ON THE ASYMPTOTIC BEHAVIOR OF UNIONS OF SETS OF LENGTHS IN ATOMIC MONOIDS

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Abstract. Let M be a commutative cancellative atomic monoid. We use unions of sets of lengths in M to construct the $\mathcal{V}$-Delta set of M. We first derive some basic properties of $\mathcal{V}$-Delta sets and then show how they offer a method to investigate the asymptotic behavior of the sizes of unions of sets of lengths.

A central focus of number theory is the study of number theoretic functions and their asymptotic behavior. This has led to similar investigations concerning non-unique factorizations in integral domains and moniods. Suppose that M is a commutative cancellative monoid in which each nonunit can be factored into a product of irreducible elements (such a monoid is known as *atomic*). For a nonunit x in M, let $L(x)$ represent the maximum length of a factorization of x into irreducibles and $l(x)$ the minimum such length. The functions

$$\overline{L}(x) = \lim_{k\to\infty} \frac{L(x^n)}{n} \text{ and } \bar{l}(x) = \lim_{k\to\infty} \frac{l(x^n)}{n}$$

have been studied in the literature by Anderson and Pruis in [3] and Halter-Koch and Geroldinger in [17]. In [14], Chapman and Smith defined the notion of a *generalized set of lengths*, and showed in [12] that the size of a generalized set of lengths (denoted $\Phi(n)$) satisfies

$$\overline{\Phi}(R) = \lim_{n\to\infty} \frac{\Phi(n)}{n} = \frac{D(G)^2 - 4}{2D(G)} \tag{1}$$

for a ring of algebraic integers R where $D(G)$ represents Davenport's constant of the ideal class group G of R (the Davenport constant is defined in [18, Section 3.4]). Since a generalized set of lengths is actually a union of certain length sets, we will refer to these sets with the more descriptive term *unions of sets of lengths*. The value $\overline{\Phi}(R)$ has also been explored for various semigroup rings over fields [2, Theorem 3.3]. In this note, we examine the limit $\overline{\Phi}(R)$ in greater detail. By generalizing the well known notion of the Delta set of a monoid M (see [18, Section 1.4]), we find new bounds for the

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value $\overline{\Phi}(M)$ which allows us to determine exact calculations in several instances recently addressed in the literature (see Examples 8 and 9). We will begin with a review of the necessary definitions and notations from the theory of non-unique factorizations. The reader is directed to the monograph [18] for a complete survey of recent results in this area.

Throughout our work, we assume that M is an atomic commutative cancellative monoid with sets $\mathcal{I}(M)$ of irreducible elements and $M^\bullet$ of nonunits. The set of lengths of $x \in M^\bullet$ is

$$\mathcal{L}(x) = \{n \mid x = x_1 \cdots x_n \text{ with each } x_i \in \mathcal{I}(M)\}.$$

Also, define $L(x) = \max \mathcal{L}(x)$ and $l(x) = \min \mathcal{L}(x)$. The quotient $\frac{L(x)}{l(x)}$ is called the *elasticity* of x and the constant

$$\rho(M) = \sup\{\frac{L(x)}{l(x)} \mid x \in M^\bullet\}$$

is known as the *elasticity* of M. A survey of the results in the literature concerning elasticity can be found in [1]. If

$$\mathcal{L}(x) = \{n_1, \ldots, n_t\} \tag{2}$$

with the n_i's listed in increasing order, then the delta set of x is

$$\Delta(x) = \{n_i - n_{i-1} \mid 2 \leq i \leq t\}.$$

The Delta set of M is then defined as

$$\Delta(M) = \cup_{x \in M^\bullet} \Delta(x).$$

If $d = \gcd \Delta(M)$, Geroldinger [16, Proposition 4] has shown that $d \in \Delta(M)$. Hence, it follows that

