ON CROSS NUMBERS OF MINIMAL ZERO SEQUENCES IN CERTAIN CYCLIC GROUPS

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Abstract. Let p and q be distinct primes with $p > q$ and n a positive integer. In this paper, we consider the set of possible cross numbers for the cyclic groups $\mathbb{Z}_{2p^n}$ and $\mathbb{Z}_{pq}$. We completely determine this set for $\mathbb{Z}_{2p^n}$ and also $\mathbb{Z}_{pq}$ for $q = 3$, $q = 5$ and the case where p is sufficiently larger than q. We view the latter result in terms of an upper bound for this set developed in a paper of Geroldinger and Schneider [8] and show precisely when this upper bound is an equality.

1. Introduction

A nonempty sequence $S = \{g_1, \ldots, g_n\}$ of not necessarily distinct elements of an additive group G is called a zero sequence if $\sum_{i=1}^{n} g_i = 0$. A zero sequence with no proper nonempty zero subsequence is called a minimal zero sequence. If a sequence contains no zero subsequence, it is known as zero free. We define

$$\mathcal{U}(G) = \{T \mid T \text{ is a minimal zero sequence in } G\}.$$

Several constants can be extracted from the study of such sequences. The Davenport constant, $D(G)$ is the maximum length of a minimal zero sequence in G. The Davenport constant is at most the order of G, and in the case of cyclic groups it indeed attains that value. We additionally define the cross number of a sequence as

$$k(S) = \sum_{i=1}^{n} \frac{1}{|g_i|}$$

and the cross number of a group as

$$K(G) = \max\{k(S) \mid S \in \mathcal{U}(G)\}.$$

A formula for the cross number of a finite abelian group is not known in general. Formulas are known for particular classes of groups, including p-groups. We will use the following well-known results in our work.

(i) $K(G) = 1$ if and only if $G \cong \mathbb{Z}_{p^n}$ for p prime and $n \in \mathbb{N}$ [9].

(ii) If p and q are distinct primes then $K(\mathbb{Z}_{pq}) = \frac{2pq-p-q+1}{pq}$ [10, Theorem 3].

(iii) If p is an odd prime and $n \in \mathbb{N}$, then $K(\mathbb{Z}_{2p^n}) = \frac{3p^n-1}{2p^n}$ [10, Theorem 3].

We mention some additional facts about such sequences that will be used frequently throughout this paper:

1. If $S = \{g_1, g_2, \ldots, g_n\}$ is a zero, minimal zero, or zero free sequence, then $T = \{g_1 + g_2, \ldots, g_n\}$ also is a zero, minimal zero, or zero free sequence. We call the process of attaining T from S *amalgamation* of elements of S.
2. If $S = \{g_1, g_2, \ldots, g_n\}$ is a zero free sequence, then $T = S \cup \{-\sum_{i=1}^{n} g_i\}$ is a minimal zero sequence.

A particularly important inference from the last fact is that any sequence of length greater than or equal to $D(G)$ must contain a zero subsequence.

In this paper, we extend the work presented in [3] and consider the set

$$W(G) = \{k(T) \mid T \in \mathcal{U}(G)\}$$

of cross numbers of all the minimal zero sequences in a finite abelian group G. We consider in particular the case $G \cong \mathbb{Z}_{pq}$ where $p > q$ are primes. Several general results appear in the literature concerning the set $W(G)$. If $G \cong \sum_{i=1}^{t} \mathbb{Z}_{p^{n_i}}$ is a p-group where p is an odd prime and $n_i \leq n_j$ for all $1 \leq i \leq j \leq t$, then the main theorem of [5] implies that

$$K(G) = \left[\sum_{i=1}^{t} \frac{p^{n_i} - 1}{p^{n_i}} \right] + \frac{1}{p^{n_t}}.$$

Setting $K(G) = \frac{x}{p^{n_t}}$, [3, Theorem 4] shows that $W(G) = \{\frac{\lambda}{p^{n_t}} \mid 2 \leq \lambda \leq x\}$. Moreover, [3, Theorem 2] also shows if $W^*(G) = \{k(S) \mid S \in \mathcal{U}(G), k(S) \leq 1\}$, then $W^*(G) = \{\frac{\lambda}{\exp(G)} \mid 2 \leq \lambda \leq \exp(G)\}$, for every abelian group G of odd order.

Using the main theorem of [8], which describes the structure of zero free sequences with large cross number in the group $G = \mathbb{Z}_p^r \oplus \mathbb{Z}_q^s$, the authors obtain an upper bound for the set $W(G)$ [8, Corollary 2]. In particular, if p and q are distinct odd primes and $p > q$, then their result reduces to

$$(*) \qquad W(\mathbb{Z}_{pq}) \subseteq \{\frac{\lambda}{pq} \mid 2 \leq \lambda \leq 2pq - p - 2q + 2 \text{ or } \lambda = 2pq - p - q + 1\}.$$

Our interest in the set $W(\mathbb{Z}_{pq})$ originated in our attempt to determine when the containment in $(*)$ is actually an equality. A combination of Example 5.2 and Corollaries 5.1, 5.4, 5.9 and 5.11 yields a proof of the following proposition.

