there exists an infinite $S' \subseteq S^+$ all of whose elements lie in the same congruence class modulo m . Choose $n \in S'$ such that $n \geq m$ and let $e = \text{lcm}\{n, m\}$.
- Fix $-M \in S^-$ such that $M > ej$.
- Finally, fix $N \in S'$ so that $N > 2eM$.
- Let α be the least positive integer such that $\alpha M \equiv bn \pmod{m}$ for some $b \in \mathbb{Z}$, and let β be the least such b which also satisfies $bn \geq \alpha M$. Define k to be the least nonnegative integer such that $(\beta + 1)n \leq e(j + k)$.

We infer a number of immediate consequences from these definitions:

- α is the order of M in the group $(\mathbb{Z}/m\mathbb{Z})/(n \cdot \mathbb{Z}/m\mathbb{Z})$, so $\alpha \mid m$. In particular, $\alpha \leq e$.
- $\beta > e/n$, since $\beta \geq \alpha M/n > \alpha ej/n \geq e/n$ and $\alpha, j \geq 1$.
- By the choice of k , $0 \leq e(j + k)/n - \beta - 1 < e/n$.
- We have $\beta n - \alpha M < 2e$. Otherwise we could replace β with the smaller positive integer $\beta' = \beta - e/n$ which satisfies the same conditions.
- Because $N \equiv n \pmod{m}$ and $m \mid e$,

$$q = \frac{(N - eM) + e(j + k) - n}{m}$$

is an integer. In fact, q is a positive integer satisfying $q > eM/m$ and $q > j + k$.

By this last remark,

$$B = [N][-M]^e [n]^{e(j+k)/n-1} [-m]^q$$

is an element of $\mathcal{B}(\mathbb{Z}, S)$. We will show that $\mathcal{L}(B) = \{2, j + k\}$. The following observations will be used repeatedly in the remainder of the proof:

- (i) $P = [n]^{e/n} [-m]^{e/m}$ is the unique atom of $\mathcal{B}(\mathbb{Z}, S)$ supported on $\{n, -m\}$. Any element of $\mathcal{B}(\mathbb{Z}, S)$ which is supported on $\{n, -m\}$ factors uniquely as a power of P .

(ii) Suppose that $X \in \mathcal{B}(\mathbb{Z}, S)$ is supported on $\{n, -M, -m\}$. We claim the following:

$$\text{If } v_{[-M]}(X) > 0 \text{ then } v_{[n]}(X) \geq \beta.$$

To prove this, write $X = [n]^x[-M]^y[-m]^z$ where x, y, z are nonnegative integers satisfying $xn = yM + zm$. Then $xn \equiv yM \pmod{m}$, so since y is positive, $y \geq \alpha$ by the choice of α . Since we also have $xn \geq \alpha M$, it follows that $x \geq \beta$, as desired.

Now, let F be an atomic factorization of B as an element of $\mathcal{B}(\mathbb{Z}, S)$. Since $v_{[N]}(B) = 1$, there is exactly one atom A of $\mathcal{B}(\mathbb{Z}, S)$ appearing in F which satisfies $v_{[N]}(A) = 1$. F now falls into one of two cases, depending on the value of $v_{[-M]}(A)$.

CASE 1: Suppose that $v_{[-M]}(A) = e$. Then B/A is supported on $\{n, -m\}$ so by (i), B/A factors uniquely as P^r .

Let us compute r . Write $A = [N][-M]^e[n]^f[-m]^g$ where f, g are nonnegative integers and $N + fn = eM + gm$. Since $N \equiv n \pmod{m}$ and $m \mid e$, we have $(f + 1)n \equiv 0 \pmod{m}$. Hence $e \mid (f + 1)n$, so $f \geq e/n - 1$. We claim that equality holds: $f \geq e/n$ and $N > 2eM$ together would imply $gm = (N - eM) + fn > e$. But then $g > e/m$, so P would be a proper divisor of A , contradicting the hypothesis that A is irreducible. Thus $v_{[n]}(A) = e/n - 1$ and

$$r = \frac{v_{[n]}(B/A)}{e/n} = \frac{e(j+k)/n - 1 - (e/n - 1)}{e/n} = j + k - 1.$$

So in this case, $F : B = AP^{j+k-1}$ and $|F| = j + k$.