$$\{d, qd\} \subseteq \Delta(M) \subseteq \{d, 2d, \ldots, qd\} \tag{3}$$

for some positive integer q. While the concept of the Delta set of a monoid M has been widely studied, there are few exact computations of specific Delta sets in the literature. If $\mathcal{B}(\mathbb{Z}_n)$ represents the block monoid (see [18] or Example 2) on the cyclic group of order n, then $\Delta(\mathcal{B}(\mathbb{Z}_n)) = \{1, 2, \ldots, n-2\}$ [18, Theorem 6.7.1]. The Delta sets of several numerical monoids (see [7]) and several congruence monoids (see [4]) have been computed under restricted conditions. In particular, an example is constructed in [7, Proposition 4.9] where both containments in (3) are strict.

The notion of a set of lengths was generalized in [14] as follows. With M as above, for each $n \in \mathbb{N}$ set

$$\mathcal{W}(n) = \{m \in M \mid n \in \mathcal{L}(m)\}$$

and

$$\mathcal{V}(n) = \bigcup_{m \in \mathcal{W}(n)} \mathcal{L}(m).$$

We refer to the set $\mathcal{V}(n)$ as a *union of sets of lengths*. In [14], the basic properties of these sets are determined. Moreover, for block monoids

$\mathcal{B}(G)$ where G is a finite abelian group, the authors argue that the sequence $\{\mathcal{V}(n)\}_{n=1}^{\infty}$ does not uniquely characterize G. We will often need to refer to the maximum and minimum values in $\mathcal{V}(n)$, hence for each $n \in \mathbb{N}$ we set

$$\lambda_n(M) = \min \mathcal{V}(n) \text{ and } \rho_n(M) = \sup \mathcal{V}(n).$$

When the monoid M is understood, we will merely use the notation λ_n and ρ_n. The sequence $\{\rho_n\}_{n=1}^{\infty}$ has been an object of study in its own right (see [18, Section 1.4] and [19]) and it is shown in [18, Proposition 1.4.2] that

$$\rho(M) = \lim_{n\to\infty} \frac{\rho_n(M)}{n}.$$

Finally, for each $n \in \mathbb{N}$, set

$$\Phi(n) = |\mathcal{V}(n)|.$$

Some basic properties of the Φ-function are explored in [11, Section 2] and several additional computations of the limit

$$\overline{\Phi}(M) = \lim_{n\to\infty} \frac{\Phi(n)}{n}$$

can be found in the literature [13, Theorem 2.7 and Theorem 2.10].

For our purposes, we extend the notion of the Delta set to unions of sets of lengths as follows. For a fixed monoid M, suppose for each $n \in \mathbb{N}$ that $\mathcal{V}(n) = \{v_{1,n}, \ldots, v_{t,n}\}$ where $v_{i,n} < v_{i+1,n}$ for $1 \leq i < t$. Define the $\mathcal{V}(n)$-Delta set of M to be

$$\Delta(\mathcal{V}(n)) = \{v_{i,n} - v_{i-1,n} \mid 2 \leq i \leq t\}$$

and the $\mathcal{V}$-Delta set of M to be

$$\Delta_{\mathcal{V}}(M) = \bigcup_{n\in\mathbb{N}} \Delta(\mathcal{V}(n)).$$

In addition, set $\mathcal{V}^*(M) = \sup \Delta_{\mathcal{V}}(M)$ and $\mathcal{V}_*(M) = \min \Delta_{\mathcal{V}}(M)$. Clearly $\Delta(\mathcal{V}(1)) = \emptyset$.