Proposition 1.1. *If $p > q$ are odd primes, then*

$$W(\mathbb{Z}_{pq}) = \{\frac{\lambda}{pq} \mid 2 \leq \lambda \leq 2pq - p - 2q + 2 \text{ or } \lambda = 2pq - p - q + 1\}$$

if and only if

1. $q = 3$, *or*
2. $q = 5$ *and* $p = 7$.

Our work shows in the case where $G = \mathbb{Z}_{pq}$ that the main result in [8] is essentially the best possible. While there is no "global" result for the structure of zero free sequences in $\mathbb{Z}_{pq}$ of cross number less than $(2pq - p - 2q + 1)/pq$, Theorems 5.3 5.8 and 5.10 show that the structure of such sequences is dependent on the size of p relative to q. In the case where p is large relative to q (in fact, when $p > q^2 - 3q + 2$) we prove a stronger version of the main theorem of [8] which allows us to determine the set $W(\mathbb{Z}_{pq})$. We show for q fixed that the size of $W(\mathbb{Z}_{pq})$ stabilizes as p increases (see Corollary 5.6). In the other cases $(2p - 4 \geq 2q \geq p + 3)$, we prove a modified form of the main result in [8] with which we are able to show the existence of a second "gap" in $W(\mathbb{Z}_{pq})$.

2. The Case $q = 2$

We begin by considering $W(\mathbb{Z}_{2p})$ and, using (iii) from the Introduction, actually determine this set for a slightly larger class of cyclic groups. If S is a finite sequence which contains exactly n copies of the element g, then we will represent this with the notation g^n.

Theorem 2.1. *Let p be an odd prime and $n \in \mathbb{N}$. Then*

$$W(\mathbb{Z}_{2p^n}) = \{\frac{k}{2p^n} | \ k \text{ even},\ 2 \leq k \leq 3p^n - 1\}$$

Proof. By [3] and [10], we have the following inclusions:

$$\{\frac{k}{2p^n} \mid k \text{ even},\ 2 \leq k \leq 2p^n\} \subseteq W(\mathbb{Z}_{2p^n}) \subseteq$$
$$\subseteq \{\frac{k}{2p^n} \mid k \text{ even},\ 2 \leq k \leq 2p^n\} \cup \{\frac{k}{2p^n} \mid 2p^n \leq k \leq 3p^n - 1\}$$

Let $1 \leq m \leq \frac{p^n-1}{2}$ be given. We wish to show that $\frac{3p^n-2m+1}{2p^n} \in W(\mathbb{Z}_{2p^n})$. View $\mathbb{Z}_{2p^n}$ as $\mathbb{Z}_2 \oplus \mathbb{Z}_{p^n}$.

Case 1: If p does not divide m, then, $B = \{(1,0),(0,m),(0,1)^{p^n-m-1},(1,1)\} \in \mathcal{U}(\mathbb{Z}_{2p^n})$ and $k(B) = \frac{3p^n-2m+1}{2p^n}$.

Case 2: If $p \mid m$, then, $B = \{(1,0),(0,2),(0,m-1),(0,1)^{p^n-m-2},(1,1)\} \in \mathcal{U}(\mathbb{Z}_{2p^n})$ and $k(B) = \frac{3p^n-2m+1}{2p^n}$.

Thus, $\{\frac{k}{2p^n} \mid k \text{ even},\ 2 \leq k \leq 3p^n - 1\} \subseteq W(\mathbb{Z}_{2p^n})$. To complete the argument, let x odd, with $2 < x < 3p^n - 1$, be given. Assume $\frac{x}{2p^n} \in W(\mathbb{Z}_{2p^n})$. Thus, there exists $S = \{g_1, \ldots, g_t\} \in \mathcal{U}(\mathbb{Z}_{2p^n})$ such that $\sum_{i=1}^{t} \frac{1}{|g_i|} = \frac{x}{2p^n}$. The only possible values for $\frac{1}{|g_i|}$ are $\frac{p^k}{2p^n}$ and $\frac{2p^k}{2p^n}$, for $0 \leq k \leq n$. Note that if $\frac{1}{|g_i|} = \frac{2p^k}{2p^n}$, then g_i (when viewed as its least positive residue in $\mathbb{Z}$) is necessarily even, while if $\frac{1}{|g_i|} = \frac{p^k}{2p^n}$, then g_i is necessarily odd. Thus,

$$\frac{x}{2p^n} = \sum_{i=1}^{t} \frac{1}{\mid g_i \mid} = \sum_{g_i \text{ even}} \frac{1}{\mid g_i \mid} + \sum_{g_i \text{ odd}} \frac{1}{\mid g_i \mid} = \frac{2m}{2p^n} + \sum_{g_i \text{ odd}} \frac{1}{\mid g_i \mid}$$

for some positive integer m. Since x is odd, it must be that the number of odd g_i is odd. But, since $S \in \mathcal{U}(\mathbb{Z}_{2p^n})$, we must have for some positive integer k,

$$2p^n k = \sum_{i=1}^{t} g_i = \sum_{g_i \text{ even}} g_i + \sum_{g_i \text{ odd}} g_i = 2l_1 + (2l_2 + 1)$$

which is a contradiction. Thus, $\frac{x}{2p^n} \notin W(\mathbb{Z}_{2p^n})$ when x is odd and $2 < x < 3p^n - 1$, completing the proof. $\square$

