A GENERALIZATION OF A RAMSEY-TYPE THEOREM ON HYPERMATCHINGS

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Abstract. For an r-uniform hypergraph G define $N(G,l;2)$ ($N(G,l;\mathbb{Z}_n)$) as the smallest integer for which there exists an r-uniform hypergraph H on $N(G,l;2)$ ($N(G,l;\mathbb{Z}_n)$) vertices with clique(H)$< l$ such that every 2-coloring ($\mathbb{Z}_n$-coloring) of the edges of H implies a monochromatic (zero-sum) copy of G. Our results strengthen a Ramsey-type theorem of Bialostocki and Dierker on zero-sum hypermatchings. As a consequence we show that for any $n \geq 2$, $r \geq 2$, and $l > r+1$, $N(n\mathcal{K}_r^r,\ l;2) = N(n\mathcal{K}_r^r,\ l;\mathbb{Z}_n) = (r+1)n-1$.

1. Introduction

Let G be an r-uniform hypergraph with n edges. The 2-color (zero-sum) Ramsey number, denoted by $R(G,2)$ ($R(G,\mathbb{Z}_n)$), is the smallest integer s such that for every 2-coloring ($\mathbb{Z}_n$-coloring) of the edges of the r-uniform complete hypergraph on s vertices, there exists a subhypergraph isomorphic to G all of whose edges have the same color (all of whose edges sum to zero); such a subhypergraph is called monochromatic (zero-sum). Clearly, $R(G,2) \leq R(G,\mathbb{Z}_n)$, and if equality holds, we say that $R(G,2)$ admits an Erdös-Ginzburg-Ziv (EGZ) generalization. Such generalizations have been proven in [2], [3], [8], and [14]. Surveys of zero-sum problems appear in [1] and [5].

Along different lines, a problem posed by P. Erdös and solved in [9] motivated R. Graham and J. Spencer [10] to introduce the following definiton. Denote by $N(k,l;s)$ the smallest integer for which there exists a graph G on $N(k,l;s)$ vertices

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with clique(G)$< l$, i.e. G does not contain a complete graph on l vertices, but such that every s-coloring of the edges of G implies a monochromatic complete subgraph on k vertices. The function $N(k, l; s)$ was investigated in several papers, among them [10], [11], and [12], which led to the more general Nešetřil-Rödl Theorem [13] concerning hypergraphs. Continuing this trend, we introduce the following definition.

Definition 1.1. Let G be an r-uniform hypergraph. Denote by $N(G, l; 2)$ (by $N(G, l; \mathbb{Z}_n)$) the smallest integer for which there exists an r-uniform hypergraph H on $N(G, l; 2)$ ($N(G, l; \mathbb{Z}_n)$) vertices with clique(H)$< l$ and such that every 2-coloring ($\mathbb{Z}_n$-coloring) of the edges of H implies a monochromatic (zero-sum) copy of G.

It is easy to see that $R(G, 2) \leq N(G, l; 2) \leq N(G, l; \mathbb{Z}_n)$ and $R(G, \mathbb{Z}_n) \leq N(G, l; \mathbb{Z}_n)$. Furthermore, if $l_1 \geq l_2$ then $N(G, l_1; 2) \leq N(G, l_2; 2)$ and $N(G, l_1; \mathbb{Z}_n) \leq N(G, l_2; \mathbb{Z}_n)$. Our paper focuses on the case where $G = n\mathcal{K}_r^r$ is a hypermatching, i.e. n pairwise disjoint r-uniform hyperedges. In [3], Bialostocki and Dierker showed that $R(n\mathcal{K}_r^r, 2) = R(n\mathcal{K}_r^r, \mathbb{Z}_n) = (r+1)n - 1$. Moreover, it follows from Lemma 3.5 in [4] that $N(n\mathcal{K}_2,\, 2n+1;\, 2) = 3n - 1$. In this paper we generalize both of these two results and prove that for any $n \geq 2$, $r \geq 2$, and $l > r+1$ we have $N(n\mathcal{K}_r^r,\, l; 2) = N(n\mathcal{K}_r^r,\, l; \mathbb{Z}_n) = (r+1)n - 1$. In particular, we will show that this equality is witnessed by the complete $(r+1)$-partite r-uniform hypergraph consisting of r vertex classes of size n and one additional class which has size $n-1$.