Example 1. Let $\mathbb{N}_0$ represent the nonnegative integers. Consider the additive submonoid

$$M = \{(x_1, x_2, x_3) \mid x_1 + 3x_2 = 4x_3 \text{ with each } x_i \in \mathbb{N}_0\}$$

of $\mathbb{N}_0^3$. Such a monoid is known as a *Diophantine monoid* (see [10]). A characterization of Diophantine monoids can be found in [18, Theorem 2.7.14]. It follows from [8, Proposition 4.8], that $\Delta(M) = \{2\}$. Using elementary number theory, it follows that the irreducible elements of M are $v_1 = (4, 0, 1)$, $v_2 = (0, 4, 3)$ and $v_3 = (1, 1, 1)$. The following two facts will be key in determining $\Delta_{\mathcal{V}}(M)$:

- using the relation $v_1 + v_2 = 4v_3$, it is clear that an irreducible factorization in M which contains both v_1 and v_2 can be increased in length by 2,
- by [13, Lemma 2.8], if a and b are in $\mathcal{V}(n)$, then $a \equiv b \pmod 2$.

By observing that λ_n is obtained by factoring nv_3 and ρ_n by factoring $2nv_3$ (if n is even) or $(2n-1)v_3$ if n is odd, we obtain the following values:

	λ_n	ρ_n
$n \equiv 0 \pmod 4$	$2\lfloor \frac{n}{4} \rfloor$	$2n$
$n \equiv 1 \pmod 4$	$2\lfloor \frac{n-1}{4} \rfloor + 1$	$2n-1$
$n \equiv 2 \pmod 4$	$2\lfloor \frac{n}{4} \rfloor + 2$	$2n$
$n \equiv 3 \pmod 4$	$2\lfloor \frac{n-1}{4} \rfloor + 3$	$2n-1$

We list the first few values of $\mathcal{V}(n)$ below:

$$\begin{array}{ll} \mathcal{V}(1) = \{1\} & \mathcal{V}(5) = \{3,5,7,9\} \\ \mathcal{V}(2) = \{2,4\} & \mathcal{V}(6) = \{4,6,8,10,12\} \\ \mathcal{V}(3) = \{3,5\} & \mathcal{V}(7) = \{5,7,9,11,13\} \\ \mathcal{V}(4) = \{2,4,6,8\} & \mathcal{V}(8) = \{4,6,8,10,12,14,16\} \end{array}$$

We have that $\Delta(\mathcal{V}(n)) = \{2\}$ for all n and hence $\Delta_{\mathcal{V}}(M) = \{2\}$. Notice here that $\Delta_{\mathcal{V}}(M) = \Delta(M)$. □

Example 2. Let G be an abelian group and $\mathcal{F}(G)$ represent the free abelian monoid on G. Set

$$\mathcal{B}(G) = \left\{ \prod_{g_i \in G} g_i^{n_i} \mid \sum_{g_i \in G} n_i g_i = 0 \right\}.$$

$\mathcal{B}(G)$ is a submonoid of $\mathcal{F}(G)$ known as the *block monoid* on G. Its irreducible elements are known as *minimal zero-sequences*. Using the results of [14], we can write out the unions of sets of lengths, and in turn the $\mathcal{V}(n)$-Delta sets of block monoids on relatively simple groups. For instance, if $G = \mathbb{Z}_5$, then [14, Example 5.4] yields:

- $\rho_n = \lfloor \frac{5n}{2} \rfloor$ for $n \geq 2$,
- $\lambda_1 = 1$, $\lambda_k = 2$ for $k = 2, 3, 4$ and 5, and $\lambda_k = \lambda_{(k-5)} + 2$ for $k \geq 6$,
- for all $n \geq 1$, $\mathcal{V}(n) = [\lambda_n, \rho_n] \cap \mathbb{Z}$.

Hence, $\Delta(\mathcal{V}(n)) = \{1\}$ for each $n > 1$ in $\mathbb{N}$ and thus $\Delta_{\mathcal{V}}(\mathcal{B}(\mathbb{Z}_5)) = \{1\}$. Notice that our previous remark yields that $\Delta(\mathcal{B}(\mathbb{Z}_5)) = \{1, 2, 3\}$. □

We consider some basic properties of the $\mathcal{V}$-Delta set of M in the following lemma.