To show that $j + k \in \mathcal{L}(B)$, it remains to show that such a factorization F actually exists. That is, we must show that there is an atomic factor A_1 of B which satisfies the conditions of this case. By the preceding arguments, the only candidate is $A_1 = [N][-M]^e[n]^{e/n-1}[-m]^{q-(j+k-1)}$. Since $q > j + k$ by an earlier remark, $A_1 \in \mathcal{B}(\mathbb{Z}, S)$. Suppose that $X \in \mathcal{B}(\mathbb{Z}, S)$ is a divisor of A_1 such that $v_{[N]}(X) = 0$. Since $v_{[n]}(A_1) < e/n \leq \beta$, observation (ii) guarantees that $v_{[-M]}(X) = 0$. Thus by (i), $X = P^d$ for some $d \geq 0$, but $v_{[n]}(X) < e/n$, so $d = 0$, $X = 1$, and A_1 is irreducible.

CASE 2: Suppose instead that $v_{[-M]}(A) < e$. Then B/A has an irreducible factor Y such that $v_{[-M]}(Y) > 0$. Since we also have $v_{[N]}(Y) = 0$, (ii) implies that $v_{[n]}(AY) \geq \beta$, so

$$v_{[n]}(B/AY) \leq \frac{e(j+k)}{n} - 1 - \beta < e/n.$$

Moreover, $v_{[N]}(B/AY) = 0$, so (ii) guarantees that $v_{[-M]}(B/AY) = 0$, and hence by (i), $B/AY = P^d$ for some nonnegative integer d . However, since $v_{[n]}(B/AY) < e/n$, we must have $d = 0$. So $B/A = Y$ is irreducible, $F : B = A(B/A)$ and any factorization obtained in this case has length 2.

It remains to show that such a factorization exists. Let $r = (\beta n - \alpha M)/m$. By the definitions of α and β , r is a positive integer, so $Y' = [-M]^\alpha[n]^\beta[-m]^r$ is an element of $\mathcal{B}(\mathbb{Z}, S)$. By earlier remarks, $\alpha \leq e$, $\beta \leq e(j+k)/n - 1$, and $r < 2e/m \leq eM/m < q$. It follows that Y' divides B , so there is an atom A_2 dividing B/Y' such that $v_{[N]}(A_2) = 1$. Since $\alpha > 0$, we have $v_{[-M]}(A_2) < e$, so $A = A_2$ satisfies the conditions of this case and it follows that B has a factorization of length 2.

By Case 1 and Case 2, we have $\mathcal{L}(B) \subseteq \{2, j+k\}$. Conversely, we have shown that appropriate factorizations exist in both cases, so $\mathcal{L}(B) = \{2, j+k\}$. From the length set, we calculate $\Delta(B) = \{j+k-2\}$ and, since distinct factorizations of length 2 and $j+k \geq 2$ cannot share any common factors, $c(B) = j+k$. Since j may be taken arbitrarily large, the result follows. \square

3. WHEN EITHER S^+ OR S^- IS FINITE

The main theorem of this section is the following:

Theorem 3.1. *Suppose that $S' = \{-m_r, \dots, -m_1, n_1, \dots, n_k\} \subseteq S$ where $m_i, n_i > 0$ for all i and $-m_r = \min S'$. If B is an element of $\mathcal{B}(\mathbb{Z}, S)$ supported on S' , then*

$$\max \Delta(B) \leq m_r(m_r + r^2) - 2$$

(if $\Delta(B)$ is nonempty) and

$$c(B) \leq m_r(m_r + r^2).$$

The above is sufficient to complete the proof of Theorem 1.1. If S is bounded from either above or below, we may assume that S^- is finite, possibly after replacing S with $-S$. From the above it follows then that if $\Delta(\mathcal{B}(\mathbb{Z}, S))$ is nonempty, then it is bounded above by $M(M + |S^-|^2) - 2$ where $-M = \min(S^-)$. Similarly, we obtain $c(\mathcal{B}(\mathbb{Z}, S)) \leq M(M + |S^-|^2)$.

The proof is based on a result of Lambert, which we now present adapted to the language of block monoids. Since the proof makes use of elements of $\mathcal{F}(\mathbb{Z})$ outside $\mathcal{B}(\mathbb{Z})$, we define some convenient notation. As before, we let $\theta : \mathcal{F}(\mathbb{Z}) \rightarrow \mathbb{Z}$ be the obvious homomorphism (so that $\mathcal{B}(\mathbb{Z}) = \ker \theta$). We define another homomorphism $\varphi : \mathcal{F}(\mathbb{Z}) \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\varphi : X \mapsto \sum_{n \in S^+} v_{[n]}(X)$$

If $X, Y \in \mathcal{F}(\mathbb{Z})$, then we write $X \leq Y$ whenever $v_{[z]}(X) \leq v_{[z]}(Y)$ for all $z \in \mathbb{Z}$.