2. Main Result

First we will establish some notation. For a given positive integer r and a set S, we will use $\mathcal{K}^r(S)$ to denote the collection of r-subsets of S.

Let $r \geq 2$ and disjoint sets $H_1 = \{z_1, \ldots, z_r\}$ and $H_2 = \{y_1, \ldots, y_{r+1}\}$ be given. Then for each $I \subseteq \{1, \ldots, r\}$ define

$$H_1^I = \{z_i \,|\, i \in \{1, \ldots, r\} \backslash I\} \cup \{y_i \,|\, i \in I\}$$
$$H_2^I = \{y_i \,|\, i \in \{1, \ldots, r+1\} \backslash I\} \cup \{z_i \,|\, i \in I\}.$$

Note that $|H_1^I| = r$ and $|H_2^I| = r+1$ and $H_1^I \cap H_2^I = \emptyset$.

Lemma 2.1. *Let $r \geq 2$ and disjoint sets $H_1 = \{z_1, \dots, z_r\}$ and $H_2 = \{y_1, \dots, y_{r+1}\}$ be given. If we partition $\mathcal{K}^r(H_1 \cup H_2)$, then either*

(1) *every r-subset of H_2 belongs to the same class as H_1, or*
(2) *there exists $I \subseteq \{1, \dots, r\}$ such that two r-subsets Y_1 and Y_2 of H_2^I belong to different classes.*

Proof. Define $I_0 = \emptyset$ and for each $1 \leq i \leq r$, put $I_i = \{1, \dots, i\}$ and $A_i = H_2^{I_i}$. If for some i there are two r-subsets Y_1 and Y_2 of A_i that belong to different classes, then we are done. Otherwise, for each i all the r-subsets of A_i belong to the same class . But since $|A_i \cap A_{i+1}| = r$ we have that every r-subset of A_i belongs to the same class as every r-subset of A_{i+1}. By induction, every r-subset of $A_0 = H_2$ belongs to the same class as every r-subset of A_r. But by our construction, H_1 is an r-subset of A_r, so we are done. □

Lemma 2.2. *Let $n \geq 2$, $r \geq 2$. Let a set S of cardinality $(r+1)n - 1$ be given and let $T_1, \dots, T_r, T_{r+1}$ be a partition of S such that for $1 \leq i \leq r$, $|T_i| = n$ and $|T_{r+1}| = n - 1$. Put $W = \mathcal{K}^r(S) \backslash (\bigcup_{i=1}^{r+1} \mathcal{K}^r(T_i))$. Then for any partition of W, we have either:*

(1) *there are n elements of W, say $A_1, A_2, \dots, A_n$, which all belong to the same class and such that for all i, j, with $1 \leq i \leq r+1$ and $1 \leq j \leq n$, we have that $|T_i \cap A_j| \leq 1$; or*
(2) *there are $2n-1$ elements of W, say $A_1, \dots, A_{n-1}$, $B_1, \dots, B_{n-1}, C$, such that for all i, j, and k, where $1 \leq i \leq j \leq n-1$ and $1 \leq k \leq r+1$, we have*
    (a) $|A_i \cap B_i| = r - 1$,
    (b) *A_i and B_i belong to different classes,*
    (c) *$(A_i \cup B_i) \cap (A_j \cup B_j) = \emptyset$ when $i \neq j$,*
    (d) $|(A_i \cup B_i) \cap T_k| = 1$,
    (e) *$(A_i \cup B_i) \cap C = \emptyset$, and*
    (f) $|C \cap T_k| \leq 1$.

Proof. Proof by induction on n.