Lemma 3. *Let M be an atomic monoid with* $\min \Delta(M) = d$ *and* $\max \Delta(M) = qd$ *for $q \geq 1$.*

1) $\mathcal{V}_*(M) = d$.
2) $\mathcal{V}^*(M) \leq qd$.
3) $\{d\} \subseteq \Delta_{\mathcal{V}}(M) \subseteq \{d, 2d, \ldots, qd\}$.

Proof. Choose $n \in \mathbb{N}$ and let $v_{i+1,n}$, $v_{i,n}$ be in $\mathcal{V}(n)$. We may choose x_1 and x_2 in $M^\bullet$ such that $\{n, v_{i+1,n}\} \subseteq \mathcal{L}(x_1)$ and $\{n, v_{i,n}\} \subseteq \mathcal{L}(x_2)$. By (3), $\mathcal{L}(x_1)$ is a subset of $n + d\mathbb{Z}$ which contains n and whose consecutive elements are at most qd apart. The same statement holds for $\mathcal{L}(x_2)$, therefore the union, $\mathcal{L}(x_1) \cup \mathcal{L}(x_2)$, also possesses all these properties. Note that the union is a subset of $\mathcal{V}(n)$, so since $v_{i+1,n}$ and $v_{i,n}$ are consecutive elements of $\mathcal{V}(n)$, they in particular must be consecutive elements of $\mathcal{L}(x_1) \cup \mathcal{L}(x_2)$. Therefore $v_{i+1,n} - v_{i,n} = td$ for some $1 \leq t \leq q$. This shows that $\Delta(\mathcal{V}(n)) \subseteq \{d, 2d, \ldots, qd\}$, which in turn implies 2) and 3). It also determines that $\mathcal{V}_*(M) \geq d$, so we are left with just showing $d \in \Delta_{\mathcal{V}}(M)$.

Since $d \in \Delta(M)$, there is an $x \in M$ and $l_1, l_2 \in \mathcal{L}(x)$ with $l_2 - l_1 = d$. Consider $\mathcal{V}(l_1)$, to which both l_1 and l_2 belong. They must be consecutive elements of $\mathcal{V}(l_1)$ since we have just shown that consecutive elements are at least d apart. Hence $d \in \Delta(\mathcal{V}(l_1)) \subset \Delta_{\mathcal{V}}(M)$. $\square$

Note that Example 2 indicates that the inequality in Lemma 3 regarding $\mathcal{V}^*(M)$ may be strict. The next corollary will later be useful and follows immediately from Lemma 3.

Corollary 4. *If $\Delta(M) = \{d\}$, then $\Delta_{\mathcal{V}}(M) = \{d\}$.*

We apply the $\mathcal{V}$-Delta set to limits of the form (1). Unlike the $\overline{L}(x)$ and $\bar{l}(x)$ functions, there is no known argument that $\overline{\Phi}(M)$ exists for a general atomic monoid M. Hence, our analysis of (1) will involve the use of $\liminf$ and $\limsup$. Moreover, we must assume that $\Phi(n)$ is finite for all n, since this is necessary for $\limsup_{n\to\infty}$ to be finite. Indeed, if $\Phi(n)$ were infinite for some n, then so would be $\Phi(kn)$ for all k: if x has a factorization of length n and of length m, then x^k has factorizations of lengths kn and km. In [11], an atomic monoid which statisfies $\Phi(n) < \infty$ for all nonnegative n is called Φ-*finite*.

Our main theorem will use the stronger hypothesis that M has finite elasticity. The following proposition shows this is a necessary condition for $\limsup_{n\to\infty} \Phi(n)/n$ to be finite, and the main theorem shows that it is sufficient as well.