Theorem 3.2 (Lambert [14]). *If S^- is finite with $\min(S^-) = -M$, then $\varphi(A) \leq M$ for all atoms A of $\mathcal{B}(\mathbb{Z}, S)$.*

Proof. Let $A = [n_1]^{a_1} \cdots [n_k]^{a_k} [-m_1]^{b_1} \cdots [-m_r]^{b_r}$ be an atom of $\mathcal{B}(\mathbb{Z}, S)$. An element X of $\mathcal{F}(\mathbb{Z})$ is called a *positive subsequence* of A if $X \leq A$ and $\theta(X) \geq 0$.

We construct a strictly increasing chain $1 = X_0 < \cdots < X_{\varphi(A)} = A$ of positive subsequences of A . Given X_i when $0 < i < \varphi(A)$, consider all positive subsequences X of A such that $\varphi(X) = \varphi(X_i) + 1$. Such an X must exist, since $\varphi(X_i) < \varphi(A)$ implies that there is some j with $X_i[n_j] \leq A$, and $\varphi(X_i[n_j]) = \varphi(X_i) + 1$. Let X_{i+1} be such an X with $\theta(X_{i+1})$ as small as possible.

We claim that $\theta(X_i) < M$ for all i . This is clear when $i = 0$ or $i = \varphi(A)$. When $0 < i < \varphi(A)$, then since X_i is a positive subsequence, there exists j such that $X_i[-m_j] \leq A$. If we had $\theta(X_i) \geq M$, then $X_i[-m_j]$ would be a positive subsequence of A , but then $0 \leq \theta(X_i[-m_j]) < \theta(X_i)$, which contradicts the choice of X_i .

If $0 < i < \varphi(A)$, then $\theta(X_i) > 0$ as otherwise X_i would be a proper nontrivial divisor of A in $\mathcal{B}(\mathbb{Z}, S)$ which contradicts the hypothesis that A is irreducible. Thus the set

$\{\theta(X_1), \dots, \theta(X_{\varphi(A)-1})\}$ is a set of size $\varphi(A) - 1$ contained in $\{1, \dots, M - 1\}$. Hence, if $\varphi(A) > M$ there would exist distinct i, j with $i < j$ such that $\theta(X_i) = \theta(X_j)$. But then $A' = X_j/X_i \in \mathcal{B}(\mathbb{Z}, S)$ and A' is a proper nontrivial divisor of A , which is impossible. Thus $\varphi(A) \leq M$ as desired. \square

Proof of Theorem 3.1. We proceed by induction on $\max \mathcal{L}(B)$. In the base case B is irreducible, so $\Delta(B)$ is empty and $c(B) = 0$.

Suppose then that B is not irreducible, let $L = \max \mathcal{L}(B) \geq 2$ and let $\ell \in \mathcal{L}(B)$. Write $B = [n_1]^{d_1} \cdots [n_k]^{d_k} [-m_1]^{e_1} \cdots [-m_r]^{e_r}$. Fix an atomic factorization $F^* : B = A_1 \cdots A_L$ of maximal length, and let $F_* : B = C_1 \cdots C_\ell$ be any atomic factorization of B . For i, j , set $e_{ij} = v_{[m_j]}(A_i)$.

Claim: There is an index f and a subset $I \subseteq \{1, \dots, \ell\}$ of cardinality at most $m_r + r^2$ such that A_f divides $\prod_{i \in I} C_i$.

Proof of the Claim. If $r > \ell$, then by taking any index f and $I = \{1, \dots, \ell\}$ satisfies the claim. Hence, let us assume $r \leq \ell$. We will find f with $1 \leq f \leq L$ and $e_{fj}/e_j \leq r/L$ for all j . Note that

$$\sum_{i=1}^L \max_j \frac{e_{ij}}{e_j} \leq \sum_{i=1}^L \sum_{j=1}^r \frac{e_{ij}}{e_j} = \sum_{j=1}^r \left(\frac{1}{e_j} \sum_{i=1}^L e_{ij} \right) = r$$

since $\sum_i e_{ij} = e_j$. If for all i we had $\max_j(e_{ij}/e_j) > r/L$, then

$$\sum_{i=1}^L \max_j \frac{e_{ij}}{e_j} > \sum_{i=1}^L \frac{r}{L} = r$$

which is a contradiction, so a desired f must exist. After reordering, we may assume $f = L$.