Consider n=2. For each $1 \leq i \leq r$ enumerate T_i as $\{a_{i,1}, a_{i,2}\}$, then define $H_1 = \{a_{i,1} \,|\, 1 \leq i \leq r\}$ and $H_2 = \{a_{i,2} \,|\, 1 \leq i \leq r\} \cup T_{r+1}$. Note that $H_1, H_2 \in W$, $|H_1| = r$, $|H_2| = r + 1$ and $H_1 \cap H_2 = \emptyset$, so we may apply Lemma 2.1. If we have that every r-subset of H_2 belongs to the same class as H_1, then H_1 and any r-subset of H_2 will satisfy the first assertion of the lemma. Otherwise we have that for some $I \subseteq \{1, \ldots, r\}$ there are r-subsets A and B of H_2^I which belong to different classes. Taking $C = H_1^I$ yields sets satisfying the second assertion.

Assume the lemma holds for $n - 1$.

Case 1. For every $x_1 \in T_1$, $x_2 \in T_2$, $\ldots$, $x_{r+1} \in T_{r+1}$ and every two r-subsets Y_1, Y_2 of $\{x_1, x_2, \ldots, x_{r+1}\}$ we have that Y_1 and Y_2 belong to the same class. It follows easily that there is one class containing all r-subsets Y with the property that $|Y \cap T_i| \leq 1$ for all i with $1 \leq i \leq r+1$. For each $1 \leq j \leq r$ enumerate T_j as $\{a_{j,1}, a_{j,2}, \ldots, a_{j,n}\}$. Then defining $A_i = \{a_{j,i} \,|\, 1 \leq j \leq r\}$ for each $1 \leq i \leq n$ yields elements of W which satisfy the first assertion of the lemma.

Case 2. There are $x_1 \in T_1$, $x_2 \in T_2$, $\ldots$, $x_{r+1} \in T_{r+1}$ and there are two r-subsets Y_1, Y_2 of $\{x_1, x_2, \ldots, x_{r+1}\}$ such that Y_1 and Y_2 belong to different classes. Then we apply the induction hypothesis to $S' = S \backslash \{x_1, x_2, \ldots, x_{r+1}\}$ and $T_i' = T_i \backslash \{x_i\}$ for each $1 \leq i \leq r+1$. If the induction hypothesis yields $A_1, A_2, \ldots, A_{n-2}$, $B_1, B_2, \ldots, B_{n-2}, C$ satisfying the second assertion of the lemma for $n-1$, then adding $A_{n-1} = Y_1$ and $B_{n-1} = Y_2$ yields $2n-1$ elements of W which satisfy the second assertion of the lemma for n.

Otherwise, we have $n-1$ pairwise disjoint elements of W, say $A_1, \ldots, A_{n-1}$, which belong to the same class and such that for all $1 \leq i \leq r+1$ and $1 \leq j \leq n-1$ $|T_i \cap A_j| \leq 1$. This property, in light of the fact that $|A_j| = r$, implies that each A_j has empty intersection with precisely one of the T_i, which we shall label $T_{\sigma(j)}$. Set

$$V = \bigcup_{j=1}^{n-1} A_j$$

and for each $1 \leq i \leq r+1$ put $d_i = |T_i \cap V|$. Since $|T_i \cap A_j| \leq 1$ for all $1 \leq i \leq r+1$ and $1 \leq j \leq n-1$, d_i represents the number of A_j that have nonempty intersection with T_i. By definition then, $n - 1 - d_i = |T_i \backslash V|$ is the number of A_j for which $T_i = T_{\sigma(j)}$. Thus, for each $1 \leq j \leq n-1$, we may choose an element $w_j \in T_{\sigma(j)} \backslash V$, such that $j \neq j'$ implies $w_j \neq w_{j'}$. The resulting pairwise disjoint sets $F_j = A_j \cup \{w_j\}$, where $1 \leq j \leq n-1$, have cardinality $r+1$ and the property that $|F_j \cap T_i| = 1$ for all i, where $1 \leq i \leq r+1$. Enumerate each F_j as $\{z_1^j, z_2^j, \ldots, z_{r+1}^j\}$, where for each $1 \leq i \leq r+1$, we have $z_i^j \in T_i$. Set