3. Zero free Sequences in $\mathbb{Z}_{pq}$

We prove two lemmas which will later be useful. When considering zero free sequences in $\mathbb{Z}_{pq}$, we will use the following notation. If S is such a sequence, write $S = A \cup B \cup C$ where A consists of elements of order p, B consists of elements of order q and C consists of elements of order pq. For ease of notation, viewing $\mathbb{Z}_{pq}$ as $\mathbb{Z}_p \oplus \mathbb{Z}_q$ we will say that $A \subseteq \mathbb{Z}_p$, $B \subseteq \mathbb{Z}_q$ and $C \subseteq \mathbb{Z}_{pq} \backslash \mathbb{Z}_p \cup \mathbb{Z}_q$. Moreover, if A is a finite sequence from an abelian group G, then let $\sum A$ represent the set of all nonempty subsums of elements in A.

Lemma 3.1. *Let $S = A \cup B \cup C$ be a zero free sequence in $\mathbb{Z}_{pq}$ such that there exist integers $1 \leq i \leq p-1$ and $1 \leq j \leq q-1$ with $|A| = p - i$, $|B| = q - j$.*

(a) $|B \cup C| < iq$ *and* $|A \cup C| < jp$.
(b) $|C| < \min\{(i-1)q + j, (j-1)p + i\}$.
(c) *If* $\min\{i, j\} = 1$ *then* $|C| < \max\{i, j\}$.

Proof. For (a), we show that $|B \cup C| < iq$ as the argument for the second inequality is similar. If $|B \cup C| \geq iq$, then there are i nonempty, non-overlapping subsequences in $B \cup C$ which sum to zero in $\mathbb{Z}_q$. Let the sums of these i subsequences be given by $(y_t, 0)$ for $1 \leq t \leq i$. Then $|A \cup \{(y_t, 0)\}_{t=1}^{i}| = p$ and so $A \cup \{(y_t, 0)\}_{t=1}^{i}$ is not a zero free sequence, contradicting the zero freeness of S. For (b), (a) implies that

$$iq > |B \cup C| = |B| + |C| = q - j + |C|$$

and

$$jp > |A \cup C| = |A| + |C| = p - i + |C|$$

and from this the result follows. Part (c) now follows directly from part (b). $\square$

Lemma 3.2. *Let $S = A \cup B \cup C$ be a zero free sequence in $\mathbb{Z}_{pq}$. If $|A| = p - 2$ and $|B| = q - 2$, then $|C| \leq 2$.*

Proof. By [4, Lemma 13],

$$\left|\sum A\right| = \begin{cases} p-2 & \text{if} \quad A = \{g^{p-2}\} \\ p-1 & \text{if} \quad A = \{g^{p-3}, 2g\} \end{cases}$$

and

$$\left|\sum B\right| = \begin{cases} q-2 & \text{if} \quad B = \{h^{p-2}\} \\ q-1 & \text{if} \quad B = \{h^{p-3}, 2h\} \end{cases}$$

where $g \neq 0$ in $\mathbb{Z}_p$ and $h \neq 0$ in $\mathbb{Z}_q$. Suppose that $|C| > 2$ and that (a_1, b_1), (a_2, b_2), (a_3, b_3) are in C with each a_i and b_j nonzero. Suppose $|\sum A| = p - 1$. Let T be a sequence in $B \cup C$ with sum $(x, 0)$. Now, $-x \in \sum A$ and if V is a sequence in A with sum $(-x, 0)$, then $V \cup T$ is a zero sequence in $\mathbb{Z}_{pq}$. A similar argument holds if $|\sum B| = q - 1$. Hence, if S is zero free then $|\sum A| = p - 2$ and $|\sum B| = q - 2$. Thus $A = \{(g, 0)^{p-2}\}$, $B = \{(0, h)^{q-2}\}$ and hence $\sum A = \{(x, 0) \,|\, x \neq -g,\, x \neq 0\}$, $\sum B = \{(0, y) \,|\, y \neq -h,\, y \neq 0\}$. If T is a sequence in $B \cup C$ which sums to zero in $\mathbb{Z}_q$, then its sum is $(g, 0)$ (otherwise, using A we can construct a zero sequence in S). We consider two cases.