If $J \subseteq \{1, \dots, \ell\}$ we set $C_J = \prod_{i \in J} C_i$. By the pigeonhole principle and the fact that $r \leq \ell$, for each $j = 1, \dots, r$ there is a subset $I_j \subseteq \{1, \dots, \ell\}$ with $|I_j| \leq r$ such that $v_{[-m_j]}(C_{I_j}) \geq r e_j / \ell$. But since $\ell \leq L$, this means that $v_{[-m_j]}(C_{I_j}) \geq e_{Lj}$. Furthermore, we know by Theorem 3.2 that $\varphi(A_L) \leq m_r$, so we may choose $I_0 \subseteq \{1, \dots, \ell\}$ such that $|I_0| \leq m_r$ and $v_{[n]}(A_L) \leq v_{[n]}(C_{I_0})$ for all $n \in S^+$. Finally, take $I = I_0 \cup I_1 \cup \cdots \cup I_r$. Then by construction, C_I is divisible by A_L . Moreover, $|I| \leq |I_0| + \sum_j |I_j| \leq m_r + r^2$. This completes the proof of the claim and after reordering the C_i 's, we may assume $I = \{1, \dots, q\}$ for some

$$(*) \quad q \leq m_r + r^2.$$

We have

$$B = A_1 \cdots A_L = C_1 \cdots C_\ell = (A_L D_1 \cdots D_t)(C_{q+1} \cdots C_\ell).$$

where $A_L D_1 \cdots D_t$ is a factorization of $C_1 \cdots C_q$ in which A_L appears. We denote the factorization $A_L D_1 \cdots D_t C_{q+1} \cdots C_\ell$ by F .

Applying Theorem 3.2 to the identity $\varphi(A_L D_1 \cdots D_t) = \varphi(C_1 \cdots C_q)$, we find

$$(**) \quad t + 1 \leq \sum_{i=1}^q \varphi(C_i) \leq q m_r \leq m_r (m_r + r^2)$$

and $q \leq \varphi(A_L) + \sum_{i=1}^t \varphi(D_i) \leq (t + 1) m_r$.

We now demonstrate the desired bounds on the Delta set. Note that if $q = 1$ or $t + 1 = 1$ then $F = F_*$. Otherwise, $||F| - |F_*|| = |(t+1) - q| \leq \max\{t-1, q-2\} \leq m_r(m_r + r^2) - 2$ (that $t - 1 \leq m_r(m_r + r^2) - 2$ follows from (**), similarly $q - 2 \leq m_r(m_r + r^2) - 2$ by (*)). Let \tilde{F} and \tilde{F}^* denote the factorizations of B/A_L obtained from F and F^* (the long factorization of B) by removing the irreducible factor A_L . By induction, if $\Delta(B/A_L)$ is nonempty, $\max \Delta(B/A_L) \leq m_r(m_r + r^2) - 2$. Hence, there is an increasing sequence $|\tilde{F}| = \ell_0, \dots, \ell_s = |\tilde{F}^*|$ of lengths of atomic factorizations of B/A_L such that $\ell_{i+1} - \ell_i \leq m_r(m_r + r^2) - 2$ for all $i < s$ (if $\Delta(B/A_L)$ is empty, it follows that F and F^* have the same length). By concatenating the corresponding factorizations with A_L , we find $\{\ell, |F| = \ell_0 + 1, \dots, \ell_s + 1\} \subseteq \mathcal{L}(B)$. As listed, the consecutive terms differ by no more than $m_r(m_r + r^2)$, with the first term being $|F_*| = \ell$ and the last being $|F^*| = \ell_s + 1$. Hence if $\Delta(B)$ is nonempty, the arbitrary choice of F_* has shown $\max \Delta(B) \leq m_r(m_r + r^2) - 2$.

To show the bound on the catenary degree, we again pass to factorizations of B/A_L and use the induction hypothesis. Thus we have an $m_r(m_r + r^2)$ -chain from \tilde{F} to \tilde{F}^* , and this lifts to a $m_r(m_r + r^2)$ -chain from F to F^* after multiplying every term in the chain by A_L . Finally, $d(F_*, F) \leq t + 1 \leq m_r(m_r + r^2)$, so appending F_* to this chain proves $c(B) \leq m_r(m_r + r^2)$. \square

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