$V_1 = S \setminus \bigcup_{j=1}^{n-1} F_j$ and observe that since $|F_j \cap T_i| = 1$ for all $1 \le i \le r+1$, we have that $|V_1 \cap T_i| = 1$ for all $1 \le i \le r$ and $V_1 \cap T_{r+1} = \emptyset$. Thus, we may enumerate V_1 as $\{y_1^1, \ldots, y_r^1\}$ such that for each $1 \le i \le r$, we have that $y_i^1 \in T_i$. Apply Lemma 2.1 to the sets V_1 and F_1. If every r-subset of F_1 belongs to the same class as V_1, then in particular A_1 belongs to the same class as V_1. In this case $A_1, \ldots, A_{n-1}, V_1$ satisfy the first assertion of the lemma and we are done. Otherwise, there is some $I_1 \subseteq \{1, \ldots, r\}$ and two r-subsets A_1' and B_1' of $F_1^{I_1}$ that belong to different classes. By the specification of the enumerations of F_1 and V_1, we have that for all $1 \le i \le r+1$, $|(A_1' \cup B_1') \cap T_i| = |F_1^{I_1} \cap T_i| = 1$.

We proceed recursively for $1 \le j \le n-2$ as follows. Set $V_{j+1} = V_j^{I_j}$. By the specification of the enumerations of F_j and V_j, for all $1 \le i \le r$ we have $|V_{j+1} \cap T_i| = 1$ and $V_{j+1} \cap T_{r+1} = \emptyset$. Thus we may enumerate V_{j+1} as $\{y_1^{j+1}, \ldots, y_r^{j+1}\}$ such that for each $1 \le i \le r$, we have that $y_i^{j+1} \in T_i$. We now apply Lemma 2.1 to the sets V_{j+1} and F_{j+1}. If every r-subset of F_{j+1} belongs to the same class as V_{j+1}, then in particular A_{j+1} belongs to the same class as V_{j+1}. In this case $A_1, \ldots, A_{n-1}, V_{j+1}$ satisfy the first assertion of the lemma and we are done with the proof. Otherwise, there is some $I_{j+1} \subseteq \{1, \ldots, r\}$ and two r-subsets A'_{j+1} and B'_{j+1} of $F_{j+1}^{I_{j+1}}$ that belong to different classes. By the specification of the enumerations of F_{j+1} and V_{j+1}, we have that for all $1 \le i \le r+1$,

$$|(A'_{j+1} \cup B'_{j+1}) \cap T_i| = |F_{j+1}^{I_{j+1}} \cap T_i| = 1.$$

Thus, at the end of the recursion, we will have produced elements $A_1', \ldots, A_{n-1}'$, $B_1', \ldots, B_{n-1}'$ of W which satisfy conditions (a)-(d) of the lemma. Put $C = V_{n-1}^{I_{n-1}}$. By the specification of the enumeration of F_{n-1} and V_{n-1}, for all $1 \le i \le r$ we have $|C \cap T_i| = 1$ and $C \cap T_{r+1} = \emptyset$. Thus $A_1', \ldots, A_{n-1}'$, $B_1', \ldots, B_{n-1}', C$ satisfy the second assertion of the lemma.

$\square$

Theorem 2.3. *Let $n \ge 2$, $r \ge 2$. Let a set S of cardinality $(r+1)n - 1$ be given and let $T_1, \ldots, T_r, T_{r+1}$ be a partition of S such that for $1 \le i \le r$, $|T_i| = n$ and $|T_{r+1}| = n - 1$. Put $W = \mathcal{K}^r(S) \setminus (\bigcup_{i=1}^{r+1} \mathcal{K}^r(T_i))$. Then for any mapping $\alpha : W \to \mathbb{Z}_n$, there are pairwise disjoint elements of W, say $Z_1, \ldots, Z_n$, such that*

$$\alpha(Z_1) + \ldots + \alpha(Z_n) = 0.$$

Moreover, we may choose these Z_j *such that for all* $1 \le i \le r+1$, $|Z_j \cap T_i| \le 1$.