Case 1: Suppose that $b_1+b_2+b_3 \neq h$. Then $a_1+a_2+a_3=g$. If for $1 \leq s < t \leq 3$ we have $b_s+b_t \neq h$, then $a_s+a_t=g$ which yields $a_r=0$ for some r, a contradiction. Thus $b_1+b_2=b_2+b_3=b_1+b_3=h$. Hence, $b_1=b_2=b_3$ and each $b_s \neq h$. Now, $g=a_1=a_2=a_3$ and our previous observation implies that $a_1=a_1+a_1+a_1$. Thus $a_1=0$, a contradiction.

Case 2: Suppose that $b_1+b_2+b_3=h$. Then $b_1+b_2 \neq h$, $b_1+b_3 \neq h$ and $b_2+b_3 \neq h$ and hence $a_1+a_2=a_1+a_3=a_2+a_3=g$. Thus $a_1=a_2=a_3 \neq g$ and hence $b_1=b_2=b_3=h$. In a manner similar to Case 1, we obtain that $b_1=0$, a contradiction. □

4. Some Minimal Zero Sequences in $\mathbb{Z}_{pq}$

We begin by generating a family of minimal zero sequences in $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$. For $1 \leq k < q$, define

$$T^{kp} = \{(1,1)^k, (k,0)^{p-1}, (0,1)^{q-k}\}, \tag{1}$$

$$T^{kq} = \{(1,1)^k, (1,0)^{p-k}, (0,k)^{q-1}\} \tag{2}$$

and for $q \leq k < p$, define

$$T^{kp} = \{(1,1)^{q-1}, (1+k-q,\ 2), (0,1)^{q-1}, (1,0)^{p-k}\}. \tag{3}$$

Elementary arguments show that each T^{kp} and $T^{kq} \in \mathcal{U}(\mathbb{Z}_{pq})$ and clearly

$$k(T^{kp}) = \tfrac{2pq-kp-q+k}{pq} \text{ or } \tfrac{2pq+q-kq-p}{pq},$$
$$k(T^{kq}) = \tfrac{2pq-p-kq+k}{pq}.$$

By amalgamating elements of the same order within (1), (2) and (3), we obtain new minimal zero sequences, each of which has a cross number easily calculated from the list above. We compile these numbers in the following lemma.

Lemma 4.1. *Let p and q be odd primes with $p > q$. The following sets are subsets of $W(\mathbb{Z}_{pq})$:*

1. $\{\frac{2pq-m_3p-m_2q+m_1}{pq} \mid 1 \leq k < q,\ 1 \leq m_1 \leq k,\ 1 \leq m_2 < p,\ k \leq m_3 < q\}$,
2. $\{\frac{2pq-m_2p-m_3q+m_1}{pq} \mid 1 \leq k < q,\ 1 \leq m_1 \leq k,\ 1 \leq m_2 < q,\ k \leq m_3 < p\}$ *and*
3. $\{\frac{2pq+q-m_3q-m_2p-m_1}{pq} \mid q \leq k < p,\ 0 \leq m_1 \leq q-2,\ 1 \leq m_2 < q,\ k \leq m_3 < p\}$.

The lemma implies a general result.

Proposition 4.2. *For any $p > q$ odd primes, we have the following inclusion:*

$$W(\mathbb{Z}_{pq}) \supseteq \{\frac{x}{pq} \mid 2 \leq x \leq 2pq-(q-1)q-p+(q-1)\} \cup \tag{4}$$
$$\cup \bigcup_{t=1}^{q-2} \{\frac{x}{pq} \mid 2pq-tq-p+1 \leq x \leq 2pq-tq-p+t\}$$

Proof. [3, Theorem 2] shows that if $2 \leq x \leq pq$, then $\frac{x}{pq} \in W(\mathbb{Z}_{pq})$. Consider x with $2pq-tq-p+1 \leq x \leq 2pq-tq-p+t$ for some $1 \leq t \leq q-2$. Then $x = 2pq-tq-p+\gamma$ for some $1 \leq \gamma \leq t$. We may generate a sequence with cross number $\frac{x}{pq}$ by considering sequence 2 with $k=m_3=t, m_1=\gamma$, and $m_2=1$, namely the minimal zero sequence $S=\{(t+1-\gamma, t+1-\gamma), (1,1)^{\gamma-1}, (1,0)^{p-t}, (0,t)^{q-1}\}$.

Now consider x with $2pq-p-q^2+q<x\leq 2pq-p-q^2+2q-1$. Thus, $x=2pq-p-(q-1)q+\gamma$ for some $1\leq\gamma\leq q-1$. We may generate a sequence with cross number $\frac{x}{pq}$ by considering sequence 2 with $k=m_1=\gamma$, $m_2=1$, and $m_3=q-1$, namely the minimal zero sequence $S=\{(1,1)^{\gamma},(q-\gamma,0),(1,0)^{p-q},(0,\gamma)^{q-1}\}$.