Proposition 5. *Let M be an atomic Φ-finite monoid. If $\rho(M) = \infty$, then*

$$\limsup_{n\to\infty} \frac{\Phi(n)}{n} = \infty$$

Proof. Since $\rho(M) = \infty$, there are x_t such that $a_t = L(x_t)$ and $b_t = l(x_t)$ satisfying $\lim_{t\to\infty} \frac{a_t}{b_t} = \infty$. But all the $\mathcal{V}(n)$ are finite and $a_t \in \mathcal{V}(b_t)$, implying that for every $M > 0$ there is an $N > 0$ such that for all $t > N$, $b_t > M$. Therefore we may assume that the sequence is chosen such that the b_t are strictly increasing.

Since $\Phi(n)$ is finite for each n, $\mathcal{V}^*(b_t)$ exists and $\mathcal{V}^*(b_t) \geq a_t$. Pruning the sequence if necessary, we may assume that the b_t are chosen such that

$$\lim_{t\to\infty} \frac{\mathcal{V}^*(b_t)}{b_t} = \infty.$$

We may estimate

$$\Phi(b_t) \geq \frac{\mathcal{V}^*(b_t) - \mathcal{V}_*(b_t) + 1}{qd}.$$

Since $\mathcal{V}_*(b_t) \leq b_t$, we find that

$$\frac{\Phi(b_t)}{b_t} \geq \frac{\mathcal{V}^*(b_t)}{b_t qd} - \frac{1}{qd} + \frac{1}{b_t qd}.$$

Taking $\liminf$ of both sides, we see that $\liminf_{t\to\infty} \Phi(b_t)/b_t \geq \infty$, since the b_t are strictly increasing. Therefore $\limsup_{n\to\infty} \Phi(n)/n = \infty$. $\square$

Now our main theorem.

Theorem 6. *Let M be an atomic monoid with $\rho(M) < \infty$. Then M is Φ-finite and moreover*

$$\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}^*(M)} \leq \liminf_{n\to\infty} \frac{\Phi(n)}{n} \leq \limsup_{n\to\infty} \frac{\Phi(n)}{n} \leq \frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}_*(M)}. \tag{4}$$

Proof. Let $n \in \mathbb{N}$ and suppose that $m \in \mathcal{V}(n)$. It follows that

$$\frac{1}{\rho(M)} \leq \frac{m}{n} \leq \rho(M)$$

and hence

$$\frac{n}{\rho(M)} \leq m \leq n\rho(M)$$

which shows that M is Φ-finite. We further obtain that

$$\frac{(\rho(M) - \frac{1}{\rho(M)})n + 1}{\mathcal{V}^*(M)} \leq \Phi(n) \leq \frac{(\rho(M) - \frac{1}{\rho(M)})n + 1}{\mathcal{V}_*(M)}.$$

Thus,

$$\left(\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}^*(M)}\right) n + \frac{1}{\mathcal{V}^*(M)} \leq \Phi(n) \leq \left(\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}_*(M)}\right) n + \frac{1}{\mathcal{V}_*(M)}.$$

After dividing by n and taking the respective $\liminf$ and $\limsup$, we get that

$$\frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}^*(M)} \leq \liminf_{n\to\infty} \frac{\Phi(n)}{n} \leq \limsup_{n\to\infty} \frac{\Phi(n)}{n} \leq \frac{\rho(M)^2 - 1}{\rho(M)\mathcal{V}_*(M)}.$$

$\square$

If $\Delta(M) = \{d\}$, then Corollary 4 implies that $\mathcal{V}^*(M) = \mathcal{V}_*(M) = d$ and Theorem 6 reduces to the following.

Corollary 7. *Let M be an atomic monoid with $\rho(M) < \infty$. If $\Delta(M) = \{d\}$, then*

$$\overline{\Phi}(M) = \frac{\rho(M)^2 - 1}{\rho(M)d}. \tag{5}$$

Corollary 7 immediately has some nice applications.