Proof. We prove the theorem first for $n = p$ prime by applying Lemma 2.2. If we have p pairwise disjoint elements of W, say $A_1, A_2, \dots, A_p$, such that $\alpha(A_1) = \cdots = \alpha(A_p)$ and for all $1 \le i \le r+1$ and $1 \le j \le p$, $|T_i \cap A_j| \le 1$, then we are done. Otherwise we have $A_1, \dots, A_{p-1}$, $B_1, \dots, B_{p-1}, C \in W$ satisfying the second assertion of Lemma 2.2. For each $1 \le i \le p-1$ put $D_i = \{\alpha(A_i), \alpha(B_i)\}$ and put $D_p = \{\alpha(C)\}$. By repeated use of the Cauchy-Davenport Theorem ([6]) we get all the elements of $\mathbb{Z}_p$ in the set $D_1 + D_2 + \cdots + D_p$. Consequently, 0 is among them and the theorem is proven for p prime.

Assume the theorem holds for $n = m_1$ and $n = m_2$. Let $r \ge 2$ and a set S of cardinality $(r+1)m_1m_2 - 1$ be given. Let $T_1, \dots, T_r, T_{r+1}$ be a partition of S such that for $1 \le i \le r$, $|T_i| = m_1m_2$ and $|T_{r+1}| = m_1m_2 - 1$. Put

$$W = \mathcal{K}^r(S) \backslash (\bigcup_{i=1}^{r+1} \mathcal{K}^r(T_i))$$

and consider a mapping $\alpha : W \to \{0, 1, \dots, m_1m_2 - 1\}$. This induces a map $\alpha' : W \to \{0, 1, \dots, m_1 - 1\}$ by defining $\alpha'(X)$ to be the least residue of $\alpha(X)$ modulo m_1. Let V be a subset of S of cardinality $(r+1)m_1 - 1$ such that for some $j^* \in \{1, \dots, r+1\}$ we have $|V \cap T_{j^*}| = m_1 - 1$ and $|V \cap T_j| = m_1$ for every $j \ne j^*$. Put

$$W_V = \mathcal{K}^r(V) \backslash (\bigcup_{i=1}^{r+1} \mathcal{K}^r(V \cap T_i)).$$

We may apply the theorem for m_1 to pick disjoint elements $B_1^V, B_2^V, \dots, B_{m_1}^V \in W_V$ such that

$$\sum_{j=1}^{m_1} \alpha'(B_j^V) = 0 \pmod{m_1}$$

and such that $1 \ge |B_j^V \cap (V \cap T_i)| = |B_j^V \cap T_i|$ for all $1 \le i \le r+1$. Thus, for some integer $k_V \ge 0$ we have $\sum_{j=1}^{m_1} \alpha(B_j^V) = k_V m_1$.

We now partition T_i, where $1 \le i \le r$, into subsets $\Gamma_{(i-1)m_1+1}, \Gamma_{(i-1)m_1+2}, \dots,$ Γ_{im_1} such that for all $1 \le j \le m_1$, $|\Gamma_{(i-1)m_1+j}| = m_2$. Additionally, we partition T_{r+1} as $\Gamma_{rm_1+1}, \Gamma_{rm_1+2}, \dots, \Gamma_{(r+1)m_1}$, where for all $1 \le j < m_1$, $|\Gamma_{rm_1+j}| = m_2$ and $|\Gamma_{(r+1)m_1}| = m_2 - 1$. In this manner, we have formed a subpartition

$\Gamma_1, \Gamma_2, \ldots, \Gamma_{(r+1)m_1}$ of S. Put

$$W' = \mathcal{K}^{(r+1)m_1-1}(S) \setminus \left(\bigcup_{i=1}^{(r+1)m_1} \mathcal{K}^{(r+1)m_1-1}(\Gamma_i) \right).$$