Finally, we must consider x with $pq<x\leq 2pq-p-q^2+q$. Evidently, $x=2pq-\alpha p-\beta q-\gamma$ for some $1\leq\alpha\leq q-1$, $1\leq\beta\leq p-1$, $0\leq\gamma\leq q-1$. However, in order for $pq<x\leq 2pq-p-q^2+q$ to hold, we must have $q-1\leq\beta\leq p-2$. If $\gamma\neq q-1$, then we may rewrite x as $x=2pq+q-\alpha p-(\beta+1)q-\gamma$ and generate a sequence with cross number $\frac{x}{pq}$ by taking sequence 3 with $k=m_3=\beta+1$, $m_1=\gamma$, and $m_2=\alpha$, namely

$$S=\{(1,1)^{q-\gamma-2},(\gamma+1,\gamma+1),(\beta+2-q,2),(0,\alpha+1),(0,1)^{q-\alpha-2},(1,0)^{p-\beta-1}\}.$$

Otherwise, if $\gamma=q-1$, then $x=2pq-\alpha p-(\beta+1)q+1$ and we may use sequence 1 with $k=m_1=1$, $m_2=\beta+1$, and $m_3=\alpha$, namely $S=\{(1,1),(\beta+1,0),(1,0)^{p-\beta-2},(0,\alpha),(1,0)^{q-\alpha-1}\}$. $\square$

5. Particular cases

5.1. **When q = 3.** Setting $q=3$, Proposition 4.2 and $(*)$ imply the following.

Corollary 5.1. *If $p>3$ is an odd prime, then*

$$W(\mathbb{Z}_{3p})=\{\frac{\lambda}{3p}\mid 2\leq\lambda\leq 6p-p-4 \text{ or } \lambda=6p-p-2\}.$$

The containment in (4) is not always an equality. Here is an example which will later be useful in our work.

Example 5.2. Let $p=7$ and $q=5$. The containment in (4) yields

$$W(\mathbb{Z}_{35})\supseteq\{\frac{2}{35},\dots,\frac{47}{35}\}\cup\{\frac{49}{35},\frac{50}{35},\frac{51}{35}\}\cup\{\frac{54}{35},\frac{55}{35}\}\cup\{\frac{59}{35}\}.$$

Now, the minimal zero sequence T^{2p} yields that $\frac{53}{35}$ is in $W(\mathbb{Z}_{35})$. Also, the minimal zero sequences T^{2q} and T^p can be amalgamated to $\{(1,1)^2,(0,2)^2,(0,4),(1,0)^5\}$, and $\{(1,1),(0,2),(1,0)^6,(0,1)^2\}$ which implies that $\frac{48}{35}$ and $\frac{52}{35}$ are in $W(\mathbb{Z}_{35})$. Hence, $(*)$ implies that

$$W(\mathbb{Z}_{35})=\{\frac{2}{35},\dots,\frac{55}{35},\frac{59}{35}\}.$$

5.2. **When p is large relative to q.** As p becomes large relative to q, we are able to show that the gaps which appear in the right hand side of the containment (4) are actual gaps in $W(\mathbb{Z}_{pq})$.

Theorem 5.3. *Let $p>q>3$ be odd primes with $p+k>kq$ for some $1\leq k<q-1$. If S is a zero free sequence in $\mathbb{Z}_{pq}$ and*

$$\frac{2pq-(k+1)q-p+(k+1)}{pq}\leq k(S)\leq\frac{2pq-kq-p}{pq} \tag{5}$$

then $S=A\cup B$, where $A\subseteq\mathbb{Z}_p$, $B\subseteq\mathbb{Z}_q$, $|A|=p-k$ and $|B|=q-1$.

Proof. Let S be as in the hypothesis. Clearly, since S is zero free and $k(S) > 1$, we have $1 \le |A| < p$, $1 \le |B| < q$, and $0 \le |C| < pq$. Let $0 < i < p$ and $0 < j < q$ be integers such that $|A| = p - i$ and $|B| = q - j$ and write $|C| = c$. Then,

$$k(S) = \frac{2pq - iq - jp + c}{pq}$$

First, we show the theorem holds if $c = 0$. In this case, (5) indicates that

$$k + 1 \le (k + 1 - i)q + (1 - j)p \le q.$$

Assume $j > 1$. Then

$$k + 1 \le (k + 1 - i)q + (1 - j)p \le kq - p + (1 - i)q < k + (1 - i)q \le k,$$

which is clearly a contradiction. Thus, $j = 1$ and the previous inequality reduces to $(k + 1) \le (k + 1 - i)q \le q$. However, since $0 < k + 1 \le q$, it must be the case that $i = k$. We now assume that $c > 0$ and consider two cases:

Case 1: $j = 1$. Notice that if $i \le k$, then (5) implies

$$\frac{2pq - kq - p}{pq} \ge k(S) = \frac{2pq - iq - p + c}{pq} \ge \frac{2pq - kq - p + c}{pq}$$

which yields $c \le 0$, a contradiction. Thus $i \ge k + 1$. From (5) one also has

$$\frac{2pq - (k + 1)q - p + k + 1}{pq} \le k(S) = \frac{2pq - iq - p + c}{pq}$$

which implies $c \ge k + 1 + (i - k - 1)q$. Consequently,

$$|A \cup C| = (p - i) + c \ge p + (q - 1)i + (1 - q)(k + 1)$$

which is at least p since $i \ge k + 1$. The result now follows from Lemma 3.1 (a).