Example 8. A numerical monoid is an additive submonoid of the non-negative integers. Every numerical monoid S has a unique minimal set of generators, and we will use the notation $S = \langle a_1, a_2, \dots, a_t\rangle$ to represent the minimal generating set (which we assume is written in linear order). S is *primitive* if $1 = \gcd\{s \mid s \in S\}$. Every numerical monoid S is isomorphic to a unique primitive numerical monoid, so when working with numerical monoids, we can always assume that S is a primitive numerical monoid. By [7], there exists a method for calculating $\max \Delta(S)$ in finite time and

$$\min \Delta(S) = \gcd\{a_i - a_{i-1} \mid i \in \{2, 3, \dots, t\}\} = d.$$

By [9, Theorem 2.1], $\rho(S) = \frac{a_t}{a_1}$. Hence for a numerical monoid, (4) reduces to

$$\frac{a_t^2 - a_1^2}{\mathcal{V}^* a_1 a_t} \le \lim_{n\to\infty} \inf \frac{\Phi(n)}{n} \le \lim_{n\to\infty} \sup \frac{\Phi(n)}{n} \le \frac{a_t^2 - a_1^2}{\mathcal{V}_* a_1 a_t}$$

If we know further that the generators of S form an arithmetic sequence (i.e., $S = \langle a, a+d, a+2d, \dots, a+kd\rangle$ for some positive integers d and k), then [7, Theorem 3.9] indicates that $\Delta(S) = \{d\}$. In this case we obtain an exact calculation of $\overline{\Phi}(S)$ as

$$\overline{\Phi}(S) = \frac{k(2a+kd)}{a(a+kd)} = k\left(\frac{1}{a} + \frac{1}{a+kd}\right). \quad \square$$

Example 9. Let a and b be positive integers with $a \le b$ and $a^2 \equiv a \pmod b$. The set of numbers

$$M(a,b) = \{x \mid x \in \mathbb{N} \text{ and } x \equiv a \pmod b\} \cup \{1\}$$

forms a multiplicative monoid known as an *arithmetical congruence monoid* (or ACM). ACMs have been the focus of three recent papers in the literature ([4], [5] and [6]). An ACM is called *local* if $\gcd(a,b) = p^\alpha$ for some prime number p and positive integer α. It follows from elementary number theory that a local ACM $M(a,b)$ has a minimal index, which we denote by β, for which $p^\beta \in M(a,b)$. There are two relevant known results for a local ACM $M(a,b)$:

- $\rho(M(a,b)) = \frac{\alpha+\beta-1}{\alpha}$ [6, Theorem 2.4],
- if $\alpha = \beta > 1$, then $\Delta(M(a,b)) = \{1\}$ [4, Theorem 3.1].

Hence, for an ACM as above where $\alpha = \beta > 1$ (for instance, $M(4,12)$), (5) reduces to

$$\overline{\Phi}(M(a,b)) = \frac{(2\alpha-1)^2 - \alpha^2}{\alpha(2\alpha-1)}.$$

$\square$

We close with a few comments.

- The proof in [12] of (1) relies on a different technique than that used above. The proof relies on knowing the exact structure of the sets in an infinite subsequence of the sequence $\mathcal{V}(1)$, $\mathcal{V}(2), \ldots$.
- We note that Theorem 6 cannot be used to verify (1) since it is not known that $\mathcal{V}^*(\mathcal{B}(G)) = \mathcal{V}_*(\mathcal{B}(G))$ for a finite abelian group G. It has been conjectured for such G (see [14]) that $\Delta_{\mathcal{V}}(\mathcal{B}(G)) = \{1\}$. This is known to be true for all finite abelian G with $\mid G \mid \leq 8$ (see [14]). In fact, it is known for finite abelian groups G that $\Delta(\mathcal{V}(n)) = \{1\}$ for infinitely many n (see [12, Lemma 3]).
- Connected to the last remark is an open problem which has appeared in the literature [14, Section 5]: for $\mathcal{B}(\mathbb{Z}_n)$ does $\rho_3 = \max \mathcal{V}(3) = n + 1$?

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