Note that if $V \in W'$ has the property that $|V \cap \Gamma_i| \leq 1$ for all $1 \leq i \leq (r+1)m_1$, then there is some $j^* \in \{1, \ldots, r+1\}$ such that $|V \cap T_{j^*}| = m_1 - 1$ and $|V \cap T_j| = m_1$ for every $j \neq j^*$. Thus, for such a $V \in W'$ the integer k_V is well-defined. We may now construct a mapping $\alpha'' : W' \to \{0, 1, \ldots, m_2 - 1\}$ as follows: for a given $V \in W'$

$$\alpha''(V) = \begin{cases} k_V \pmod{m_2} & \text{if } |V \cap \Gamma_i| \leq 1 \text{ for all } 1 \leq i \leq (r+1)m_1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying the theorem for m_2 yields $V_1, V_2, \ldots, V_{m_2} \in W'$ such that $\sum_{j=1}^{m_2} \alpha''(V_i) = 0 \pmod{m_2}$. Furthermore, the theorem permits us to choose these V_j such that $|V_j \cap \Gamma_i| \leq 1$ for all i, j, where $1 \leq i \leq (r+1)m_1$ and $1 \leq j \leq m_2$. Thus

$$\sum_{j=1}^{m_2} k_{V_j} = 0 \pmod{m_2}$$

and so the elements $B_i^{V_j} \in W$, where $1 \leq i \leq m_1$ and $1 \leq j \leq m_2$, satisfy the assertion of the theorem for $n = m_1 m_2$.

□

Immediate consequences of Theorem 2.3 are the following two corollaries.

Corollary 2.4. *If $\alpha : e(\mathcal{K}^r_{\underbrace{n, n, \ldots, n}_{r \text{ times}}, n-1}) \to \mathbb{Z}_n$ is a $\mathbb{Z}_n$-coloring of the edges of the complete $(r+1)$-partite r-uniform hypergraph $\mathcal{K}^r_{\underbrace{n, n, \ldots, n}_{r \text{ times}}, n-1}$ then there exist n pairwise disjoint hypermatchings, say $Z_1, \ldots, Z_n$, such that $\sum_{i=1}^{n} \alpha(Z_i) = 0$.*

Corollary 2.5. *For any $n \geq 2$, $r \geq 2$, and $l > r + 1$,*

$$N(n\mathcal{K}^r_r, l; 2) = N(n\mathcal{K}^r_r, l; \mathbb{Z}_n) = (r+1)n - 1.$$

Proof. Consider the complete $(r+1)$-partite r-uniform hypergraph $\mathcal{K}^r_{\underbrace{n, n, \ldots, n}_{\text{r times}}, n-1}$. This graph has clique number $r + 1$ (indeed, any set of $r + 2$ vertices contains two vertices in the same vertex class; hence there is no hyperedge through these two vertices). Thus the previous corollary restated becomes

$$N(n\mathcal{K}_r^r,\, r+2; 2) = N(n\mathcal{K}_r^r,\, r+2; \mathbb{Z}_n) = (r+1)n - 1.$$

Combining our observations in the last paragraph of the introduction, the equality follows for all $l > r+1$. $\square$

Our results have shown the existence of an r-uniform hypergraph on $(r+1)n-1$ vertices, namely $\mathcal{K}^r_{\underbrace{n, n, \ldots, n}_{\text{r times}}, n-1}$, which witnesses the equality $N(n\mathcal{K}_r^r,\, l; 2) = N(n\mathcal{K}_r^r,\, l; \mathbb{Z}_n) = (r+1)n - 1$. However, we were unable to answer the question of uniqueness. Specifically, we have the following:

Open Question: For a given integers $n, r \geq 2$, how many non-isomorphic r-uniform hypergraphs H possess the following three properties:

(1) H has $(r+1)n-1$ vertices,

(2) H has clique number $\leq r+1$, and

(3) for every coloring α of the hyperedges of H by $\mathbb{Z}_n$, there exists n pairwise disjoint hyperedges, say $Z_1, \ldots, Z_n$, such that $\sum_{i=1}^n \alpha(Z_i) = 0$.

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