Case 2: $j > 1$. Using (5), we have

$$k(S) = \frac{2pq - iq - jp + c}{pq} \ge \frac{2pq - (k + 1)q - p + (k + 1)}{pq},$$

which yields

$$c \ge (i - k - 1)q + (j - 1)p + k + 1 = (i - 1)q + (j - 2)p + (p + k + 1 - kq)$$

Since $p + k > kq$, c is nonnegative. It follows that

$$|B \cup C| \ge (q - j) + (i - k - 1)q + (j - 1)p + k + 1 \ge iq - 1 + (p + k - kq) \ge iq.$$

Thus, the result follows from Lemma 3.1 (a). □

Theorem 5.3 leads to several corollaries.

Corollary 5.4. *Let $p > q > 3$ be odd primes.*

a) *If $p + k > kq$ for some $0 < k < q - 1$, then*

$$W(\mathbb{Z}_{pq}) \subseteq \{\frac{x}{pq} \mid 2 \le x \le 2pq - (k + 1)q - p + (k + 1)\} \cup$$
$$\cup \bigcup_{t=1}^{k} \{\frac{x}{pq} \mid 2pq - tq - p + 1 \le x \le 2pq - tq - p + t\}.$$

b) *If* $p > q^2 - 3q + 2$. *Then*

$$W(\mathbb{Z}_{pq}) = \{\frac{x}{pq} \mid 2 \leq x \leq 2pq - (q-1)q - p + (q-1)\} \cup$$
$$\cup \bigcup_{t=1}^{q-2} \{\frac{x}{pq} \mid 2pq - tq - p + 1 \leq x \leq 2pq - tq - p + t\}.$$

Proof. Part a) follows directly from the Theorem 5.3 and the observation that any minimal zero sequence with cross number greater than 1 contains an element of order pq. Part b) follows from part a) and Proposition 4.2 with $k = q-2$. $\square$

Example 5.5. Part a) of Corollary 5.4 can sometimes easily be applied to determine the set $W(Z_{pq})$ when $p \leq q^2 - 3q + 2$. Let $q = 5$ and $p = 11$. Part a) above (with $k = 2$) and (4) yield

$$\{\frac{2}{55}, \dots, \frac{83}{55}, \frac{85}{55}, \frac{86}{55}, \frac{87}{55}, \frac{90}{55}, \frac{91}{55}, \frac{95}{55}\} \subseteq W(Z_{55}) \subseteq \{\frac{2}{55}, \dots, \frac{87}{55}, \frac{90}{55}, \frac{91}{55}, \frac{95}{55}\}.$$

Since these sets differ by one integer, one of these containments is an equality. The minimal zero sequence T^q can be amalgamated to $\{(1,1), (0,1)^2, (0,2), (1,0)^{10}\}$ and hence $\frac{84}{55} \in W(\mathbb{Z}_{55})$. Thus

$$W(Z_{55}) = \{\frac{2}{55}, \dots, \frac{87}{55}, \frac{90}{55}, \frac{91}{55}, \frac{95}{55}\}.$$

Define $H(G) = \{\frac{x}{\exp(G)} \mid 2 \leq x \leq \exp(G)K(G)\}$, which represents the set of potential cross numbers for a minimal zero sequence in G.

Corollary 5.6. *Let* $p > q > 3$ *be odd primes with* $p > q^2 - 3q + 2$. *The number of values in* $H(\mathbb{Z}_{pq})$ *which do not appear in* $W(\mathbb{Z}_{pq})$ *is* $\frac{(q-1)(q-2)}{2}$. *Thus, for fixed* q,

$$\limsup_{p \to \infty} \frac{|W(\mathbb{Z}_{pq})|}{|H(\mathbb{Z}_{pq})|} = 1.$$

Proof. This is a direct consequence of previous corollary, since the lengths of the contiguous gaps vary over all the integers from 1 to $q-2$ inclusive. Hence, the total number of missing values is $\sum_{i=1}^{q-2} i = \frac{(q-1)(q-2)}{2}$, whence the second result of the corollary follows immediately. $\square$

5.3. **When q** $= 5$. Examples 5.2, 5.5 and part b) of Corollary 5.4 yield a determination of $W(\mathbb{Z}_{5p})$.

Corollary 5.7. *If* $p > 5$, *then*

$$W(\mathbb{Z}_{5p}) = \{\frac{\lambda}{5p} \mid 2 \leq \lambda \leq 9p - 16, 9p - 14 \leq \lambda \leq 9p - 12,$$
$$9p - 9 \leq \lambda \leq 9p - 8, \lambda = 9p - 4\}$$

unless $p = 7$ *or* $p = 11$, *in which case*

$$W(\mathbb{Z}_{35}) = \{\frac{2}{35}, \dots, \frac{55}{35}, \frac{59}{35}\} \text{ or } W(Z_{55}) = \{\frac{2}{55}, \dots, \frac{87}{55}, \frac{90}{55}, \frac{91}{55}, \frac{95}{55}\}$$

5.4. **When $\mathbf{2p-4>2q\geq p+3}$.** Theorem 5.3 determines the location of a second gap in $W(\mathbb{Z}_{pq})$ when $p+2>2q$. We now determine the location of a second gap (provided one exists) for all other odd primes p and q other than twin primes.

Theorem 5.8. *Suppose $p>q>5$ are odd primes with $2p-4>2q\geq p+3$. If S is a zero free sequence in $\mathbb{Z}_{pq}$ and*

$$\frac{p-1}{p}+\frac{q-2}{q}+\frac{1}{pq}<k(S)\leq\frac{p-2}{p}+\frac{q-1}{q} \tag{6}$$

then $S=A\cup B$, where $A\subseteq\mathbb{Z}_p$, $B\subseteq\mathbb{Z}_q$, $|A|=p-2$, and $|B|=q-1$

Proof. Let $S=A\cup B\cup C$ with the same assumptions on A, B and C as in the proof of Theorem 5.3. Suppose $c=0$. After some simplification (6) becomes:

$$1<(1-i)q+(2-j)p<q-3.$$

Since i and j are positive, in order for the left inequality to hold, we must have $j=1$. Thus $1<(1-i)q+p<q-3$. Clearly, for $i=1$, we obtain a contradiction of the hypothesis that $p>q$, while for $i>2$, we have $1<(1-i)q+p\leq-2q+p\leq-3$, the last inequality following from the hypothesis on p and q. This contradiction rules out all remaining possibilities except when $i=2$ and $j=1$, which are the conditions we seek. So suppose $c>0$.

Case 1: $j=1$. If $i=1$ or 2, we get $k(S)>\frac{p-2}{p}+\frac{q-1}{q}$ contradicting the conditions of the theorem. Hence $i\geq3$. Using the left inequality of (6), namely

$$\frac{p-i}{p}+\frac{q-1}{q}+\frac{c}{pq}>\frac{p-1}{p}+\frac{q-2}{q}+\frac{1}{pq},$$

we obtain that $c>(i-1)q-p+1$. Consequently,

$$\begin{aligned}|A\cup C|&=(p-i)+c>(p-i)+(i-1)q-p+1=\\&=(q-1)i-q+1\geq(q-1)3-q+1=2q-2\geq p+1\end{aligned}$$

The result now follows by Lemma 3.1 (a).

Case 2: $j>1$. Then the right inequality of (6) yields $c>(i-1)q+(j-2)p+1$, and since $i\geq1$ and $j\geq2$, we conclude that $c>0$. From this we see that

$$\begin{aligned}|B\cup C|&=(q-j)+c>(q-j)+(i-1)q+(j-2)p+1=\\&=iq-2p+1+j(p-1)\geq iq-2p+1+2(p-1)=iq-1.\end{aligned}$$

Thus $|B\cup C|\geq iq$ and the result follows from Lemma 3.1 (a). □

Corollary 5.9 now follows from Theorem 5.8 and $(*)$.

Corollary 5.9. *Suppose $p>q>5$ are odd primes with $2p-4>2q\geq p+3$. Then*

$$\begin{aligned}W(\mathbb{Z}_{pq})\subseteq\{\frac{\lambda}{pq}\mid\ &2\leq\lambda\leq 2pq-q-2p+2,\\&2pq-2q-p+1\leq\lambda\leq 2pq-2q-p+2,\ \text{or}\ \ \lambda=2pq-p-q+1\}.\end{aligned}$$

Computer calculations (see [1]) based on Lemma 4.1 indicate if $2p-4>2q\geq p+3$, then the structure of $W(\mathbb{Z}_{pq})$ beyond the two gaps exhibited in Corollary 5.9 may not be as nice as that exhibited in Corollary 5.4. We demonstrate this by illustrating in the following table the values generated by Lemma 4.1 in $W(\mathbb{Z}_{pq})$ for

$q = 7$ and $p = 11, 13, 17, 19, 23, 29$. We use the notation $[n, m]$ to represent the set of integers x such that $n \leq x \leq m$.

p	values of x such that $\frac{x}{7p}$ is known to be in $W(\mathbb{Z}_{7p})$
11	$[2, 120] \cup [123, 127] \cup [130, 131] \cup \{137\}$
13	$[2, 133] \cup [135, 139] \cup [142, 145] \cup [149, 151] \cup [156, 157] \cup \{163\}$
17	$[2, 192] \cup [194, 199] \cup [201, 203] \cup [208, 209] \cup \{215\}$
19	$[2, 211] \cup [213, 217] \cup [200, 223] \cup [227, 229] \cup [234, 235] \cup \{241\}$
23	$[2, 275] \cup [279, 281] \cup [286, 287] \cup \{293\}$
29	$[2, 347] \cup [350, 353] \cup [357, 359] \cup [364, 365] \cup \{371\}$

5.5. **When p and q are Twin Primes.**

Theorem 5.10. *Let p and q be twin primes with $q = p - 2 > 5$. Let S be a zero free sequence in $\mathbb{Z}_{pq}$ with*

$$(\#) \qquad \frac{2pq - p - 3q + 3}{pq} \leq k(S) \leq \frac{2pq - 2p - q}{pq}.$$

Then $S = A \cup B$ where $A \subseteq \mathbb{Z}_p$, $B \subseteq \mathbb{Z}_q$, $\mid A \mid = p - 1$ and $\mid B \mid = q - 2$.

Proof. Let $S = A \cup B \cup C$ and write $|C| = c$. We proceed as in the proofs of Theorems 5.3 and 5.8. Now

$$(\dagger) \qquad k(S) = \frac{2p^2 - (i + j + 4)p + 2i + c}{p(p - 2)}.$$

For $1 \leq i < p$ and $1 \leq j < q$, let $S_{i,j}$ be a zero free sequence of $\mathbb{Z}_{pq}$ with $k(S_{i,j}) = \frac{p-i}{p} + \frac{(p-2)-j}{p-2}$. Thus, $k(S_{i,j})$ decreases as i or j increases. Set $B_u = \frac{2p(p-2)-2p-(p-2)}{p(p-2)}$ and $B_l = \frac{2p(p-2)-p-3(p-2)+3}{p(p-2)}$. Suppose that $c = 0$. Calculations with $(\dagger)$ indicate that $k(S_{1,1}) > B_u$, $k(S_{1,2}) = B_u$, and $k(S_{1,3}) < B_l$. For $i = 2$, $k(S_{2,1}) > B_u$ and $k(S_{2,2}) < B_l$. Further $k(S_{3,1}) < B_l$. Hence, if S satisfies $(\#)$, then $i = 1$ and $j = 2$, which is the desired result. If $c \neq 0$, then we again consider two cases.

Case 1: $j = 1$. From the previous calculations with $(\dagger)$, we obtain that $i \geq 3$. Hence

$$k(S) = \frac{2p^2 - (i + 5)p + 2i + c}{p(p - 2)} \geq \frac{2p^2 - 4p - p - 3p + 9}{p(p - 2)}$$

and thus

$$c \geq (p - 2)i - 3p + 9.$$

So,

$$\mid A \cup C \mid = (p - i) + c > (i - 2)p + 9 - 3i \geq p$$

for $i \geq 3$. The argument now follows from Lemma 3.1 (c).

Case 2: $j > 1$. Now

$$(p - i)(p - 2) + p(p - 2 - j) + c \geq 2p(p - 2) - p - 3(p - 2) + 3.$$

Thus

$$c \geq (i + j - 4)p + 9 - 2i$$

and hence

$$\begin{aligned}(p-2)-j+c \geq ((p-2)-j)+(i+j-4)p+9-2i \\ = (p-2)i+(p-2)(j-3)+j+1.\end{aligned}$$

Hence, if $j \geq 3$, then $\mid B \cup C \mid \geq (p-2)i$ and the argument follows from Lemma 3.1 (a). Thus, we merely need to deal with the case $j = 2$.

We have shown earlier that $k(S_{1,2}) = \frac{2p(p-2)-2p-(p-2)}{p(p-2)}$. Hence, we can assume that $i \geq 2$. Now, $\mid B \mid = q-2$ and by [4, Lemma 13] we have that $\mid \sum B \mid \geq q-2$. Thus $c < p+i$ for otherwise $\mid A \cup C \mid \geq (p-i)+(p+i) = 2p$ and we can appeal to Lemma 3.1 (a). If $i > 2$,

$$(p-i)(p-2)+p(p-4)+c < 2p^2-(i+5)p+3i.$$

If $i = 3$ the result follows. If $i > 3$ then $2p^2-(i+5)p+3i < 2p^2-8p+9$ and the result follows. To complete the proof, we need to show that the result holds for $i = j = 2$. In this case, Lemma 3.2 implies that

$$2p^2-(i+j+4)p+2i+c = 2p^2-8p+4+c \leq 2p^2-8p+6 < 2p^2-8p+9,$$

completing the proof. □

Corollary 5.11. *Let p and q be twin primes with $q = p-2 > 5$. Then,*

$$\begin{aligned}W(\mathbb{Z}_{pq}) \subseteq \{\frac{\lambda}{pq} \mid 2 \leq \lambda \leq 2pq-3q-p+3, \\ 2pq-2p-q+1 \leq \lambda \leq 2pq-2q-p+2, \text{ or } \lambda = 2pq-p-q+1\}.\end{aligned}$$

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